# AN ANALYTICK TREATISE OF Conick Sections,

And their Use for Resolving of EQUATIONS in Determinate and Indeterminate PROBLEMS.

BEING

The Posthumous Work of the Marquis De L'HOSPITAL, Honorary Fellow of the Academy Royal of Sciences.

*Made* English *by* E. STONE.



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A N

# TREATISE

O F

# THE TRANSLATOR TO THE READER.

CONICK SECTIONS *being Curves, by the Knowledge whereof ſuch vaſt Diſcoveries and Improvements have been made both in* Aſtronomy, Geometry, Algebra, *and* Natural Philoſophy; *and our Illuſtrious and Learned Author finding that moſt of their uſeful Properties and Problems might be demonſtrated ſhorter and eaſier than in any publick Treatiſe of them; as alſo that there was wanting a general and eaſy way of drawing the* Conick Sections, *being the Loci of Equations of two Dimenſions, with the manner of conſtructing Equations and ſolving Geometrical Problems by the Loci, did therefore write the following Treatiſe of* Conick Sections, *and*

*their Uſe in* Geometry *and* Algebra ; *but died about the Time he intended to publiſh it, and left the Manuſcript without a Preface ; he thinking, I ſuppoſe, that the Title thereof, together with his great Character in Things of this Nature, would well enough ſupply that Deficiency.*

*This Work is divided into Ten Books, whereof the firſt Three treat ſeparately of the* Parabola, Ellipſis, *and* Hyperbola. *In theſe Books are laid down the fundamental Properties and Problems of thoſe Curves demonſtrated algebraically, after a very ſhort and eaſy manner, from their moſt ſimple and natural Deſcriptions upon a Plane. They being particularly adapted to the Capacities of thoſe who are acquainted with but a few of the Propoſitions of the firſt ſix Books of* Euclid, *and withal have a ſmall Knowledge in the Rudiments of* Algebra.

*And becauſe thoſe Properties attending the Ellipſis, do alſo in ſome wiſe appertain to the Hyperbola, and oppoſite Sections, and even ſometimes to the Parabola ; therefore our Noble Author, in the Fourth Book of the Three* Conick Sections, *lays down his Propoſitions more general than in the Three former Books ; for the Propoſitions here do contain that Property of one, both, or all three of the Curves that was ſeparately ſhewn in thoſe Three Books. Here are alſo ſeveral other Properties and Problems not in the former Three Books : All demonſtrated after a ſhort and eaſy way.*

*The Fifth Book contains* the Compariſon of the Conick Sections, and their Segments with each other, *in ſeveral Propoſitions and Corollaries, with a ſhort Specimen of the Method of drawing Tangents to Curves, demonſtrated after as ſhort and eaſy manner as the Nature of the thing*

*thing will admit, from three or four fundamental Lemmata.*

*Altho' it be harder, especially for one who has not read the* 11*th Book of* Euclid, *to comprehend the Demonstrations of the fundamental Properties of the* Conick Sections *from the Cone, than from their Description upon a Plane by means of the noted Properties of the Foci, on account of the Consideration of Planes; yet our Noble Author, thinking that some People would rather like this Way, as being easier to those who are unacquainted with* Algebra, *and that his Work would be imperfect without it, has given us a Tract in his Sixth Book,* Of the Conick Sections consider'd in the Solid. *But herein has proceeded after a different Way from other Authors: For he considers the Ellipsis as the Section of a Cylinder, and easily demonstrates the Properties of its Diameters from the Cylinder; and afterwards, by supposing Cones to have Elliptick Bases instead of Circular ones, he easily proves the same Properties of the Parabola and Hyperbola.*

*The Seventh Book is* of Geometrick Loci, *which are treated of more easy and general than elsewhere, especially the Loci, which are* Conick Sections.

*The Eighth Book is* of Indeterminate Problems, *and shews the Use of the Seventh Book of Geometrick Loci in resolving them.*

*The Ninth Book is* of the Constructon of Equations. *Wherein by the Intersection of two Loci, you are taught how to construct any Equation of what Dimension soever.*

*Lastly, the Tenth Book* of Determinate Problems *is a farther Use of the Loci in solving them. Herein are some*

*ſome pleaſant and new* Theorems *touching the Inſcription of regular Polygons in a Circle, and a Demonſtration of Sir* Iſaac Newton's Theorem for finding the Unciæ.

*Now becauſe we have nothing in* Engliſh *on this Subject but what is lame and imperfect, and there are ſeveral Perſons who would be deſirous of reading this Treatiſe in* Engliſh, *I therefore preſent them with it; and here ſhall add to the ſame the following Propoſition, with an eaſy Demonſtration, which is of great Uſe in the Doctrine of Centripetal Forces, and conſequently in Aſtronomy; and for what Reaſon, I know not, omitted by our Illuſtrious Author.*

*ANY Parallelogram* (vide *Plate* 31. *Fig.* B.C.) *deſcribed about an Ellipſis, or between the Conjugate Hyperbola's, ſo that the four Points of Contact may be join'd by two Diameters* GH, IF *only, which therefore will be Conjugates, is equal to the Parallelogram deſcrib'd about the two Axes* Aa, Bb; *and conſequently all ſuch Parallelograms are equal to one another.*

From $F$, the Extremity of one Diameter, draw the Line $FD$ parallel to the other Diameter $GH$, (continu'd out in the oppoſite Hyperbola's) meeting the Axis (produced in the Ellipſis) in the Point $T$, and from $G$ the Extremity of the Diameter $GH$ draw the Line $GD$ parallel to the Diameter $IF$ meeting $DF$ in $D$. And from the Point $F$ let fall the Perpendicular $FP$ to the Axis $Aa$. Then $GD$, $DF$ will * touch the Ellipſis, and the Hyperbola's $bG$, $aF$ in $G$, $F$ the Extremities of the Conjugate Diameters, and ſo the Parallelogram $CGDF$ will be $\frac{1}{4}$ of that deſcrib'd about the Ellipſis, or between the Conjugate Hyperbola's, having the Condition mention'd in the Theorem. Therefore, if $CE$ be drawn perpendicular to $DF$ (produced in the Conjugate Hyperbola's) we are to prove that $CG\,(=DF) \times CE$ is $= Cb \times Ca = \frac{1}{4}$ of the Rectangle under the two Axes.

* *Art.* 56, 134. *and Def.* 13. *Book* III.

Call $Ca$, $t$; $Cb$, $c$; and $Cp$, $x$; then $\overline{Ca}^2\,(tt$ *) : $AP \times Pa$ $(tt - xx$ in Ellipſis, or $xx - tt$ in Hyperbola) : : $\overline{Cb}^2\,(cc) : \overline{FP}^2 = cc - \frac{ccxx}{tt}$. In the Ellipſis, or $\frac{ccxx}{tt} - cc$ in Hyp. And $\overline{CF}^2 = xx + cc - \frac{ccxx}{tt}$ in

* *Art.* 39, *and* 79.

in Ellip. or $xx-cc+\frac{ccxx}{tt}$ in Hyp. becauſe $FPC$ is a right-angl'd Triangle. Again, * $CP\,(x):Ca\,(t)::Ca\,(t):CT=\frac{tt}{xx}$. And $\overline{PT}^2=$ * *Art.* 57, *and* 121. $xx-2tt+\frac{t^4}{xx}$, and $\overline{FT}^2=xx+cc-2tt+\frac{t^4}{xx}-\frac{ccxx}{tt}$ in Ellip. or $xx-cc-2H+\frac{t^4}{xx}+\frac{ccxx}{tt}$ in Hyp. becauſe $FPT$ is a right-angl'd Triangle. Now the Triangles $FPT$, $CET$ are ſimilar, becauſe the Angles at $E$ and $P$ are right ones, and the Angle $ETC$ in the Ellip. common (but in the Hyperbola the Angle $ETC=PTF$) whence $\overline{FT}^2\,(xx\mp cc-2tt+\frac{t^4}{xx}\mp\frac{ccxx}{tt}):\overline{FP}^2\,(cc-\frac{ccxx}{tt}$ or $\frac{ccxx}{tt}-cc)::\overline{CT}^2\,(\frac{t^4}{xx}):\overline{CE}^2=\frac{t^4c^2}{t^4-t^2x^2\mp c^2x^2}$; Farther *, the Square of the Semi- * *Art.* 53, *and* 125. conjugate, *viz.* $\overline{CG}^2\,(=\overline{Ca}^2+\overline{Cb}^2-\overline{CF}^2$ in Ellip. or $=\overline{CF}^2+\overline{Cb}^2-\overline{Ca}^2$ in Hyp.) is $=tt-xx+\frac{ccxx}{tt}$ in Ellip. or $xx+\frac{ccxx}{tt}-tt$ in Hyp. and $\overline{CE}^2\times\overline{DF}^2\,(=\overline{DG}^2)$ is $=\frac{\mp t^6c^2+t^4c^4x^2\mp t^6c^2x^2}{t^6\mp t^2c^2x^2+t^4x^2}$, which is * * *Hyp.* $=\overline{Ca}^2\times\overline{Cb}^2=\mp ttcc$; as it evidently appears by multiplying the Denominator by $\mp ttcc$. And therefore $Ca\times Cb=CG\times CE$, and ſo $4\,CE\times EG=4\,Cb\times Ca$. W. W. D.

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# AN ANALYTICK TREATISE OF CONICK SECTIONS,

## And their Uſe for ſolving of EQUATIONS in Determinate and Indeterminate PROBLEMS.

## BOOK I.

### *Of the Parabola.*

### DEFINITIONS.

**I.**

IF the Rule *BC* be placed upon a Plane, together with the Square *GDO*, in ſuch manner, that *DG*, one of its Sides, lies along the Edge of that Rule; and if you take a Thread *FMO* equal in Length to *DO*, the other Side of the Square, and fix one End thereof to *O* the Extremity of the Side *DO*, and the other, in any Point *F* taken in the Plane on the ſame Side of the Rule as the Square: This being done, if you ſlide *DG*, the Side of the Square, along the Rule *BC*, and at the ſame time keep the Thread continually tight by means of the Pin *M*, with its Part *MO* cloſe to the Side of the Square *DO*: the Curve FIG. I.



*AMX*,

*AMX*, which the Pin deſcribes by this Motion, is one Part of a Parabola.

And if the Square be turn'd about, and moves on the other ſide of the fix'd Point *F*, the other Part *AMZ* of the ſame Parabola may be deſcrib'd after the like manner; ſo that the Line *XAZ* will be one and the ſame Curve, which is call'd a *Parabola*.

2.

The Line *BC* wherein the lower Edge of the fixed Rule *BC* touches the Plane (and the Side *DG* of the Square *GDO*) is called the *Directrix*.

3.

The fix'd Point *F*, is call'd the *Focus* of the Parabola.

4.

If the right Line *FE* be drawn from the fix'd Point *F* perpendicular to the Directrix *BC*, and meeting the Parabola in the Point *A*; then the Line *AF* infinitely produced towards *F*, is called the *Axis* of the Parabola.

5.

The Line *p* being quadruple of *AF*, is call'd the *Parameter* of the Axis.

6.

All Lines, as *MP*, drawn from Points of the Parabola perpendicular to the Axis, are called *Ordinates* to the Axis.

7.

All Lines, as *MO*, drawn from Points of the Parabola, parallel to the Axis, are called *Diameters* of the Parabola.

8.

A right Line meeting the Parabola but in one Point, and which both ways continued does not fall within the Parabola, is call'd a *Tangent* in that Point.

## COROLLARY I.

1. IT follows from the Definition of the Parabola, if a right Line *MF* be drawn from *M*, any Point thereof to the Focus *F*, and another Line *MD* perpendicular to the Directrix *BC*; the right Lines *MF*, *MD*, will always be equal between themſelves. For if the common Part *OM* be taken from *OD* the Side of the Square, and from the Thread *OMF*, which * is equal to it; then it is manifeſt, that

* *Def.* 1.

that the remaining Parts $MD$, $MF$, will always be equal to one another.

### COROLLARY II.

2. HENCE, if any right Line $KK$ be drawn parallel to the Directrix $BC$, and if from any Point $M$ of the Parabola, there be drawn alſo $MK$ perpendicular to that Line, and the right Line $MF$ to the Focus; then the Difference or Sum ($KD$) of the two right Lines $MF$, $MK$, will always be the ſame: *viz.* the Difference, when the Point $M$ falls below $KK$; and the Sum, when it falls above.

### COROLLARY III.

3. IT is evident, that $FE$ is biſected by the Parabola in the Point $A$. For when the Point $M$ falls on the Point $A$, the Line $MF$ falls on $AF$, and the Line $MD$ on $AE$, which conſequently will be equal to each other; ſince $MF$ is always equal * to $MD$, let the Point $M$ be any how taken in the Parabola. * *Art.* 1.

### COROLLARY IV.

4. HENCE you may perceive, how a Parabola $XAZ$ may be deſcribed, by having the Axis $AP$, (whoſe Origin or Vertex is $A$,) and its Parameter $p$ given. For having firſt aſſumed the Parts $AF$, $AE$ (on the Axis $AP$) on both Sides of the Origin $A$, each equal to the Parameter $p$, and drawn the indefinite Line $BC$ from the Point $E$ perpendicular to $FE$; then if the under Edge of a Rule be laid along the ſaid Line $BC$, (which is the Directrix) and the Parabola $XAZ$ be deſcrib'd, (as is directed in *Def.* 1.) by means of the Square $ODG$, and a Thread $FMO$, equal in Length to the Side $OD$, (one of whoſe Ends is fixed to the Focus $F$, and the other to $O$, the Extremity of the ſame Side,) it is manifeſt, that this Parabola is that required.

It is alſo manifeſt, that the longer the Side ($OD$) of the Square, and the Thread $OMF$ (which * muſt be equal to it) is, the longer likewiſe will the Portion of the Parabola deſcribed be; ſo that it may be augmented at pleaſure, by augmenting equally the Side ($OD$) of the Square and the Thread $OMF$. * *Def.* 1.

### COROLLARY V.

5. IF $MP$, an Ordinate to the Axis, be drawn from any Point $M$ in the Parabola, together with the right Line $MF$ to the Focus; it is manifeſt, that the Line $MF$ is $= AP + AF$, becauſe $MF = MD = AP + AE$, and * $AF = AE$. * *Art.* 3.

## PROPOSITION I.

### Theorem.

FIG. 1. 6. *THE Square of* MP, *any ordinate to the Axis* AP, *is equal to the Rectangle under the Parameter* p; *and the Part* (AP) *of the Axis taken from* A, *the Origin thereof to* P, *the Point of Concurrence of the Ordinate.*

*We are to prove that* $\overline{MP}^2 = p \times AP$.

If the given Quantity $AF$ be called $m$, and the indeterminate ones $AP$, $x$; $PM$, $y$; we shall have $MF =^* m + x$, and $PF = x - m$ or $m - x$, according as the Point $p$ happens below or above the Focus $F$. Now the right angled Triangle $MPF$, in both Cases, gives $\overline{MF}^2 (mm + 2mx + xx) = \overline{MP}^2 (yy) + \overline{PF}^2 (mm - 2mx + xx)$; from whence arises $4mx = yy$. Then since $p = 4m$ (by *Def.* 5.) we have also $yy = px$. *W. W. D.*

* *Art.* 5.

### *The Fundamental* COROLLARY I.

7. IT is now manifest, if the Parameter of the Axis ($AP$) be called $p$; each of its Parts $AP$, $x$; and every of the correspondent Ordinates $PM$, $y$; that we shall have always $yy = px$. And since this Property agrees to all Points of the Parabola, and determines the Position thereof with respect to its Axis $AP$; it follows, that the Equation $yy = px$ expresses perfectly the Nature of the Parabola with regard to its Axis.

### COROLLARY II.

FIG. 2. 8. IF there be drawn $MP$, $N\mathcal{Q}$, any two Ordinates to the Axis $AP$, their Squares are to each other as the Parts $AP$, $A\mathcal{Q}$ of the Axis, taken from its Origin $A$, to $P$ and $\mathcal{Q}$, the Points of Concurrence of the same Ordinates. For $^* \overline{PM}^2 : \overline{\mathcal{Q}N}^2 :: p \times AP : p \times A\mathcal{Q} :: AP : A\mathcal{Q}$.

* *Art.* 6, 7.

### COROLLARY III.

9. IF the Line $M\,P\,M$ be drawn thro' any Point $P$ in the Axis parallel to its Ordinates; that Line will meet the Parabola in only two Points $M$, $M$, equally distant on both Sides from the Point $P$; for in order that the Points $M$ and $M$ be in the Parabola, it

is neceſſary * that the Squares of each $P M$, taken on both Sides the Point $P$, be equal to the ſame Rectangle $p x$. * *Art.* 7.

## COROLLARY IV.

10. BEcauſe * $y y$ is $= p x$, it follows that the greater $A P (x)$ is, the more likewiſe will the Ordinates $P M (y)$ increaſe on both Sides the Axis $A P$, even to Infinity; and contrariwiſe, the leſſer $A P (x)$ is, the leſſer likewiſe will $P M (y)$ be: So that when $A P (x)$ is nothing, both the $P M' (y)$ taken on each Side the Axis $A P$, will be nothing; that is, when the Point $P$ falls in $A$, the two Points of Concurrence $M$ and $M$ will alſo coincide in the Point $A$. From whence it is manifeſt, * *Art.* 7.

1°. That if the Line $L L$ be drawn parallel to the Ordinates thro' $A$ the Origin of the Axis, it will be a Tangent in $A$.

2°. That the Parabola infinitely extends itſelf more and more on each Side the Axis $A P$, beginning from the Origin $A$; and likewiſe that every Parallel to the Axis, as $L M$, meets the Parabola but in one Point, $M$ only, and falls within the ſame; ſince its Diſtance from the Axis remains always the ſame.

## COROLLARY V.

11. IF the Line $M L$ be drawn from any Point $(M)$ of the Parabola, parallel to the Axis $A P$, meeting the Parallel $(AL)$ to its Ordinates in $L$; it is manifeſt, if the Ordinate $M P$ be drawn, that $A L = P M (y)$, and $M L = A P (x) \frac{y y}{p}$, becauſe * $p x = y y$; whence the right Lines $M L\left(\frac{y y}{p}\right)$, $M L\left(\frac{y y}{p}\right)$ taken on both Sides the Axis $A P$, are equal to each other, when the Points $L$, $L$, are equally diſtant from the Point $A$; and therefore if any right Line $M M$ terminated by the Parabola be biſected by the Axis in $P$, it will be parallel to the Line $L L$, that is, it will be an Ordinate on both Sides to the Axis. For when the Parallels $M L$, $M L$, are drawn to the Axis $A P$, it is manifeſt that $L L$ will be cut into two equal Parts in $A$, becauſe $M M$ is ſo cut in $P$. Therefore the right Lines $M L$, $M L$, will be equal between themſelves, as we have prov'd; and conſequently the Line $M M$ will be parallel to $L L$. * *Art.* 7.

COROL-

### COROLLARY VI.

12. HEnce it follows that all the Perpendiculars (*M P M*) to the Axis, terminating on both Sides in the Parabola, † are biſected in *P*; and that the Axis divides the Parabola into two equal Parts, ſimilarly ſituate on each Side thereof. For if the two Parts of the Plane on which the Parabola is drawn, on each Side the Axis, be ſuppoſed to be folded together, it is manifeſt that they will exactly agree, or fall upon one another.

† *Art.* 9.

## PROPOSITION II.

### Theorem.

FIG. 3. 13. *IF any right Line* A M *be drawn from* A, *the Origin of the Axis* A P, *in either of the Angles* P A L, P A L, *made by the Axis, and the Line* L L *which is parallel to the Ordinates; I ſay that Line will meet the Parabola* M A M *in ſome other Point* M.

Take *A G* upon the Line *A L*, on either Side the Point *A*, equal to *p* the Parameter of the Axis, and draw *G F* parallel to the Axis, meeting the Line *A M* (produced if neceſſary) in the Point *F*. Moreover, take the Part *A L*, upon the Line *A L* (on the ſame Side of the Axis as *A M* is) equal to *G F*, and draw *L M* parallel to the Axis; then I ſay the Point *M* wherein *L M* meets *A M*, will be in the Parabola *M A M*.

For drawing *M P* parallel to *A L* the ſimilar Triangles *F G A*, *A P M* will give this proportion, *F G* or *A L* or *P M* : *G A* : : *A P* : *P M*; and therefore $\overline{PM}^2 = G\ A\ (p) \times P\ A$. And ſo the Line *P M* will * be an Ordinate to the Axis *A P*. *W. W. D.*

* *Art.* 7.

### COROLLARY I.

14. HEnce, if *A P* the Axis of a Parabola, as alſo its Parameter, be given, and any right Line *A M* be drawn, from *A* the Origin of the Axis, in either of the Angles *P A L*, *P A L*, made by the Axis *A P* and the Line *L L*, which is parallel to the Ordinates; the Point *M* wherein the Line *A M* meets the Parabola *M A M* may be found.

### COROLLARY II.

15. IT is manifeſt * that no other Line but $L A L$ which is parallel to the Ordinates to the Axis $A P$, can be a Tangent to the Parabola $M A M$ in $A$ the Origin of the Axis; becauſe there is no Line but that only, which being drawn thro' the Point $A$, and both ways continued, but what will fall within the Parabola. * *Art.* 10. 13.

### DEFINITIONS.

9.

If thro' any Point $M$ in a Parabola, there be drawn a Diameter $M O$, an Ordinate ($M P$) to the Axis $A P$, and a right Line $M T$ cutting the Axis produced (beyond $A$) in the Point $T$, ſo that $A T$ be equal to $A P$: Then all right Lines, as $N O$, drawn from any Points in the Parabola parallel to $M T$, and terminating in the Diameter $N O$, are call'd *Ordinates* to that Diameter, FIG. 4. & 5.

10.

If the Line $q$ be taken a third proportional to $A T$, $M T$; this Line $q$ is call'd the *Parameter* of the Diameter $M O$.

### COROLLARY I.

16. IF either of the indeterminate Lines $A P$ or $A T$, be called $x$; then it is manifeſt that $\overline{M T}^2 = q x$, becauſe $A T (x) : M T :: M T : q$.

### COROLLARY II.

17. BEcauſe $M P T$ is a right angl'd Triangle, the Square $\overline{M T}^2 (q x)$ is $= \overline{P T}^2 (4 x x) + \overline{M P}^2$ * $(p x)$; whence, dividing by $x$, we have $q = 4 x + p$. * *Art.* 7.

That is, $q$ the Parameter of any Diameter $M O$, exceeds $p$ the Parameter of the Axis by quadruple the Axis $A P (x)$.

### COROLLARY III.

18. IF the right Line $M F$ be drawn from the Point $M$ to the Focus $F$, we ſhall have * $M F = A P + A F$. But becauſe (by *Def.* 5.) the Parameter of the Axis is $p = 4 A F$, the Parameter of the Diameter $M O$ will be † $q = 4 A P + 4 A F$. And therefore $q$ the Parameter of any Diameter $M O$, will be four times the Line $M F$, drawn from $M$ the Origin thereof to the Focus $F$. * *Art.* 5. † *Art.* 17.

PROPO-

## PROPOSITION III.

### Theorem.

FIG. 4. 5. 16. *THE Square of* (O N) *any Ordinate to the Diameter* M O, *is equal to the Rectangle under the Parameter* q; *and* M O, *that Part of the Diameter taken from its Origin* M *to* O, *the Point of Concurrence of the Ordinate.*

*We are to prove that* $\overline{ON}^2 = q \times MO$.

Draw $N\mathcal{Q}$, an Ordinate, to the Axis $AP$, meeting the Diameter $MO$ in the Point $R$, and draw $OH$ parallel to $MP$; then call the given Quantities $AP$ or $AT$, $x$; $PM$ or $R\mathcal{Q}$, $y$; and the indeterminate ones $OR$ or $\mathcal{Q}H$, $a$; $MO$ or $PH$, $b$; and then the similar Triangles $TPM$, $ORN$ will give this Proportion, *viz.* $TP\ (2x) : PM\ (y) :: OR\ (a) : RN = \frac{ay}{xx}$. This being laid down:

Becauſe (*Fig.* 4.) $N\mathcal{Q} = R\mathcal{Q}\ (y) - RN\left(\frac{ay}{2x}\right)$, or $RN\left(\frac{ay}{2x}\right) - R\mathcal{Q}\ (y)$, and $A\mathcal{Q} = AH\ (x+b) - H\mathcal{Q}\ (a)$, when the Point $N$ falls on the ſame Side of the Axis, in reſpect of the Diameter $MO$; and contrariwiſe (*Fig.* 5.) $N\mathcal{Q} = R\mathcal{Q}\ (y) + RN\left(\frac{ay}{xx}\right)$, and $A\mathcal{Q} = AH\ (x+b) + H\mathcal{Q}\ (a)$, when it falls on the other Side: We ſhall have $\overline{\mathcal{Q}N}^2 = yy \pm \frac{ayy}{x} + \frac{aayy}{4xx}$, and $A\mathcal{Q} = x + b \pm a$, *viz.* — in the firſt Caſe, and + in the ſecond. But $AP$ * *Art.* 8. $(x) : A\mathcal{Q}\ (x + b \pm a) :: \overline{PM}^2\ (yy) : \overline{\mathcal{Q}N}^2 = yy + \frac{byy}{x} \pm \frac{ayy}{x}$. Whence by comparing the two Values of $\overline{\mathcal{Q}N}^2$ together, we ſhall have this Equation, *viz.* $yy + \frac{byy}{x} \pm \frac{ayy}{x} = yy \pm \frac{ayy}{x} + \frac{aayy}{4xx}$; and ſtriking out $yy \pm \frac{ayy}{x}$, from each Side of the Equation, dividing by $yy$, and multiplying by $4xx$, we ſhall have $\overline{OR}^2\ (aa) = 4bx$; but becauſe the Triangles $MPT$, * *Art.* 16. $NRO$ are ſimilar, therefore $\overline{PT}^2\ (4xx) : \overline{OR}^2\ (4bx) :: \overline{MT}^2 \pm (qx) : \overline{ON}^2 = bq = q \times MO\ (b)$. *W. W. D.*

### *A General* COROLLARY. I.

20. IT is manifeſt (by the laſt *Prop.*) that what has been demonſtrated in *Prop.* 1. with regard to the Axis $AP$, its Ordinates ($PM$), and Parameter $p$, is equally true of any Diameter $MO$, its Ordi-

Ordinates (*O N*) and Parameter *q*. Now ſince the 7th, 8th, 9th, 10th, 11th, 12th, 13th, 14th and 15th Articles ariſe from Prop. 1. and are true, whether the Angles *APM* be right ones, or not; it follows, that if the Line *A P* in thoſe Articles be ſuppoſed any other Diameter inſtead of the Axis, whoſe Ordinates are the right Lines *P M*, *Q N*, and Parameter the Line *p*, thoſe Articles according to this Suppoſition, will be ſtill true; for their Demonſtration remains always the ſame, and there is nothing more neceſſary for making this appear, but reading them over again, and uſing the Word Diameter for Axis.

## COROLLARY II.

21. BEcauſe the 10th and 15th Articles are equally true, whether the Line *A P* be the Axis, or any other Diameter, as *M O*; therefore the Line *M T* parallel to *O N* the Ordinates to that Diameter, touches the Parabola in *M*, and no other Line can touch it in that Point. FIG. 4. & 5.

Whence there can be drawn but one right Line from a given Point in a Parabola, to touch the ſame.

## COROLLARY III.

22. HENCE it is manifeſt (according to Def. 9.) if (*M P*) an Ordinate to the Axis *A P* be drawn from any Point *M* of a Parabola, as alſo another right Line *M T* cutting the Axis (produced beyond *A*) ſo, that *A T* be equal to *A P*; that this Line *M T* will touch the Parabola in *M*. And contrariwiſe, if the Line *M T* touches the Parabola in *M*, and *M P* an Ordinate to the Axis be drawn; then the Parts of the Axis *A T*, *A P* will be equal to one another.

## COROLLARY IV.

23. IF you ſuppoſe in the 9th and 10th Definitions, as alſo in the laſt Propoſition, the Line *A P* to be any other Diameter inſtead of the Axis, whoſe Ordinates are the right Lines *F M*, *Q N*; that Propoſition will yet appear true; becauſe it may be demonſtrated in the ſame Manner as before, as is evident by contemplating the 6th Figure, where the ſimilar Triangles give the ſame Proportions as in the Caſe of the Axis. FIG. 6.

Whence it follows, 1. That the laſt Corollary ought ſtill to take place, when the Line *A P* is any other Diameter, as well as the Axis. 2. That according to that Suppoſition, the Diameter *M O* may be the Axis; and ſo the Axis may be eſteem'd ſuch a Diameter that makes right Angles with the Ordinates thereof.

## PROPOSITION IV.

### Theorem.

FIG. 7. 24. IF M P *an Ordinate to the Axis be drawn thro' any Point* M *in a Parabola, and alſo* M G *a perpendicular to the Tangent* M T *which paſſes through the Point* M; *I ſay the Part of the Axis* P G *will always be equal to the half of* p, *the Parameter thereof.*

*We are to prove that* P G *is* $= \frac{1}{2}$ p.

For becauſe *T P M*, *T M G*, are right Angles, therefore *T P* $(2x)$ : *P M* $(y)$ :: *P M* $(y)$ : *P G* $= \frac{y\,y}{2\,x} = \frac{1}{2}\,p$, by ſubſtituting $p\,x$ * the Value of $y\,y$ for the ſame.

* *Art.* 7.

## PROPOSITION V.

### Theorem.

FIG. 7. 25. IF *the right Line* M F *be drawn from the Focus* F *to any Point* M *in a Parabola, as alſo a Diameter* M O, *and a Tangent* T M S; *I ſay the Angles* F M T, O M S, *made by the Tangent* T M S, *and right Line* M F *on one Side, and the Diameter* M O *on the other, are equal to each other.*

For if the Axis *A P* be drawn, meeting the Tangent *T M S* in T, and *M P* an Ordinate to the Axis; we ſhall have † *T A* + *A F* or *T F* = *A P* + *A F* or * *M F*. Therefore *T F M* will be an Iſoſceles Triangle; and conſequently the Angle *F T M* or *O M S* which is equal to it, will be equal to the Angle *F M T*. *W. W. D.*

† *Art.* 22.

* *Art.* 8.

### COROLLARY.

26. HENCE it is evident that the Tangent *T M S* infinitely produced both ways from *M*, the Point of Contact, leaves the Parabola wholly next to its Focus *F*. And ſince this every where happens in whatſoever place of the Parabola the Point of Contact *M* be taken; it follows, that the Parabola being extended never ſo much, is Concave next to its Focus *F*.

PROP.

## PROPOSITION VI.

### Problem.

27. *A Diameter* A P, *together with the Tangent* L A L, *paſſing through* A *the Origin thereof, as alſo its Parameter being given; to find a Diameter* B Q, *its Origin* B, *and Parameter, which ſhall contain an Angle either way with its Ordinates, equal to a given Angle* K. FIG. 8,9.

Draw the right Line *A E* through *A* the Origin of the given Diameter, making the Angle *P A E* with this Diameter equal to the given Angle *K*, and find *M* a Point * in the ſaid *A E* (produced on the other Side of *A*, when it does not fall in either of the Angles *P A L*, *P A L*,) which may meet the Parabola. This being done, draw *Q D* through *Q* the middle Point of the Line *A M*, parallel to the Diameter *A P*, meeting the Tangent *A L* in the Point D, and biſect *Q D* in *B*. I ſay, the Line *B Q* is the Diameter requir'd, and the Origin thereof is the Point *B*, and its Parameter is a third proportional to *B Q*, and *Q A*. * *Art.* 14. 20.

For 1. becauſe the Line *A M* is divided by the Diameter *B Q* into two equal Parts in the Point *Q*, the ſaid Line will be *an Ordinate both ways to that Diameter; and ſince the Lines *B Q*, *A P*, are parallel to each other, the Angle *B Q A*, made by the Diameter *B Q*, and its Ordinates *Q A*, will be equal to the Angle *P A M*, equal to the given Angle *K* or its Complement to two right Angles. 2. The Point *B* the middle of the Line *Q D* will be † the Origin of that Diameter. 3. The Parameter of the Diameter *B Q* is ‖ a third Proportional to *B Q*, *Q A*. * *Art.* 11. 20. † *Art.* 22, 23. ‖ *Art.* 19.

When rhe given Angle *K* is not a right Angle, it is manifeſt that two different Lines *A E* can be drawn on each ſide the Diameter *A P*, making Angles with that Diameter equal to the given Angle *K*; and ſo we may always have two different Solutions; but you muſt obſerve that when *A E* one of the two Lines falls in the Tangent, then the Diameter *A P* itſelf will ſatisfy the Queſtion. But when the Angle *K* is not a right one, ſince there can be drawn but one Line *A E* only, at right Angles to the Diameter; therefore in this Caſe the Problem will have but one Solution; and the Diameter ſought will be the Axis. FIG. 8.

It muſt be obſerv'd, that the two Diameters *B Q*, *B Q*, which anſwer the Problem, when the Angle *K* is not a right one, are alike ſituate on each ſide the Axis *A P*, and their Parameters are equal: This appears from the Conſtruction itſelf, if the given Diameter *A P* be ſuppoſed the Axis, and two Lines *A E*, *A E*, are drawn on each FIG. 10.

Side thereof. For becauſe the Right-angled Triangles $A\,L\,M$, $A\,L\,M$, and $A\,D\,Q$, $A\,D\,Q$, are equal, and ſimilar to one another, the Lines $A\,D$, $A\,D$; $D\,Q$, $D\,Q$; their halfs $B\,Q$, $B\,Q$; and the Ordinates $Q\,A$, $Q\,A$ will be equal to one another; and conſequently * the Parameters are ſo too.

* *Art.* 19.

### COROLLARY.

28. HENCE it is manifeſt, 1. That there is but one Diameter only which is at right Angles with its Ordinates; and ſo a Parabola has but one Axis. 2. That there may be always found two different Diameters, making Angles with their Ordinates equal to an Angle given, provided the Angle be not a right one; and that the ſaid two Diameters are alike ſituate on each ſide the Axis, and have equal Parameters.

## PROPOSITION VII.

### Problem.

29. *A Diameter, the Parameter thereof, and a Tangent paſſing through its Origin, being given, to deſcribe the Parabola by a continued Motion.*

FIG. 11. If the Diameter given be the Axis, the Parabola may be deſcribed by *Art.* 4. but if it be not, let $M\,O$ be the Diameter given, and $T\,M\,S$ the Tangent drawn through $M$ the Origin thereof.

Aſſume $M\,D$ in the Diameter $M\,O$ (continued out beyond $M$,) equal to a fourth Part of the Parameter given; and draw $D\,E$ of an indefinite Length perpendicular to $M\,D$. Again, draw $M\,F$, making the Angle $F\,M\,T$ with the Tangent $T\,M\,S$ equal to the Angle $O\,M\,S$, and aſſume $M\,F$ equal to $M\,D$. This being done, if a Parabola be deſcrib'd by Def. 1. with the Directrix $D\,E$, and the Focus $F$; I ſay, this will be that requir'd.

For, 1. Becauſe the Line $M\,O$ is perpendicular to the Directrix $D\,E$, it will be parallel to the Axis; and conſequently will be a Diameter by Def. 7. 2. The Line $T\,M\,S$ will be * a Tangent in $M$. 3. The Parameter of the Diameter $M\,O$ will be † quadruple of $M\,F$.

* *Art.* 25.

† *Art.* 18.

### *Another Way.*

FIG. 12. Let $A\,P$ be the given Diameter, and $L\,A\,L$ the Tangent paſſing through $A$ the Origin thereof.

Aſſume

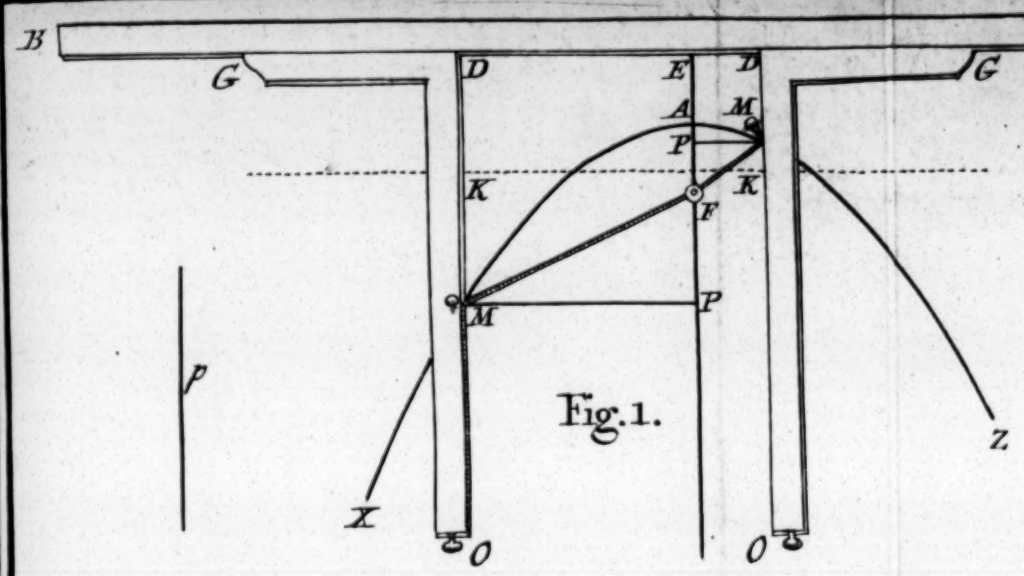
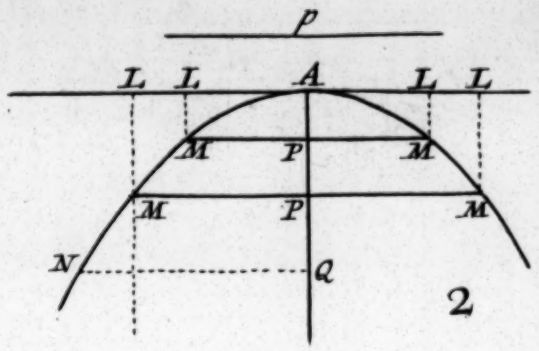
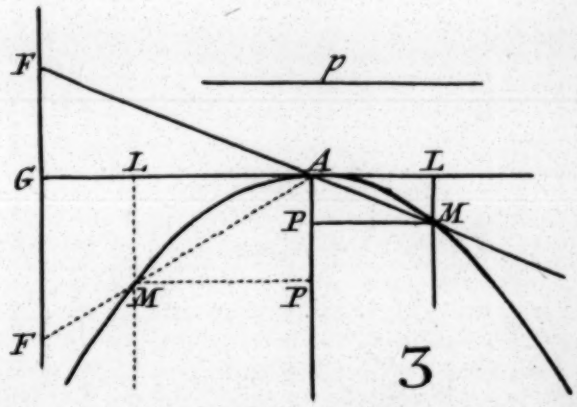


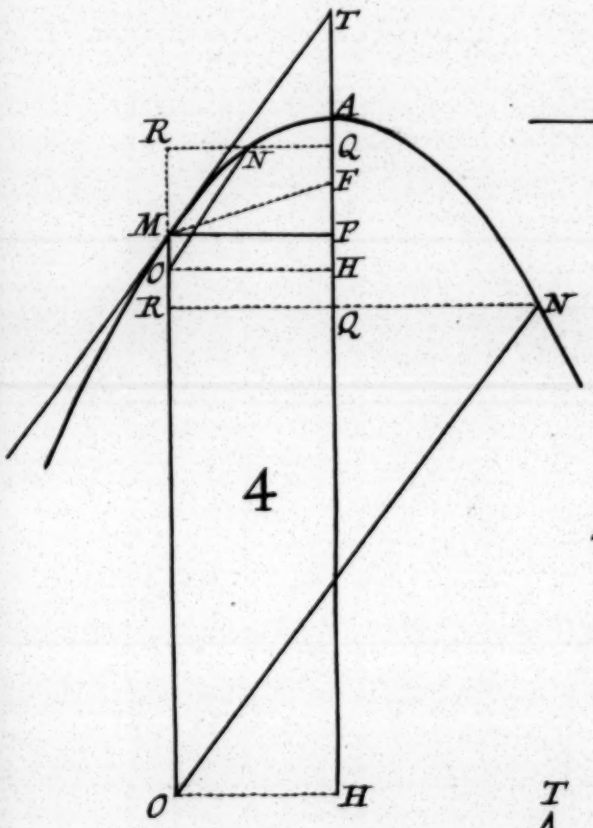

Fig. 1.



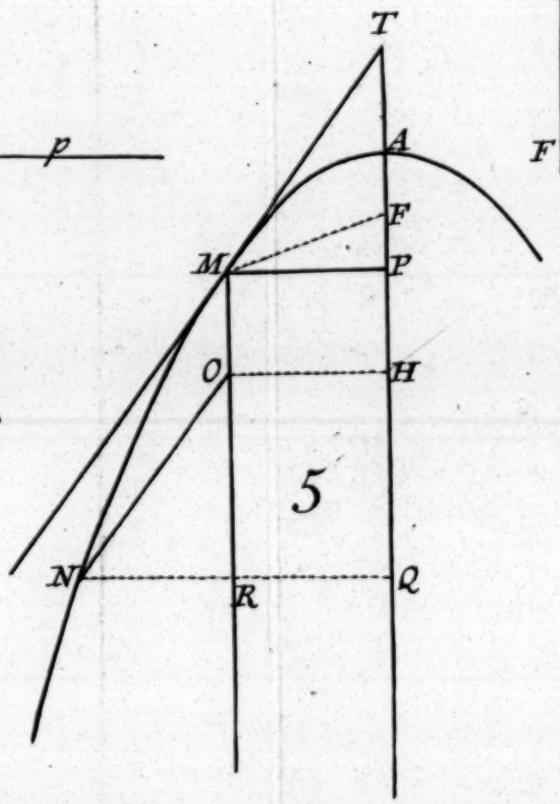

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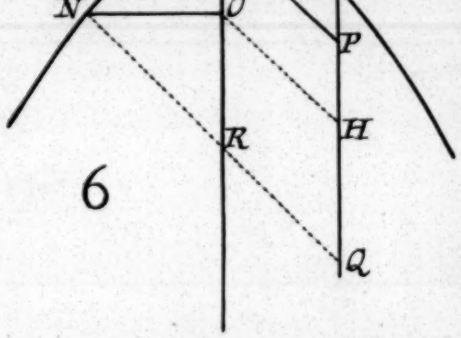

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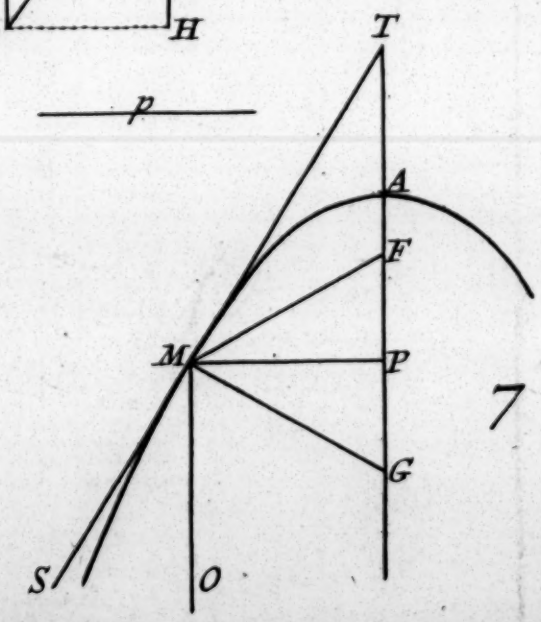

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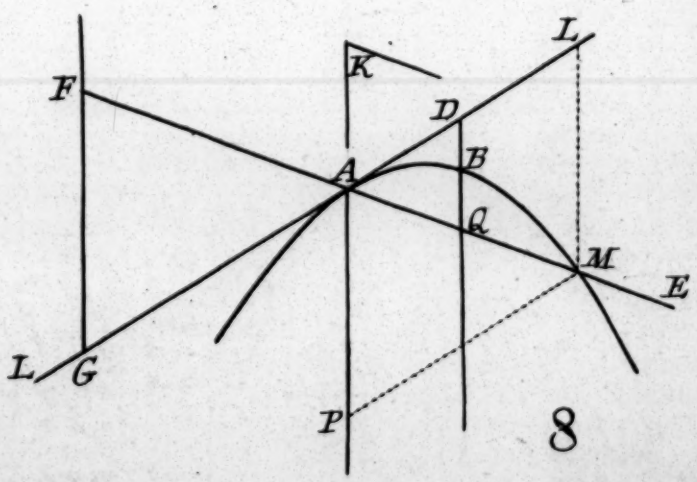

5




6




7




8

Aſſume $AG$ in the Diameter $AP$ (continu'd out beyond $A$) equal to the given Parameter, and draw an indefinite right Line $DGD$, making with $AG$ the Angle $AGD$ equal to the Angle $GAL$ taken on the ſame Side; then if an indefinite right Line $DM$ moves along the Line $DG$ always parallel to $AG$; and at the ſame time the Extremity $D$ thereof carries along with it the Side ($DA$) of the Angle $DAM$, equal to $GAL$, whoſe Vertex is moveable about the fixed Point $A$: I ſay, the Point $M$, the continual Interſection of the Line $DM$ and the Side $AM$, will by this Motion deſcribe the Parabola ſought.

For if $MP$ be drawn parallel to $AL$, the Lines $MP$, $GD$ will be equal to one another; ſince the Angle $APM$ or $GAL$ being equal to the Angle $AGD$, they are equally inclin'd between the Parallels $GP$, $DM$. Now the Triangles $AGD$, $MPA$, are ſimilar: for the Angle $MPA$ or $GAL$ is equal to the Angle $AGD$; and the Angle $PMA$ or $MAL$, equal to the Angle $GAD$, becauſe if the ſame Angle $DAL$ be taken from each of the equal Angles $GAL$, $DAM$, the remaining Angles are equal. Therefore we ſhall have $AG : GD$, or $PM :: PM : AP$; and ſo $GA \times AP = \overline{PM}^2$; whence it is manifeſt *, that $PM$ is an Ordinate to the Diameter $AP$, whoſe Origin is the Point $A$, the Line $LAL$ is a Tangent in that Point, and the Line $AG$ the Parameter to the ſame. *W. W. D.* * *Art.* 19, *and* 21.

If the Diameter $AP$ be the Axis; then the Lines $GD$, $AL$ will be parallel, and the Demonſtration will become more eaſy; for it is immediately perceiv'd that $GD$ is equal to $PM$, and that the Right-angled Triangles $AGD$, $MPA$ are ſimilar; whence $AG : GD$, or $PM :: PM : AP$. And therefore $AG \times AP = \overline{PM}^2$, &c. FIG. 13.

## PROPOSITION VIII.

### Problem.

30. *A Diameter* AP, *the Parameter thereof, and the Tangent* AL *paſſing thro'* A *the Origin of that Diameter, being given; to find any Number of Points of the Parabola, or* (*which is all one*) *to deſcribe it thro' ſeveral Points.*

Aſſume $AG$ in the Diameter $AP$ (continu'd out beyond $A$) equal to the given Parameter, which biſect in the Point $D$, and draw an indefinite right Line $AF$ perpendicular to $AG$; then about the Point $C$, taken every where in $DA$, produced indefinitely towards $A$, as a Centre, with the Radius $CG$, deſcribe $PF$ an Arc of a Circle, cutting the Diameter $AP$, and its Perpendicular $AF$, in two Points $P$, $F$. FIG. 14.

And

And draw $MPM$ through the Point $P$ parallel to the Tangent $AL$, and in this Line take the Parts $PM$, $PM$ both ways, each equal to $AF$: and then the Points $M$, $M$ will be two of thoſe ſought. And after the ſame manner may any Number of Pairs of Points $M$, $M$ be found; through which, if a Curve Line be drawn, it will be the Parabola ſought.

For all the Arcs ($PF$) that paſs through the ſame Point $G$, and whoſe Centres are in the Line $GA$, (produced if it be neceſſary) will have the Lines $GP$ for their Diameters; and therefore by the Property of the Circle we ſhall always have $\overline{AF}^2 = GA \times AP$. But every $PM$ is equal * to its Correſpondent $AF$; and moreover, parallel to the Tangent $AL$, paſſing through $A$, the Origin of the Diameter $AP$; and conſequently $PM$ will be † an Ordinate to that Diameter; therefore the Parabola requir'd muſt paſs through all the Points $M$, $M$, found as before directed.

* *Hyp.*

† *Art.* 19, *and* 21.

It is evident, that one may err in drawing the Parts of the Parabola, that join the Points found: But this Error will not be ſenſible, when the Points are very near to each other. Thoſe who have occaſion to often deſcribe the Conic Sections, commonly prefer the Way of deſcribing them through many Points; becauſe the Inſtruments that have been invented for deſcribing them by a continued Motion, being Compound ones, are often faulty, and not exact enough in Practice.

*Another Way.*

FIG. 15. Draw the indefinite right Line $LE$ through $L$, any Point in the Tangent $AL$, parallel to the Diameter $AP$; and in this Line and the Diameter $AP$ (continued out beyond $A$) aſſume at pleaſure the equal Parts $LE$, $EE$, $EE$, &c. $AF$, $FF$, $FF$, &c. and take the Point $M$ in $LE$, ſo that $LM$ be a third Proportional to the Parameter of the Diameter $AP$ given, and $AL$ the Part of the Tangent. Then if the Lines $AE$, $AE$, $AE$, &c. $MF$, $MF$, $MF$, &c. be drawn from the Points $A$ and $M$: I ſay, the Points of Interſection $N$, $N$, $N$, &c. of every $AE$, and its Correſpondent $MF$, will be all in the Parabola requir'd.

For if the Lines $MP$, $N\mathcal{Q}$ are drawn through the Points $M$ and $N$, parallel to the Tangent $AL$, and you call $AP$, $x$; $PM$ or $AL$, $y$; $A\mathcal{Q}$, $u$; $\mathcal{Q}N$, $z$; the ſimilar Triangles $N\mathcal{Q}A$, $ALE$ and $MPF$; $N\mathcal{Q}F$ will give theſe two Proportions, *viz.* $\mathcal{Q}N\,(z) : \mathcal{Q}A\,(u)\; AL\,(y)$. $LE$ or $AF = \frac{uy}{z}$. And $MP\,(y) : PF$ or $PA + AF\,(x + \frac{uy}{z}) :: N\mathcal{Q}\,(z) : \mathcal{Q}F$ or $\mathcal{Q}A + AF\,(u + \frac{uy}{z})$. And multiplying the

Means and Extremes, we ſhall from this Equation $uy + \frac{uyy}{z} = xz + uy$; and ſtriking out $uy$ from both Sides of the Equation, and multiplying by $z$, we ſhall have $uyy = xzz$, which may be reduced to this Proportion $AP(x) : A\mathcal{Q}(y) :: \overline{MP}^2 (yy) : \overline{N\mathcal{Q}}^2 (zz)$. But by Conſtruction, the Square of $AL$ or $PM$ is equal to the Rectangle under $AP$, the Part of the given Diameter, and the Parameter thereof. And therefore the Line $PM$ will be * an Ordinate to the Diameter $AP$; and ſo $\mathcal{Q}N$ will be † another. Conſequently the Point $N$ will be one Point of the Parabola, that falls on one Side of the Diameter $AP$: And to find the Points on the other Side, you need only take the equal Parts $LE$, $EE$, &c. $AF$, $FF$, &c. in the indefinite right Lines $LE$, $AF$, on the other Side of the Points $L$, $A$.

* *Art.* 19, *and* 21.

† *Art.* 8, *and* 20.

If inſtead of the Parameter of the Diameter $AP$, which is here ſuppos'd to be given, we have $M$ one of the Points in the Parabola, which often happens: Then you muſt draw the indefinite Line $LE$ thro' that Point parallel to the Diameter $AP$, and proceed afterwards as above.

*The End of the Firſt Book.*

BOOK

# BOOK II.

## *Of the Ellipſis.*

### DEFINITIONS.

1.

FIG. 16. IF *F*, *f* be two Points, or Nails fix'd in a Plane, and a Thread *FMf* be put about them, whoſe Length muſt be more than the Diſtance *Ff*; and if then you put the Pin *M* to the Thread, ſo as always to keep the ſame ſtrain'd tight; and move the Pin round theſe two Points, till it be come to the ſame Place from whence it went; the ſaid Pin, by this Motion, will deſcribe a Curve, which is called an *Ellipſis*.

2.

The Points *F*, *f*, are called the *Foci* of the Ellipſis.

3.

The Line *Aa*, which paſſes thro' the two Foci *F*, *f* terminating both ways in the Ellipſis, is called the *firſt* or *great Axis* [or *tranſverſe Axis*.]

4.

The Point *C*, which divides the firſt Axis *Aa* in half, is called the *Centre* of the Ellipſis.

5.

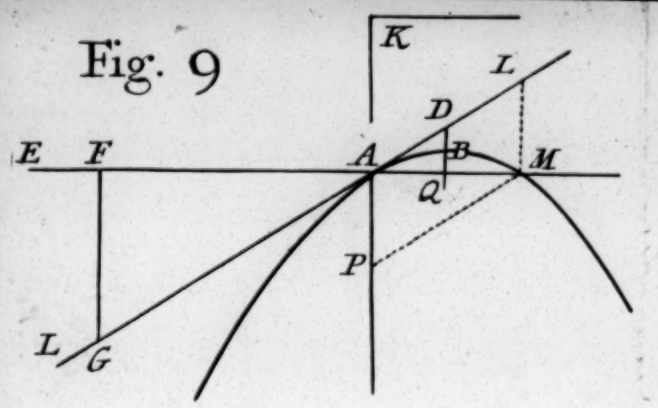
The Line *Bb* drawn through the Centre *C*, perpendicular to the firſt Axis *Aa*, and terminating both ways in the Ellipſis, is called the *ſecond* or *ſmall Axis* [or Conjugate Axis.]

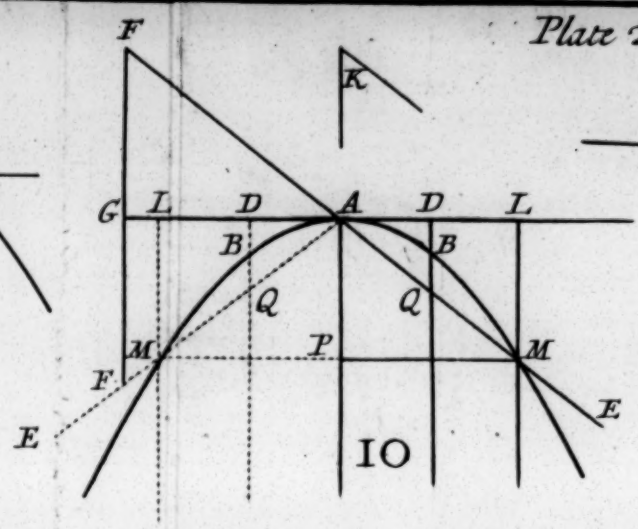
6.

The two Axes *Aa*, *Bb*, are called together, *Conjugate Axes*: So that the firſt Axis *Aa* is ſaid to be a Conjugate to the ſecond *Bb*; and reciprocally the ſecond *Bb*, a Conjugate to the firſt *Aa*.

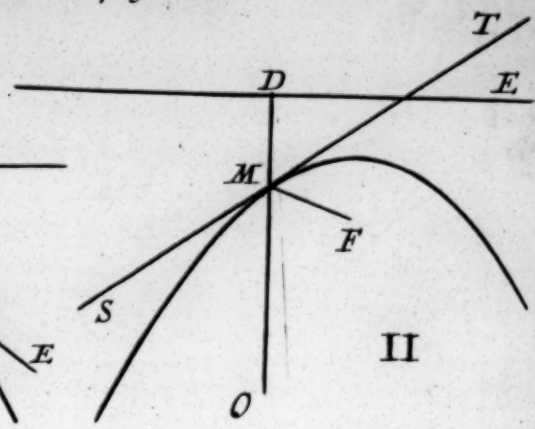
7. The

Fig. 9

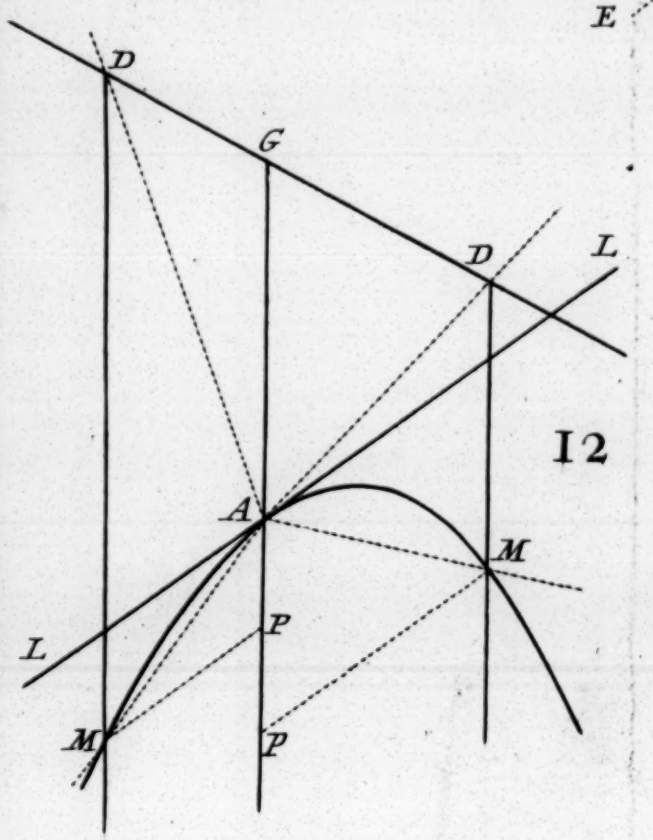


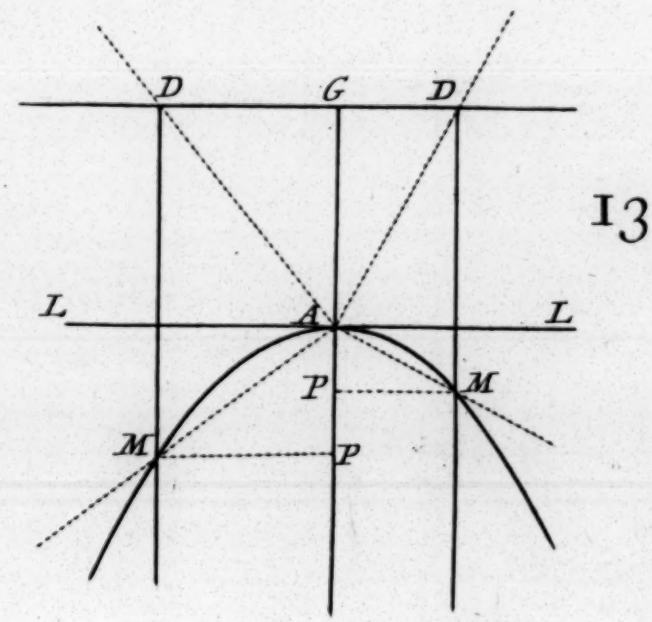

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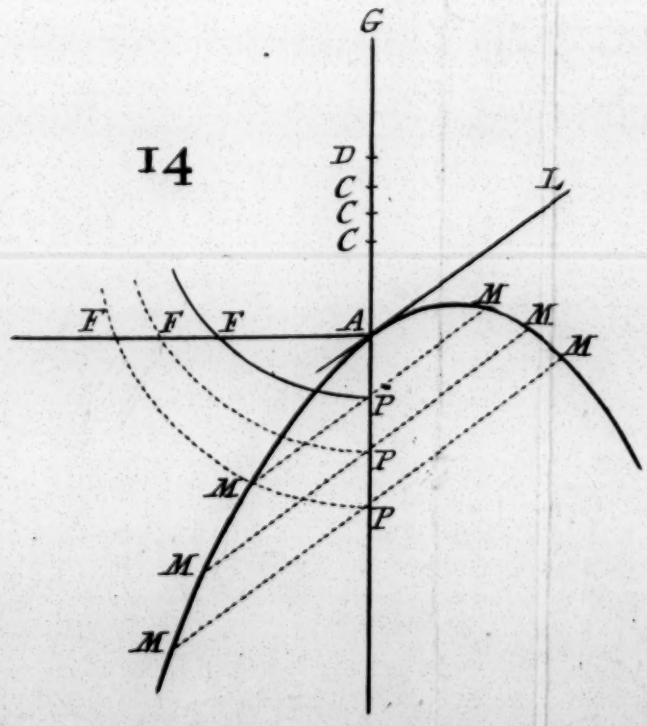

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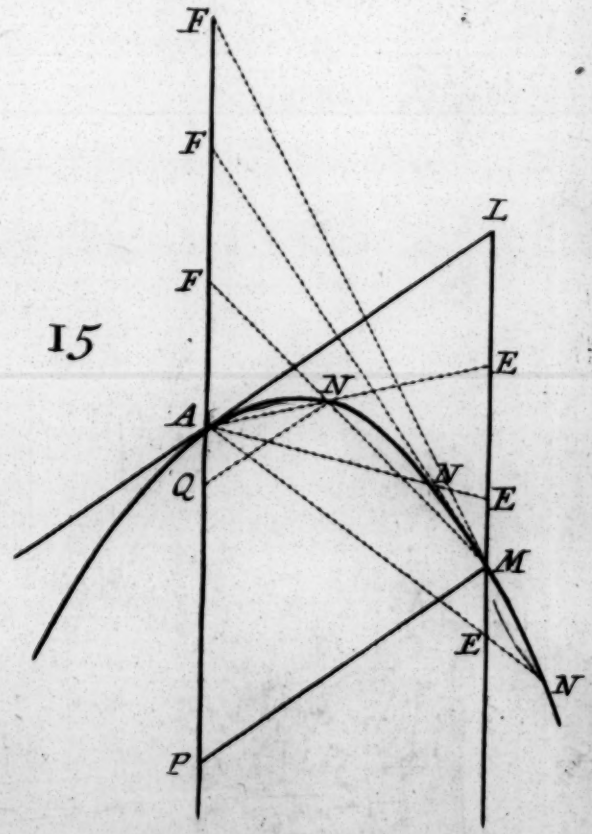

12




13




14




15

7.

The Lines *MP*, *MK* drawn from Points (*M*) of the Ellipſis, parallel to one of the Axes, and terminated by the other, are call'd Ordinates to that other Axis: So *MP* is an Ordinate to the Axis *Aa*, and *MK* to the Axis *Bb*.

8.

A third Proportional to the two Axes, is called the *Parameter* of that which is the firſt Term of the Proportion: So if it be made as the firſt Axis *Aa* to the ſecond *Bb*, ſo is the ſecond *Bb* to a third Proportional *p*; this Line *p* will be the Parameter of the firſt Axis.

9.

All right Lines paſſing through the Centre *C*, and terminated both ways by the Ellipſis, are called *Diameters*.

10.

A right Line meeting the Ellipſis but in one Point, and being both ways continued does not fall within the ſame, is called a *Tangent* in that Point.

SCHOLIUM.

31. IF the two Foci *F*, *f*, and the Centre *C* be ſuppoſed to be united in one Point, it is manifeſt then, that the Ellipſis will be changed into a Circle, having the right Line *CM* for the Radius thereof, equal to one half of the Thread *CMC*, put about the Point *C*, the Centre of the Circle. Hence a Circle may be conſider'd as a particular Species of an Ellipſis, wherein the Diſtance of the Foci is nothing: And therefore, whatever in the following Treatiſe is demonſtrated of Ellipſes, be their focal Diſtance what it will, may be alſo apply'd to a Circle, by ſuppoſing that Diſtance to become nothing. FIG. 17.

COROLLARY I.

32. IT follows from the firſt Definition, that if the right Lines *MF*, *Mf* be drawn from any Point *M* of the Ellipſis; their Sum will be always the ſame. FIG. 16.

COROLLARY II.

33. IT is manifeſt, that when the Point *M* falls in *A*, the Line *MF* will become *AF*, and *Mf* the Line *Af*: It is moreover evident, that when the Point *M* falls in *a*, the Line *MF* will become

$aF$, and $Mf$ will become the Line $af$. And therefore we ſhall have $AF + Af$, or $2AF + Ff = aF + af$, or $2af + fF$; and conſequently $AF = af$. Whence it follows:

1. That the Sum of the two right Lines $MF$, $Mf$, is always equal to the great Axis $Aa$, ſince $Mf + MF = Af + AF = Af + fa$.

2. That the focal Diſtance $Ff$ is divided into two equal Parts by the Centre $C$, becauſe $CA - AF$, or $CF = Ca - af$, or $Cf$.

## COROLLARY III.

34. IF the right Lines $BF$, $Bf$ be drawn from $B$, the Extremity of the ſecond Axis $Bb$ to the two Foci $F, f$; it is then manifeſt, that the right-angled Triangles $BCF$, $BCf$ are equal to each other; and ſo $BF$ the Hypothenuſe of the one, equal to $Bf$ the Hypothenuſe of the other; therefore $BF$, or $Bf = CA$, or $Ca$, becauſe * $BF + Bf = Aa$. And after the ſame manner we prove, that $Fb$, or $fb$ is $= CA$ or $Ca$. Whence it appears:

* *Art.* 33.

1. The ſecond Axis $Bb$ is divided into two equal Parts by the Centre $C$; for the right-angled Triangles $FCB$, $FCb$ are equal; becauſe their Hypothenuſes $FB$, $Fb$ are equal, and the Side $FC$ is common.

2. The ſecond Axis $Bb$ is always leſs than the firſt one $Aa$; becauſe $BC$, the half thereof, being one of the Sides of the right-angled Triangle $BCF$, will be leſs than the Hypothenuſe $BF$, which is equal to $CA$, the half of the firſt Axis $Aa$.

3. If a Circle be deſcribed about $B$, one of the Extremes of the ſecond Axis $Bb$, with the Radius $BF$, equal to $CA$, the half of the firſt or great Axis $Aa$; then that Circle will cut the great Axis in the two Points $F, f$, which will be the two Foci of the Ellipſis.

## COROLLARY IV.

35. THE ſame Things being premiſed, if $CA$ or $BF$ be called $t$; and $CF$, $m$; then the right-angled Triangle $BCF$, will give $\overline{BC}^2 = tt - mm$. But $AF = t - m$, and $Fa = t + m$, and therefore $AF \times Fa = tt - mm$. Whence it is manifeſt, that the Square of $CB$, half of the little Axis $Bb$, is equal to the Rectangle under $AF$, and $Fa$, the two Parts of the great Axis between $A$, $a$, the Ends thereof, and the Focus $F$.

## COROLLARY V.

36. HEnce it will be eaſy to deſcribe an Ellipſis, whoſe two Axes $Aa$, $Bb$, are given: For if the two Foci $F, f$ be found in the great Axis, and you put the Thread $FMf$, whoſe Length is equal to the

the great Axis, about them; and then deſcribe an Ellipſis, according to the Directions of *Def.* 1. it is evident, that the Ellipſis deſcrib'd will be that requir'd.

## PROPOSITION I.

### Theorem.

37. *IF the Ordinate* MP *be drawn to the firſt or great Axis* Aa, *and the Part* AD *be taken upon that Axis equal to* MF; *I ſay,* CA : CF :: CP : CD. FIG. 16.

Call (as before) the given Quantities $CA$, $CF$, $t$ and $m$; and the Indeterminate ones $CP$, $PM$, $x$ and $y$; and $CD$ the unknown one $z$: Now there may happen two Caſes.

*Caſe* 1. When the Point $P$ falls above the Centre $C$. Becauſe $PF$ is always leſs than $Pf$; therefore $MF$ or $AD$ will be leſs than $Mf$ or $aD$; and ſo $AD$ or $MF = t - z$, $aD$ or $Mf = t + z$, $FP = m - x$ or $x - m$ (according as the Point $P$ falls below or above the Focus $F$), and $Pf = x + m$. But the right-angled Triangles $MPF$, $MPf$, give $tt - 2tz + zz = yy + mm - 2mx + xx$, and $tt + 2tz + zz = yy + mm + 2mx + xx$. And by ſubſtracting the former Equation from the latter, we ſhall have $4tz = 4mx$; and conſequently $CD\,(z) = \frac{mx}{t}$.

*Caſe* 2. When the Point $P$ falls below the Centre $C$, becauſe $PF$ is always greater than $Pf$, it is manifeſt, that $MF$ or $AD$ will be greater than $Mf$ or $aD$: therefore $AD$ or $MF = t + z$, $aD$ or $Mf = t - z$, $PF = x + m$, $Pf = x - m$, or $m - x$ (according as the Point $P$ falls below or above the Focus $f$) but the right-angled Triangles $MPF$, $MPf$, give $tt + 2tz + zz = yy + mm + 2mx + xx$, and $tt - 2tz + zz = yy + mm - 2mx + xx$. And by ſubſtracting this laſt Equation from the former, we ſhall have $4tz = 4mx$, and conſequently $CD\,(z) = \frac{mx}{t}$. Whence in both Caſes we have $CA\,(t) : CF\,(m) :: CP\,(x) : CD\,(z)$. *W. W. D.*

### COROLLARY.

38. HENCE it is manifeſt, if the given Quantities $CA$ or $Ca$ be called $t$; $CF$ or $Cf$, $m$; and the indeterminate one $CP$, $x$: we ſhall always have $MF = t - \frac{mx}{t}$, and $Mf = t + \frac{mx}{t}$, when the

Point *P* falls above the Centre *C*: And on the contrary, when the Point *P* falls below it, we ſhall have $MF = t + \frac{mx}{t}$, and $Mf = t - \frac{mx}{t}$.

## PROPOSITION II.

### Theorem.

39. *THE Square of* MP, *any Ordinate to the Axis* A a, *is to the Rectangle under* A P *and* P a, *the Parts of that Axis, as the Square of its Conjugate Axis* B b, *to the Square of the Axis* A a.

*We are to prove*, that $\overline{PM}^2 : AP \times Pa :: \overline{Bb}^2 : \overline{Aa}^2$.

The ſame being ſuppos'd, as in the foregoing Article, if $\frac{mx}{t}$ $(= z)$ be put for $z$ in the Equation $tt \pm 2tz + zz = yy + mm \pm 2mx + xx$, which was found * by means of the right-angled Triangle $MPF$, we ſhall always form this, *viz.* $tt\,yy = t^4 - tt\,xx - mm\,tt + mm\,xx$, which being reduced to a Proportion, gives $\overline{PM}^2\,(yy) : AP \times Pa\,(tt - xx) :: \overline{BC}^2 \dagger (tt - mm) : \overline{CA}^2\,(tt) :: \overline{Bb}^2 : \overline{Aa}^2$. *W. W. D.*

* *Art.* 37.

† *Art.* 35.

### COROLLARY I.

40. IF any Ordinate $MK$ be drawn to the other Axis $Bb$, which I call $2c$, it is manifeſt that $MK = CP\,(x)$, and $CK = PM\,(y)$. But * $\overline{PM}^2\,(yy) : AP \times Pa\,(tt - xx) :: \overline{Bb}^2\,(4cc) : \overline{Aa}^2\,(4tt)$. And therefore $4cc = xx = 4cc\,tt - 4tt\,yy$; and ſo we have this proportion $\overline{MK}^2\,(xx) : BK \times Kb\,(cc - yy) :: \overline{Aa}^2\,(4tt) : \overline{Bb}^2\,(4cc)$.

* *Art.* 39.

That is, the Square of $MK$ any Ordinate to the Axis $Bb$, is to the Rectangle under $BK$ and $Kb$, the Parts of that Axis, as the Square of its Conjugate Axis $Aa$, is to the Square of the Axis $Bb$.

### *The Fundamental* COROLLARY I.

41. IF one of the Axes, as $Aa$, be call'd $2t$; its Conjugate $Bb$, $2c$; the Parameter thereof $p$; every of the Ordinates $PM$, $y$; and each of its Parts $(CP)$ contain'd between the Centre and the Points of Concurrence of the Ordinates, $x$; we ſhall always have * $\overline{PM}^2\,(yy) : AP \times Pa\,(tt - xx) :: \overline{Bb}^2\,(4cc) : \overline{Aa}^2\,(4tt) :: p : Aa\,(2t)$

FIG. 18, *and* 19.

* *Art.* 39.

$(2t)$. Becauſe by the Definition of a Parameter, $Aa\,(2t) : Bb\,(2c) :: Bb\,(2c) : p$, therefore $p = \frac{4cc}{2t}$. Whence if the Means and Extremes of the proportion $yy : tt - xx :: 4cc : 4tt$, are firſt multiply'd into one another, and afterwards thoſe of this $yy : tt - xx :: p : 2t$. We ſhall have $yy = cc - \frac{ccxx}{tt}$, and $yy = \frac{1}{2}pt - \frac{pxx}{2t}$. Now ſince this Property equally agrees to all the Points of the Ellipſis, and determines the Poſition thereof with reſpect to the two Conjugate Axes $Aa$, $Bb$; it follows that the Equation $yy = cc - \frac{ccxx}{tt}$, or $yy = \frac{1}{2}pt - \frac{pxx}{2t}$, expreſſes the Nature of the Ellipſis with regard to the Axes thereof.

## COROLLARY III.

42. IF there be drawn $MP$, $N\mathcal{Q}$, any two Ordinates to the Axis $Aa$; their Squares will be to one another as $(AP \times Pa,\ A\mathcal{Q} \times \mathcal{Q}A)$, the Rectangles under the Parts of the Axis made by the Points of Concurrence of the ſaid Ordinates; for $\overline{Bb}^2 : \overline{Aa}^2 :: \overline{PM}^2 :: AP \times Pa :: \overline{\mathcal{Q}N}^2 : A\mathcal{Q} \times \mathcal{Q}a$. And therefore $\overline{PM}^2 : \overline{\mathcal{Q}N}^2 :: AP \times Pa : A\mathcal{Q} \times \mathcal{Q}a$. * *Art.* 39.

## COROLLARY IV.

43. IF $MM$ be drawn from any Point $P$ in one of the Conjugate Axes $Aa$, parallel to the other Axis $Bb$; that Line will meet the Ellipſis in only two Points $M$, $M$, equally diſtant on each ſide of the Point $P$, and not more. For that the Points $M$, $M$, be in the Ellipſis, it is * neceſſary that the Squares of $PM\,(y)$ taken on both Sides of the Axis $Aa$, be each equal to the ſame Quantity, *viz.* $cc - \frac{ccxx}{tt}$. * *Art.* 41.

## COROLLARY V.

44. BEcauſe * $yy$ is $= cc - \frac{ccxx}{tt}$, it follows, that the more $CP\,(x)$ taken both ways from $C$ the Centre, increaſes, the more does both the Ordinates $PM\,(y)$ taken on both Sides of either of the Axes $Aa$, diminiſh; ſo that when $CP\,(x)$ is equal to $CA$, or $Ca\,(t)$, both the Ordinates $PM\,(y)$ will then become nothing: And contrariwiſe, the more $CP\,(x)$ diminiſhes, the more will both the Ordinates $PM\,(y)$ taken on each ſide of the Axis $Aa$ increaſe; ſo that when $CP\,(x)$ becomes * *Art.* 41.

becomes nothing, both the Ordinates $PM$ ($y$), each of which then is $CB$ or $Cb$ ($c$) will be the greateſt of the Ordinates. Whence is is manifeſt,

1. That if right Lines be drawn through $B$, $b$, the Ends of one of the Conjugate Axes parallel to the other; thoſe Lines will touch the Ellipſis in them Points.

2. That the Ellipſis recedes more and more from $A$ the Extremity of either of its Axes $Aa$, until it meets the Conjugate Axis $Bb$; after which it continually accedes to the ſame Axis $Aa$, until it meets the ſame in $a$, the other End thereof.

## COROLLARY VI.

* *Art.* 41. 45. BEcauſe * $yy = cc - \frac{ccxx}{tt}$, it follows, that if the Points $P$, $P$, be taken equally diſtant both ways from the Centre $C$; the Ordinates $PM$, $PM$, will be equal. And therefore, if any right Line $MM$ terminating in the Ellipſis, be cut into two equal Parts, by one of the Conjugate Axes $Bb$, in the Point $K$, not being the Centre, that right Line will be parallel to the other Axis $Aa$: For draw $MP$, $MP$, parallel to the Axis $Bb$, and then $PP$ will be biſected in $C$, becauſe $MM$ is ſo divided in $K$; therefore the Ordinates $PM$, $PM$ will be equal: And ſo the right Line $MM$ will be parallel to the Axis $Aa$.

## COROLLARY VII.

46. IF the two Parts of the Plane whereon the Ellipſis is drawn, on each ſide of either of the Axis, as $Bb$, be ſuppoſed to be folded together; it is manifeſt that the two Semi-Ellipſes $BAb$, $Bab$, will exactly fall on each other, [*or Coincide*] *viz.* the Points $A$, $M$, &c. on $a$, $M$, &c. becauſe * all the Perpendiculars $Aa$, $MM$, &c. to that Axis, are biſected in the Points $C$, $K$, &c. Whence it appears that the Ellipſis is cut by the two Axes into four equal and uniform Parts, not at all differing but in Situation.

* *Art.* 45.

## PROPOSITION III.

### Theorem.

FIG. 20. 47. *IF any right Line* A M *be drawn thro' the* A *one of the Extremities of one of the Axes* A a, *in one of the Angles* a A L, a A L, *made by that Axis, and the Line* L A L *parallel to* B b *its Conjugate Axis; I ſay the ſaid Line will meet the Ellipſis in ſome other Point* M.

Take

Take $AG$, on either ſide the Point $A$, in the Line $AL$, equal to $p$ the Parameter of the Axis $Aa$, and draw $GF$ parallel to the Axis, meeting the Line $AM$ (produced, if neceſſary) in the Point $F$: Moreover, aſſume $AL$ in the Line $AL$, (on the ſame Side as the Line $AM$ falls with reſpect to the Axis $Aa$) equal to $GF$, and draw the right Line $aL$ through $a$ the other Extremity of the Axis $Aa$; I ſay the Point $M$ wherein this Line cuts the Line $AM$, is in the Ellipſis $MAM$.

For draw $MP$ parallel to $AL$, and call the known Lines $Aa$, $AG$, $2t$, $p$; and $GF$ or $AL$, $a$; and the unknown ones $CP$, $x$; $PM$, $y$; then the ſimilar Triangles $AGF$, $MPA$ and $LAa$, $MPa$, give this proportion, $AG\,(p) : GF\,(a) :: MP\,(y) : AP\,(t \pm x) = \frac{ay}{p}$. And $AL\,(a) : Aa\,(2t) :: PM\,(y) : aP\,(t \mp x) = \frac{2ty}{a}$. And conſequently we have always $AP \times Pa\,(tt - xx) = \frac{2tyy}{p}$, let the Point $P$ fall either above or below the Centre $C$; and therefore $yy = 2pt - \frac{pxx}{2t}$. Whence the Line $PM$ will be * an Ordinate to the Axis $Aa$; and ſo the Point $M$ will be in the Ellipſis $MAM$. *W. W. D.* * *Art.* 41.

## COROLLARY I.

48. HENCE if $Aa$ one Axis of an Ellipſis $MAM$, as alſo $p$ the Parameter thereof be given, and any right Line $AM$ be drawn through $A$ one of the Extremities of that Axis, in either of the Angles $aAL$, $aAL$ made by that Axis, and the Line $LAL$ which is parallel to the Conjugate Axis $Bb$: Then it appears how the Point $M$ wherein the Line $AM$ meets the Ellipſis $MAM$ may be found.

## COROLLARY II.

49. HENCE it is evident that there is but one Line $LAL$ parallel to the Axis $Bb$, that can touch the Ellipſis $MAM$ in the Point $A$, one of the Ends of $Aa$ the Conjugate Axis thereto; becauſe no Line but that only, which paſſing through the Point $A$, and being continued both ways, but what will fall within the Ellipſis, and meet it in another Point.

PRO-

## PROPOSITION IV.

### Theorem.

FIG. 20. 50. *ALL Diameters, as* MCm *are cut into two equal Parts by the Centre* C, *and meet the Ellipſis but in two Points* M, m.

Draw the Ordinate *MP*, and aſſume *Cp* equal to *CP*, and on the Point *p* raiſe the perpendicular *pm* terminating in the right Line *MCm*; then it is manifeſt that the Triangles *CPm*, *Cpm* are ſimilar and equal, and ſo *CM* is equal to *Cm*, and *PM* to *pm*. And thoſe Ordinates that are equally diſtant from the Centre *C* on both Sides thereof, are * equal to one another. But *PM* is an Ordinate; therefore *pm* will alſo be an Ordinate, and conſequently the Point *m* will be in the Ellipſis.

* *Art.* 45.

Moreover, if a right Line parallel to the Axis *Bb*, be ſuppoſed to move from *C* towards *A*; it is manifeſt that the Part of that Line included within the Angle *ACM*, will continually increaſe as *CP* does. And contrariwiſe the Part of that Parallel included between the Quadrant of the Ellipſis *AMB*, and the Axis *CA*, that is, the Ordinate *PM*, will * continually decreaſe; and ſo the right Line *CM*, paſſing through the Centre, meets the Ellipſis in one Point *M* only, on the ſame Side the Axis; and the ſame is to be underſtood of the Point *m* taken on the other ſide. Whence, &c.

* *Art.* 44.

## DEFINITIONS.

II.

FIG. 21, *and* 22. If through any Point *M* of an Ellipſis, be drawn a Diameter *MCm*, an Ordinate *MP* to either of the Axes, as *Aa*, and a right Line, in ſuch manner that *CT* be a third proportional to *CP* and *CA*; the Diameter *SCs* which is parallel to *MT*, is call'd the *Conjugate Diameter* to the Diameter *Mm*. And contrariwiſe, the Diameter *Mm* is ſaid to be a Conjugate to the Diameter *Ss*: So that both of them together are call'd *Conjugate Diameters*.

12.

All right Lines drawn from Points of the Ellipſis parallel to one of the ſaid two Diameters, and terminating in the other, are call'd *Ordinates* to that other Diameter. So *NO* which is parallel to the Diameter *Ss*, is an Ordinate to *Mm* the Conjugate Diameter thereto.

13. A third

13.

A third Proportional to two Conjugate Diameters, is call'd the *Parameter* of the firſt Term of the Proportion: So a third Proportional to *M m*, *S s*, is call'd the *Parameter* of the Diameter *M m*.

COROLLARY.

51. IF the given Line *CA* be called $t$; and the Indeterminate ones *CP*, *PT*, $x$, $s$; it is manifeſt (by *Def.* 11.) that $CT\ (x + s = \frac{t\,t}{x}$; and ſo $s\,x = t\,t - x\,x = AP \times Pa$.

## PROPOSITION V.

### Theorem.

52. *IF two Ordinates,* M P, S K, *to the Axis* A a, *be drawn thro'* M, S, *the Ends of the Conjugate* Diameters M m, S s; *I ſay,* C K, *the Part of the Axis between the Centre and the Point of Concurrence of one of the Ordinates* S K, *is a mean Proportional between* A P, P a, *the two Parts of the Axis made by the Concurrence of the other Ordinate* M P.

*We are to prove, that* $\overline{CK}^2 = AP \times Pa$.

Call the known Lines *CA*, $t$; *CP*, $x$; *PT*, $s$; and the unknown one *CK*, $m$; then we ſhall have $AP \times Pa = t\,t - x\,x = {}^*s\,x$, and * *Art.* 51. $AK \times Ka = t\,t - m\,m = s\,x + x\,x - m\,m$, by putting $x\,x + s\,x$ for $t\,t$, which it is equal to. Now, by the Property of the Ellipſis, † $AP$ † *Art.* 42. $\times Pa\ (s\,x) : AK \times Ka\ (s\,x + x\,x - m\,m) :: \overline{PM}^2 : \overline{KS}^2 :: \overline{TP}^2\ (s\,s) : \overline{CK}^2\ (m\,m)$. By the Similarity of the Triangles *TPM*, *CKS*; whence by multiplying the Means and Extremes, and orderly tranſpoſing, there ariſes $\overline{CK}^2\ (m\,m) = \frac{s\,x\,x + s\,s\,x}{x + s} = s\,x = AP \times Pa$. *W. W. D.*

COROLLARY.

53. BEcauſe $\overline{CK}^2 = t\,t - x\,x$, it follows, that $\overline{CA}^2 - \overline{CK}^2$, or $AK \times Ka = x\,x$. But * $\overline{CA}^2\ (t\,t) : \overline{CB}^2\ (c\,c) :: AK \times Ka\ (x\,x) :$ * *Art.* 41. $\overline{SK}^2 = \frac{c\,c\,x\,x}{t\,t}$. And $\overline{CA}^2\ (t\,t) : \overline{CB}^2\ (c\,c) :: AP \times Pa\ (t\,t - x\,x) : \overline{PM}^2 = c\,c - \frac{c\,c\,x\,x}{t\,t}$. Moreover, becauſe the Triangles *CPM*, *CKS* are

right-angled, we have the Square $\overline{CM}^2$ or $\overline{CP}^2 + \overline{PM}^2 = xx + cc - \frac{ccxx}{tt}$, and the Square $\overline{CS}^2$ or $\overline{CK}^2 + \overline{KS}^2 = tt - xx + \frac{ccxx}{tt}$. Therefore $\overline{CM}^2 + \overline{CS}^2 = tt + cc$.

That is, the Sum of the Squares of any two Conjugate Diameters *M m*, *S s*, is equal to the Sum of the Squares of the two Axes *A a*, *B b*.

## PROPOSITION VI.

### Theorem.

54. *THE Square of* ON, *any Ordinate to the Diameter* M m, *is to the Rectangle* M O, O m, *under the Parts of that Diameter, as the Square of* S s, *the Conjugate Diameter to it, to the Square of the said Diameter* M m.

*We are to prove, that* $\overline{ON}^2 : MO \times Om :: \overline{Ss}^2 : \overline{Mm}^2$.

Draw *N Q*, *O H* parallel to the Axis *B b*, and *O R* parallel to *A a*, the Conjugate thereto, meeting the Ordinate *N Q* (produced, if neceſſary) in the Point *R*; then call the given Lines *C P*, *x*; *P M*, *y*; *C A*, *t*; *P T*, *s*; and the Indeterminate ones *H Q* or *O R*, *a*; *C H*, *b*; and the ſimilar Triangles *C P M*, *C H O*, and *M P T*, *N R O*, will give theſe two Proportions $CP(x) : PM(y) :: CH(b) : HO$ or $RQ = \frac{by}{x}$. And $TP(s) : PM(y) :: OR(a) : RN = \frac{ay}{s}$.

And becauſe (*Fig.* 21.) *N Q* is always the Difference of $RQ\left(\frac{by}{x}\right)$ $RN\left(\frac{ay}{s}\right)$, and *C Q* the Sum of *C H* (*b*), *H Q* (*a*), when the Point *N* falls between the Points *M*, *S*, or *m*, *s*; and on the contrary, (*Fig.* 22.) *N Q* always the Sum of *R Q*, *R N*, and *C Q*, the Difference of *C H*, *H Q*, when the Point *N* falls otherwiſe, we have $\overline{NQ}^2 = \frac{bbyy}{xx} \mp \frac{2abyy}{sx} + \frac{aayy}{ss}$, and $\overline{CQ}^2 = aa \mp 2ab + bb$; *viz.* $-\frac{2abyy}{sx}$, and $+ 2ab$ in the firſt Caſe, and $+\frac{2abyy}{sx}$, and $- 2ab$ in the ſecond. But * $AP \times Pa\ (tt - xx) : AQ \times Qa$, or $\overline{CA}^2 - \overline{CQ}^2\ (tt - aa \mp 2ab - bb) :: \overline{PM}^2\ (yy) : \overline{QN}^2 = \frac{ttyy - aayy + 2abyy - bbyy}{tt - xx}$. And by comparing the two Values of $\overline{NQ}^2$ together, this Equation will be had $\frac{bbyy}{xx} \mp \frac{2abyy}{sx} + \frac{2ayy}{ss} = \frac{ttyy - aayy \mp 2abyy - bbyy}{tt - xx}$, and ſtriking out the Term $\mp \frac{2abyy}{sx}$, from one

* *Art.* 42.

Side of the Equation, and the Term $\pm\frac{2abyy}{tt-xx}$ from the other Side (theſe two Terms being equal, ſince, by *Art.* 51. $sx = tt - xx$) and dividing by $yy$, there will be had $\frac{bb}{xx} + \frac{aa}{ss} = \frac{tt-aa-bb}{tt-xx}$.

And multiplying by $xx$, and tranſpoſing $bb$, we ſhall have $\frac{aaxx}{ss}$, or $\frac{aax^4}{ssxx} = \frac{ttxx-aaxx-bbtt}{tt-xx}$; and again multiplying the firſt Member by $ssxx$, and the ſecond by the Square of $tt-xx$, the Value of $sx$ (which is done in only multiplying the Numerator by $tt-xx$) we ſhall have $aax^4 = t^4xx - aattxx - bbt^4 - ttx^4 + aax^4 + bbttxx$; and ſtriking out $aax^4$ from both Sides, tranſpoſing of $aattxx$, and dividing by $ttxx$, there will be had $\overline{HQ}^2$ or $\overline{OR}^2(aa) = tt - xx + bb - \frac{bbtt}{xx}$.

Now if the Semidiameter $CM$ or $Cm$ be called $z$; by the Similarity of the Triangles $CPM$, $CHO$, we ſhall have the following Proportion $CP(x) : CM(z) :: CH(b) : CO = \frac{bz}{x}$. Therefore $MO \times Om = zz - \frac{bbzz}{xx}$. But the ſimilar Triangles $ORN$, $CKS$ give this Proportion $\overline{ON}^2 : \overline{CS}^2 :: \overline{OR}^2 \left(tt - xx + bb - \frac{bbtt}{xx}\right) : \overline{CK}^2$ (*) $(tt-xx) :: MO \times Om \left(\frac{xxzz-bbzz}{xx}\right) : \overline{CM}^2$ (zz.) Since the ſame Product ariſes by multiplying the Means and the Extremes. And therefore $\overline{ON}^2 : MO \times Om : \overline{CS}^2 : \overline{CM}^2$ (*) $:: \overline{Ss}^2 : \overline{Mm}^2$. *W. W. D.*

(*) *Art.* 52.

(*) *Art.* 50.

## *A General* COROLLARY. I.

55. HENCE it is manifeſt, that what has been demonſtrated in *Prop.* 2. with reſpect to the two Axes $Aa$, $Bb$, is equally true of any two Conjugate Diameters $Mm$, $Ss$. And ſince the 40th, 41ſt, 42d, 43d, 44th, 45th, 47th, 48th and 49th Articles, ariſing from *Prop.* 2. are true, whether the Angle $ACB$ be right or not; it follows, that if the Lines $Aa$, $Bb$, are ſuppos'd in thoſe Articles, inſtead of the two Axes, any two Conjugate Diameters, thoſe Articles will ſtill be true according to this Suppoſition: For their Demonſtration will always remain the ſame; and there is nothing more requir'd to make this appear, but reading them over again, and every where uſing the word *Diameter* for *Axis*.

### COROLLARY II.

56. BEcauſe the 44th and 49th Articles are equally true, whether the Lines *Aa*, *Bb*, are any two Conjugate Diameters, as *Mm*, *Ss*, as well as the Axes; therefore the Line *MT* drawn from the Point *M*, one End of any Diameter *Mm*, parallel to *Ss* the Conjugate Diameter thereto, touches the Ellipſis in *M*; and no other Line but that can touch the Ellipſis in that Point.

Whence it appears, that there can but one Line be drawn from a given Point in an Ellipſis, to touch the ſame in that Point.

### COROLLARY III.

57. HENCE it is evident, by *Def.* 11. that if an Ordinate *MP*, be drawn from any Point *M* of an Ellipſis, to either of the Axes *Aa*; and you take *CT*, towards the Point *P*, a third Proportional to *CP*, *CA*, and draw the right Line *MT*; this Line *MT* will touch the Ellipſis in *M*. And contrariwiſe, if the Line *MT* touches the Ellipſis in *M*, and the Ordinate *MP* be drawn to the other Axis *Aa*, the Parts of that Axis *CP*, *CA*, *CT*, will be in a continual Geometrick Proportion.

### COROLLARY IV.

58. IN the 11th, 12th, and 13th Definitions and in the two laſt Propoſitions, if you ſuppoſe that the Lines *Aa*, *Bb*, inſtead of the Axes, are any two conjugate Diameters, thoſe Propoſitions will ſtill be true, becauſe they may be demonſtrated as before: as is evident by contemplating the 23d Figure, wherein the ſimilar Triangles give the ſame Proportions as in the Caſe of the Axes.

Whence it follows, 1. That the laſt Corollary ought ſtill to take place, when the Line *Aa* is any Diameter, as well as the Axis. 2. That the Conjugate Diameters *Mm*, *Ss*, may be the two Axes according to that Suppoſition; and ſo the two Axes may be eſteemed as two Conjugate Diameters, being at right Angles with each other.

## PROPOSITION VII.

### Theorem.

FIG. 24. 59. *IF from any Point of an Ellipſis, whoſe Centre is* C, *there be drawn the Ordinate* MP *to* Aa *one of the Axes, and* MG *perpendicular to the Tangent* MT *paſſing through the Point* M: *I ſay* CP *will always have*

*have the ſame Proportion to* PG, *as the Axis* Aa *has to its Parameter.*

For if the Semi-Axis $CA$ or $Ca$ be called $t$; and the indeterminate Lines $CP$, $x$; $PM$, $y$; we ſhall have * $CT = \frac{tt}{x}$; and therefore $PT = \frac{tt - xx}{x}$. But the Right-angl'd ſimilar Triangles $TPM$, $MPG$, give this proportion, $TP\left(\frac{tt-xx}{x}\right) : PM(y) :: PM(y) : PG\left(\frac{xyy}{tt-xx}\right)$. Hence we may get this Proportion $CP(x) : PG\left(\frac{xyy}{tt-xx}\right) :: AP \times Pa$ $(tt - xx) : \overline{PM}^2 (yy)$. Becauſe by multipling the Means and Extreemes, the ſame Product $xyy$ will ariſe, but the Rectangle $AP \times Pa$ is * to the Square $\overline{PM}^2$, as the Axis $Aa$ to its Parameter. Wherefore, &c.

* *Art.* 57.

* *Art.* 41.

## PROPOSITION VIII.

### Theorem.

60. IF *a Tangent* TMS *be drawn through any Point* M *of an Ellipſis, as alſo the right Lines* MF, Mf, *to the two Foci* F, f: *I ſay the Angles* FMT, fMS, *made both ways by theſe two Lines, with the Tangent* TMS, *are equal to one another.* FIG. 25.

For draw $FD$, $fd$, perpendicular to the Tangent; alſo draw the firſt Axis $Aa$ meeting it in T, and the Ordinate $MP$ to that Axis, and call $CA$ or $Ca$, $t$; $CF$ or $Cf$, $m$; and $CP$, $x$; then we ſhall have $MF * \left(t - \frac{mx}{t}\right) : Mf\left(t + \frac{mx}{t}\right) :: TF$, or $CT * \left(\frac{tt}{x}\right) - CF(m) :$ $Tf$ or $CT\left(\frac{tt}{x}\right) + Cf(m)$. Becauſe in multiplying the Means and Extremes, the ſame Product ariſes. But the ſimilar Triangles $TFD$, $Tfd$, give this Proportion, $TF : Tf :: FD : fd$. And ſo $MF$ the Hypothenuſe of the right-angled Triangle $MDF$ will be to $Mf$, the Hypothenuſe of the right-angled Triangle $Mdf$, as the Side $DF$ is to the Side $df$: and conſequently theſe two Triangles will be ſimilar: wherefore the Angles $FMD$, $fMd$, or $FMT$, $fMS$, which are oppoſite to the Homologous Sides $DF$, $df$, will be equal to one another. *W. W. D.*

* *Art.* 38.
* *Art.* 57.

COROL-

### COROLLARY.

FIG. 26. 61. HENCE it is manifeſt, that the Tangent *TMS* being both ways infinitely produced from the Point of Contact *M*, leaves the Ellipſis entirely next to its two Foci *F*, *f*. And ſince this is always ſo, let the Point *M* be where it will in the Ellipſis, it follows, that the Ellipſis will be Concave quite round about the two Foci thereof, and conſequently alſo about its Centre.

## PROPOSITION IX.

### Theorem.

FIG. 26. 62. *IF* DAE *be drawn thro'* A, *one End of any Diameter* A a, *parallel to* B b, *the Conjugate Diameter thereto, meeting any two other Conjugate Diameters* M m, S s, *in the Points* D, E; *I ſay, the Rectangle under* D A, A E, *is equal to the Square of* CB, *the one half of the Diameter* B b.

*We are to prove, that* $DA \times AE = \overline{CB}^2$.

Through *M*, *S*, the Ends of the Conjugate Diameters *M m*, *S s*, draw the Ordinates *MP*, *SK* to the Diameter *A a*, and call *CA*, *t*; *CB*, *c*; and *CP*, *x*; *PM*, *y*; then we ſhall * have $\overline{CK}^2 = AP \times Pa = tt - xx$; and conſequently $AK \times Ka$, or $\overline{CA}^2 - \overline{CK}^2 = xx$. But * $\overline{BC}^2 (cc) : \overline{CA}^2 (tt) :: \overline{MP}^2 (yy) : AP \times Pa$, or $\overline{CK}^2 = \frac{ttyy}{cc}$. And $\overline{CA}^2 (tt) : \overline{CB}^2 (cc) :: AK \times Ka (xx) : \overline{KS}^2 = \frac{ccxx}{tt}$. And by extracting the ſquare Root, we get $CK = \frac{ty}{c}$, and $KS = \frac{cx}{t}$. But the ſimilar Triangles *CPM*, *CAD*, and *CKS*, *CAE*, give theſe Proportions $CP (x) : PM (y) :: CA (t) : AD = \frac{ty}{x}$. And $CK \left(\frac{ty}{c}\right) : KS \left(\frac{cx}{t}\right) :: CA (t) : AE = \frac{ccx}{ty}$. Therefore $DA \times AE = cc = \overline{BC}^2$. *WW. D.*

* *Art.* 52.

* *Art.* 54.

## PROPOSITION X.

### Problem.

FIG. 27. 63. *TWO Conjugate Diameters* A a, B b, *of an Ellipſis being given, as alſo a right Line* MCm, *in paſſing thro' the Centre* C, *to find the Points* M, m *in that Line, wherein it meets the Ellipſis.* Draw

Draw the indefinite Line $AD$ thro' $A$, one End of the Diameter $Aa$, parallel to $Bb$ the Conjugate Diameter, meeting the Line $CM$ given in Poſition in the Point $D$; moreover, draw the Line $AO$ thro' the Point $A$, perpendicular to $AD$, and equal to $CB$, and the Line $OD$ thro' the Points $O$, $D$. This being done, about the Centre $O$, with the Radius $OA$, deſcribe a Circle $OA$, cutting the Line $OD$ in the two Points $N$, $n$; and then if $NM$, $nm$ be drawn from theſe Points parallel to the Line $OC$, which joins the Centres of the Circle and Ellipſis; I ſay the Points $M$, $m$, wherein they meet the Line $CD$, will be in the Ellipſis, and conſequently will determine the Extremities of the Diameter $MCm$ given in Poſition.

For draw the Lines $MP$, $N\mathcal{Q}$, parallel to $AD$ meeting the Line $CA$, $OA$ in the Points $P$, $\mathcal{Q}$; then the ſimilar Triangles $CDO$, $MDN$, and $CDA$, $CMP$, and $ODA$, $ON\mathcal{Q}$ will give theſe Proportions $CA : CP :: CD : CM :: OD : ON :: OA : O\mathcal{Q}$. That is, $CA : CP :: OA : O\mathcal{Q}$. And therefore if the right Line $P\mathcal{Q}$, be drawn, it will be parallel to $OC$; and conſequently alſo to $MN$ which is ſuppoſed parallel to $OC$. So the Parallels $MP$, $N\mathcal{Q}$, will be equal to each other. This being ſuppoſed, if we call the given Lines $CA$, $t$; $CB$ or $AO$ or $ON$, $c$; and the undeterminate ones $CP$, $x$; $PM$ or $N\mathcal{Q}$, $y$; we ſhall have this Proportion $CA\,(t) : CP\,(x) :: OA\,(c) : O\mathcal{Q} = \frac{cx}{t}$. And becauſe the Triangle $O\mathcal{Q}N$ is Right-angl'd at $\mathcal{Q}$, the Square $\overline{N\mathcal{Q}}^2$ or $\overline{MP}^2\,(yy) = \overline{ON}^2\,(cc) - \overline{O\mathcal{Q}}^2\left(\frac{ccxx}{tt}\right)$. Whence the Line $MP$ will be * an Ordinate to the Diameter $Aa$, and conſequently the Point $M$ will be in the Ellipſis, whoſe Conjugate Diameters are $Aa$, $Bb$. But becauſe the Lines $NM$, $OC$, $nm$, are parallel, the Line $Mm$ is biſected by the Centre $C$; ſince (by the Property of the Circle) $Nn$ is biſected in $O$. Therefore the Point $m$ will be ** likewiſe in the ſame Ellipſis.

* *Art.* 41, *and* 55.

** *Art.* 50.

If the two Conjugate Diameters $Aa$, $Bb$, ſhould happen to be the Axes, then the Parallels $CO$, $P\mathcal{Q}$, would Coincide with the Lines $CA$, $AO$, all four of which would make but one ſtreight Line. And by this Means the Conſtruction and Demonſtration would have been ſomething eaſier.

## PROPOSITION XI.

### Problem.

64. *TWO Conjugate Diameters* A a, B b, *of an Ellipſis, being given; to find the Axes* (M m, S s) *thereof: And demonſtrate that an Ellipſis can have but two Axes.* FIG. 27.

Draw

Draw the Line *D E* through *A* one End of the Diameter *A a*, parallel to *B b* the Conjugate Diameter, and the Line *A O* perpendicular to *D E*, and equal to *C B*. Then join *O C*, and draw the Line *F G* thro' *F* the middle Point thereof perpendicular to the ſame, and meeting the Line *D E* in the Point *G*, on both ſides of which Point take the equal Parts *G D*, *G E*, in the Line *DE*, each equal to *GO* or *GC*. This being done, if the right Lines *C D*, *C E* be drawn: I ſay the two Axes *M m*, *Ss*, are ſituate in theſe Lines.

† *Art.* 58. For ſince the two Axes may be eſteem'd † as Conjugate Diameters, cutting one another at right Angles, they will meet the Line *D E* in the Points *D*, *E*, ſuch, that a Circle deſcrib'd on that Line as a Diameter, will paſs through the Points *C*, *O*; becauſe the Rectangle * *Art.* 62. *D A* × *A E* being equal * to the Square of *AO*, the Angle *D O E* will be a right one, as well as the Angle *D C E*. But it is manifeſt, that this is entirely what the foregoing Conſtruction has effected; ſince the Lines *G O*, *GC*, *G E*, *G D*, being all equal to one another, are ſo many Radii of the ſame Circle. But ſince there are no other Points in the Line *D E*, but *D*, *E*, which at the ſame time can ſatisfy theſe two Conditions, *viz.* that the Angles *D C E*, *D O E* be each right Angles; therefore the two Conjugate Diameters *M m*, *Ss*, which are at right Angles to each other, will be the Axes, and there are only two of them.

Now to determine the Lengths of the Axes, you need only draw the right Lines *O D*, *O E*, meeting the Circle whoſe Radius is *O A*, in the Points *N*, *R*, and then the Parallels *N M*, *R S*. For it is evident, * *Art.* 63. * that *M*, *S*, the Points wherein theſe Parallels meet the right Lines *CD*, *C E*, appertain to the Ellipſis, whoſe Conjugate Diameters are the Lines *A a*, *B b*; and ſo the Points *M*, *S*, will be Extremities of the two Axes.

## COROLLARY.

63. IF it had been requir'd to find two Conjugate Diameters *M m*, *Ss*, that might cut each other in the Angle *M C S*, equal to a given Angle; two other Conjugate Diameters *A a*, *B b*, being given: It is plain that the Problem might have been reduced to this, *viz.* to find two Points *D*, *E*, in the Line *D E* given in poſition ſuch; that if the right Lines *D O*, *O E*, *C D*, *C E*, are drawn to the two Points *O*, *C*, given without the Line *DE*, the Angle *D O E* may be a right Angle, and the Angle *D C E*, equal to a given Angle. But ſince the Solution of this Problem is pretty difficult, I refer it to the 10th Book, and here follow another Manner, which is more ſimple; and that is, to find firſt the two Axes, and then by means of them, the two Conjugate Diameters ſought, as we are going to ſhew in the following Problem.

PRO-

## PROPOSITION XII.

### Problem.

66. *THE two Axes* A a, B b *of an Ellipſis being given, to find two Conjugate Diameters* M m, S s, *cutting one another in the Angle* M C S, *equal to an Angle given.* FIG. 28. and 29.

Let us ſuppoſe the Diameters *M m*, *Ss*, to be thoſe requir'd, and that they meet the indefinite right Line *D E* (drawn thro' *A*, the End of the little Axis *Aa*, parallel to the great Axis *Bb*) in the Points *D* and *E*. Now draw the Line *CF* from *C*, the Centre of the Ellipſis, making the Angle *CFE*, at the Point *F*, with the Line *D E*, equal to the given Angle *MCS*, and call the given Lines *CA*, *t*; *CB*, *c*; *AF*, *a*; and the unknown one *AE*, *z*; then will * $AD = \frac{cc}{z}$, and $CE = \sqrt{tt+zz}$, becauſe *CAE* is a right-angled Triangle. This being ſuppos'd, * *Art.* 62.

The Triangles *FEC*, *CED* will be ſimilar; becauſe the Angle at the Point *E* is common, and the Angle *CFE* was made equal to the Angle *MCS*; therefore $FE\,(z-a) : EC\,(\sqrt{tt+zz}) :: EC\,(\sqrt{tt+zz}) : ED\,(z+\frac{cc}{z})$. Whence by multiplying the Means and Extremes, this Equation will be formed $zz-az+cc-\frac{acc}{z} = tt+zz$, and ſtriking out *zz* from both Sides, multiplying by *z*, and dividing by *a*, there will come out $zz-\frac{cc}{a}z+\frac{tt}{a}z+cc=0$. And (for Brevity's Sake) putting $\frac{cc-tt}{a} = 2b$; the laſt Equation will become this, $zz-2bz+cc=0$, or $zz-2bz+bb=bb-cc$. And by extracting the ſquare Roots of both Sides there comes out $z-b$, or $b-z = \sqrt{bb-cc}$; and conſequently the unknown Quantity $AE\,(z) = b \pm \sqrt{bb-cc}$, which laſt Equation gives the following Conſtruction.

Produce the ſmall Axis *A a* to the Point *O*, ſo that *AO* be equal to *CB*, the half of the great Axis, and draw *CF*, making the Angle *CFE*, with the Line *DE* drawn through *A*, parallel to *B b*, equal to the given Angle; join *O F*, and draw the right Lines *O H*, *CG*, perpendicular to *O F*, *CF*, meeting the Line *D E* in the Points *H*, *G* (the Points *H*, *G*, in the 28th and 29th Figures, are not denoted in the Line *D E*; becauſe doing that would have enlarged the Figures too much,

much, and ſince it is eaſy to imagine them) This being done, about the Centre $O$, with the Radius $OK$, equal to the half of $GH$, (that Part of $AD$ produced, which is comprehended between $G$,$H$,) deſcribe an Arc of a Circle cutting $DE$ in the Points $K$,$K$; then if $KD$, $KE$, be taken in $DE$, each equal to $KO$, and the right Lines $DC$, $EC$, are drawn through $C$ the Centre of the Ellipſis; I ſay, the Diameters ſought $Mm$, $Ss$, are ſituate in $DC$, $EC$.

For becauſe $FAC$, $FCG$, and $FAO$, $FOH$, are right Angles, we ſhall have $AG = \frac{tt}{a}$, $AH = \frac{cc}{a}$; and therefore $GH = \frac{cc-tt}{a} = 2b$. Whence the Radius $OK$, which is equal to $\frac{1}{2}GH$, will be equal to $b$. And becauſe $OAK$ is a right-angled Triangle, we have $AK = \sqrt{bb-cc}$, and $AE$ or $KE \mp AK = b \mp \sqrt{bb-cc}$, and $AD$ or $KD \pm AK = b \pm \sqrt{bb-cc}$. Now this being ſuppos'd, if the Value of $AE$ be multiplied by that of $AD$, there will ariſe $AE \times AD = cc = \overline{CB}^2$; and therefore * $Mm$, $Ss$ are Conjugate Diameters. But the Rectangle under $AE + AD$ or $DE$ ($2b$) and $AE - AF$ or $EF$ ($b \mp \sqrt{bb-cc} - a$) is $= 2bb \mp 2b\sqrt{bb-cc} + 2ab = 2bb \pm 2b\sqrt{bb-cc} + tt - cc$ by putting $cc-tt$ for $2ab$ the Value thereof; and becauſe the Triangle $CAE$ is right-angled, the Square $\overline{CE}^2 = \overline{AE}^2 + \overline{CA}^2 = 2bb \mp 2b\sqrt{bb-cc} + tt - cc = DE \times EF$; and ſo $FE : EC :: EC : ED$. Therefore the Triangles $FEC$, $CED$ will be ſimilar; becauſe the Angle at the Point $E$ is common, and the Sides about that Angle are proportional. Whence the Angle $MCS$ will be equal to the given Angle $CFE$. Which is what was to be demonſtrated.

* *Art.* 62.

Now to determine the Lengths of $CM$, $CS$, the two Semi-Diameters ſought, you need only draw the Lines $OD$, $OE$, and then the Lines $NM$, $RS$, parallel to $OC$, thro' the Points $N$, $R$, wherein $OD$, $OE$, meet the Circle, whoſe Radius is $OA$. For it is manifeſt, * that the Points $M$, $S$, wherein $NM$, $MS$ meet the Lines $CD$, $CE$, will be in the Ellipſis, and conſequently do determine the Extremities of the Diameters.

* *Art.* 63.

## COROLLARY I.

67. IT follows from the foregoing Conſtruction, 1. That when the Problem is poſſible, $OK\ (\frac{cc-tt}{2a})$ muſt exceed or be equal to $AO$ ($c$); for otherwiſe the Circle deſcrib'd with the Radius $OK$, will not meet the Line $DE$, and ſo the Problem will in this Caſe be impoſſible.

2. When

2. When $OK$ exceeds $OA$, we can always find two different Pair of Conjugate Diameters $Mm$, $Ss$, by means of the Points $K$, $K$, that will anſwer the Queſtion: But then the Diameter $Ss$, of *Fig.* 29. is equal to the Diameter $Mm$ of *Fig.* 28. and alike poſited on the other Side of the Axis $Aa$; becauſe $AE$ of *Fig.* 29. is equal to $AD$ of *Fig.* 28. and moreover, the Diameter $Mm$ of *Fig.* 29. is equal to the Diameter $Ss$ of *Fig.* 28. and alike ſituate on the other Side of the Axis $Aa$; becauſe $AD$ of *Fig.* 29. is equal to $AE$ of *Fig.* 28. that is, the two different Pair of Conjugate Diameters $Mm$, $Ss$, which equally anſwer the Problem, are alike ſituate on each Side of the Axis $Aa$; and their Magnitudes will remain the ſame in thoſe two different Poſitions.

3. When $OK = OA$, the two Points of Interſection $K$, $K$ will coincide in the Point of Contact $A$; and then you need but take $AE$, $AD$, each equal to $CB$, the half of the great Axis. Whence it appears, that in this Caſe the Problem is capable but of one Solution; and the two Conjugate Diameters $Mm$, $Ss$, which ſolve it, are equal between themſelves. FIG. 30.

## COROLLARY II.

68. IT is manifeſt alſo, that the greater $AF$ ($a$) is, the greater is the given obtuſe Angle $CFE$, and contrariwiſe the leſs will the Line $OK$ $\left(\frac{cc-tt}{2a}\right)$ be: So that when $AF$ is the greateſt poſſible, the obtuſe Angle $CFE$ will be alſo the greateſt poſſible; and contrariwiſe, the Line $OK$ will be the leaſt poſſible, *viz.* equal to $AO$. But then, if the right Lines $Ba$, $ab$, are drawn, the right-angled Triangles $aCB$, $CAD$, $aCb$, $CAE$ will be all equal to one another; becauſe the Lines $AE$, $AD$ are each equal to $CB$ or $Cb$, the Halfs of the Axis $Bb$, and $CA$ is equal to $Ca$. And therefore the Angle $ACM$ will be equal to the Angle $CaB$, and $ASC = Cab$; therefore the given Angle $MCS$ or $CFE$ will be equal alſo to the Angle $Bab$. Hence it follows, FIG. 28, 29, *and* 30. FIG. 30.

1. If the Lines $aB$, $ab$ are drawn from $a$, one End of the little Axis $Aa$ to $B$, $b$, the Ends of the great one; the given obtuſe Angle $CFE$ muſt be equal or leſs than the Angle $Bab$, that ſo * the Problem may be poſſible. FIG. 28, 29, *and* 30. * *Art.* 67.

2. When the given obtuſe Angle $CFE$ is $= Bab$, as in *Fig.* 30. then there are only two Conjugate Diameters $Mm$, $Ss$, anſwering the Problem; and they are equal to one another.

3. And when the Angle $CFE$ is leſs than $Bab$, as in the 28th and 29th Figures; then there will be always two different Pair of Conjugate Diameters anſwering the Problem, being ſituate ſimilarly on both

Sides the little Axis, the ſaid Angle $CFE$ between them remaining the ſame, and their Magnitude will be equal.

## PROPOSITION XIII.

### Problem.

69. *TWO Conjugate Diameters* A a, B b, *of an Ellipſis, being given; to deſcribe the ſame by a continued Motion.*

* *Art.* 64. Firſt find * the two Axes, and then deſcribe the Ellipſis by the Directions in Art. 36.

FIG. 31, 32. But this may be done another way, which is thus. Draw the right Line $AH$ through $A$ one End of the given Diameter $Aa$, perpendicular to the other Diameter $Bb$, and take $AQ$ in the ſaid Line, on either ſide the Point $A$, equal to $CB$, and draw the Line $CQ$; then if the Line $GF$, equal to $HQ$, be mov'd ſo, that the Ends thereof be always in the Lines $Bb$, $CQ$, (produced both ways from the Centre $C$, as is neceſſary) till it has mov'd ſucceſſively through the four Angles made by the two Lines, and is come again to the ſame Situation from which it went; I ſay, if $GM$ be taken equal to $AQ$, the Point $M$ by this Motion will deſcribe the Ellipſis ſought.

For draw $GP$ parallel to $QA$, meeting the Diameter $Aa$ in $P$, and the Diameter $Bb$ in $O$; then the ſimilar Triangles $CHQ$, $COG$, and $CAQ$, $CPG$, will give this proportion, $CQ : CG :: AQ$, or $GM : GP :: HQ$, or $GF : GO$. And ſo the Line $PM$ will be parallel to the Diameter $Bb$. This being ſuppoſed,

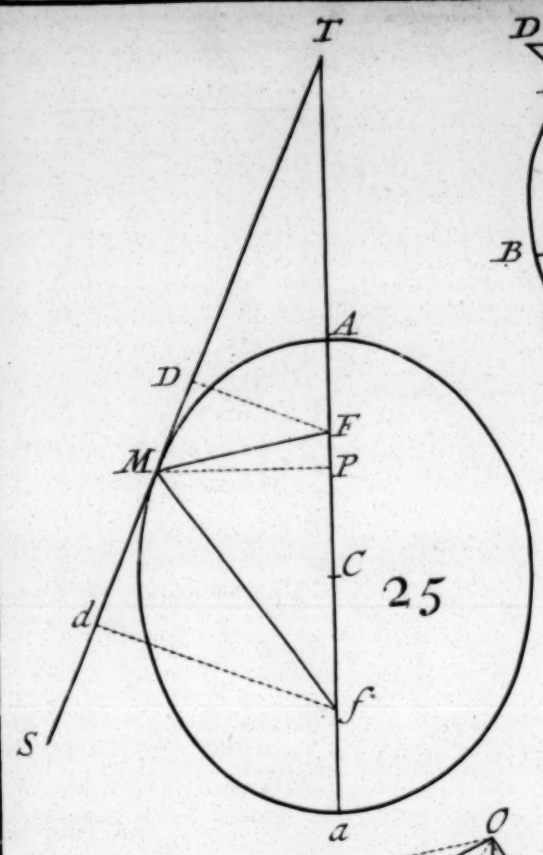
Call the given Lines $CA$, $t$; $AQ$, or $CB$, $c$; and the unknown ones $CP$, $x$; $PM$, $y$; then we ſhall have $CA(t) : CP(x) :: AQ(c) : GP = \frac{cx}{t}$. And the right-angl'd Triangle $GPM$ will give $\overline{PM}^2 = \overline{GM}^2 - \overline{GP}^2$, that is $yy = cc - \frac{ccxx}{tt}$. Whence $PM$ will be * an Ordinate to the Diameter $Aa$ in the Ellipſis, whoſe Conjugate Diameters are the Lines $Aa$, $Bb$. Therefore, &c.

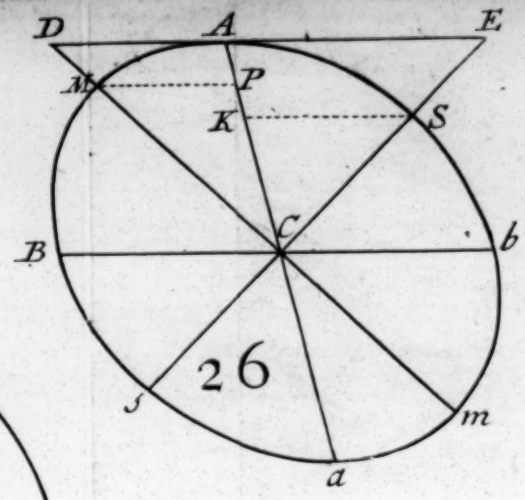
* *Art.* 41, 55.

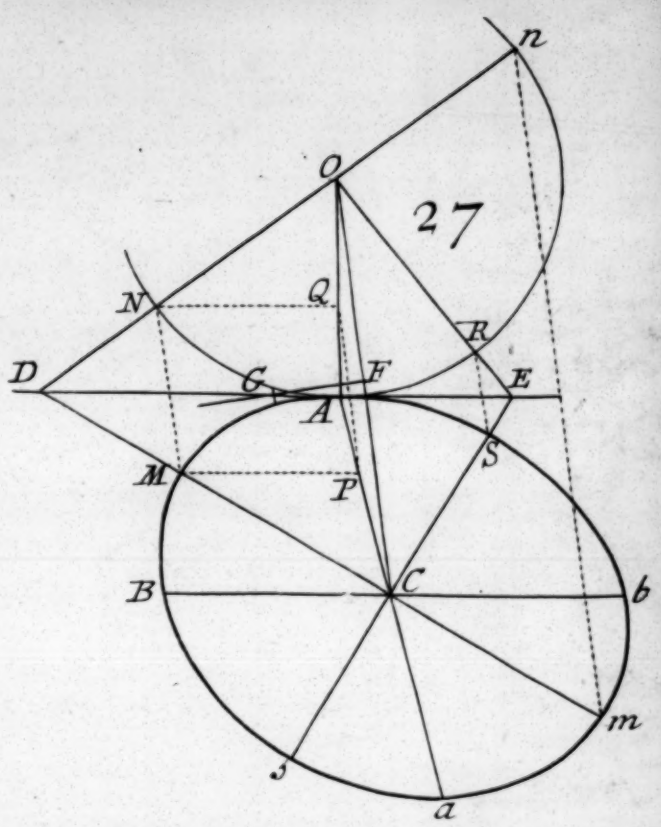
If the two Conjugate Diameters $Aa$, $Bb$, were the Axes; then it is manifeſt that the Lines $AQ$, $CQ$, would fall in the Diameter $Aa$, which would be one of the Axes, and the Point $H$ would fall in the Centre $C$. Whence it appears, that in this Caſe, $GF$ muſt have been then taken equal to $CQ$, the Sum or Difference of the two Semi-Axes $CA$, $CB$; and the Ends thereof mov'd along the Axes $Aa$, $Bb$ produced, as is neceſſary.

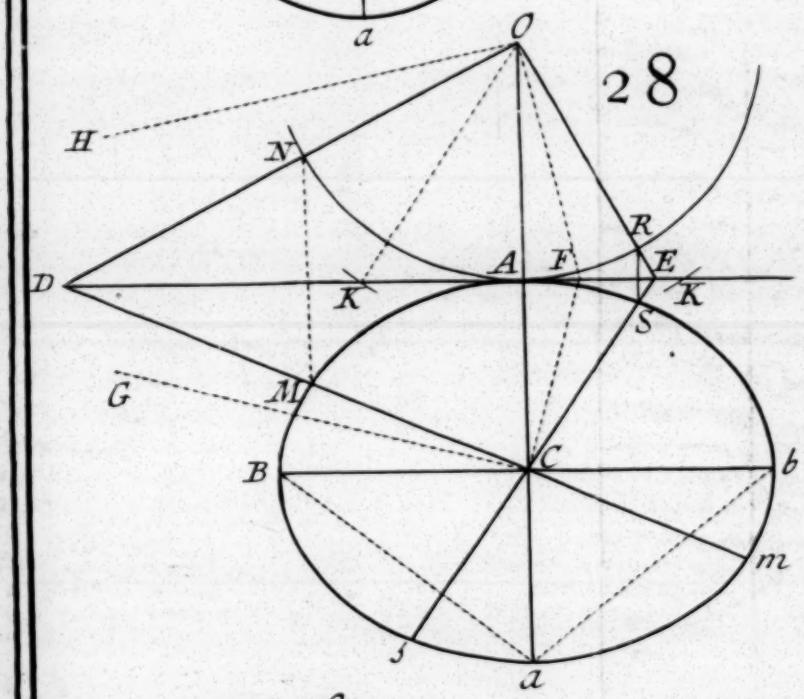
FIG. 33.

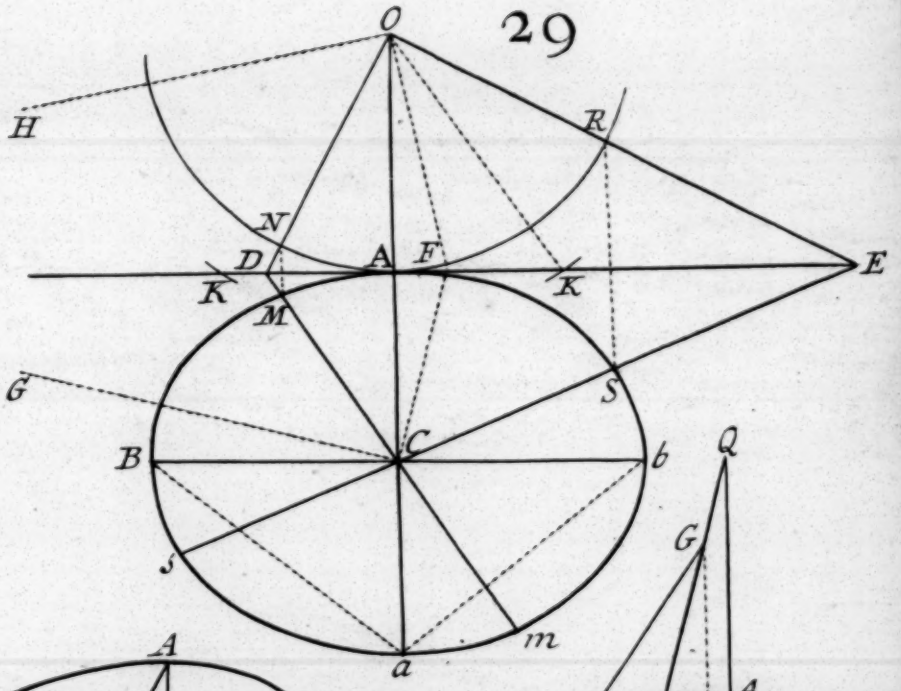
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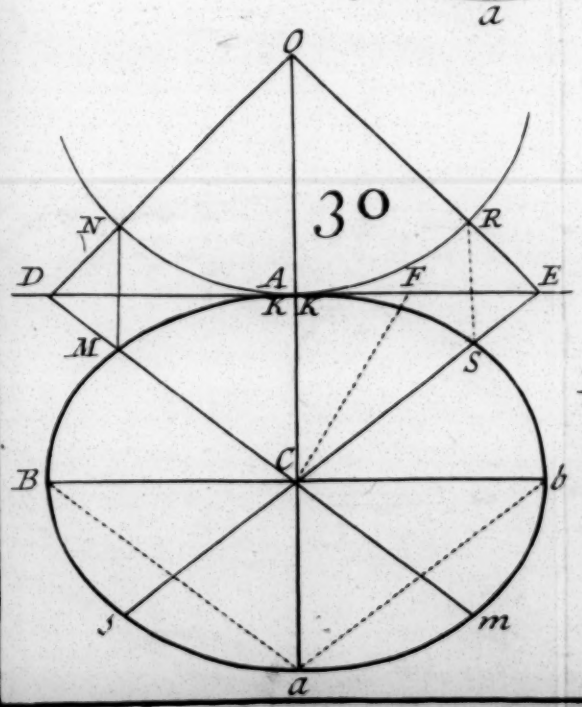

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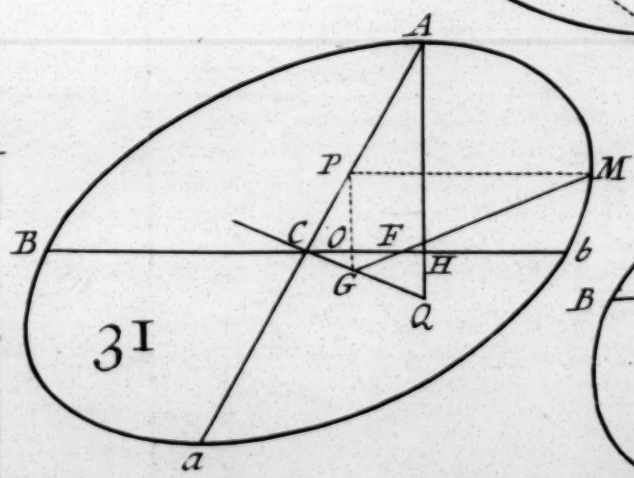

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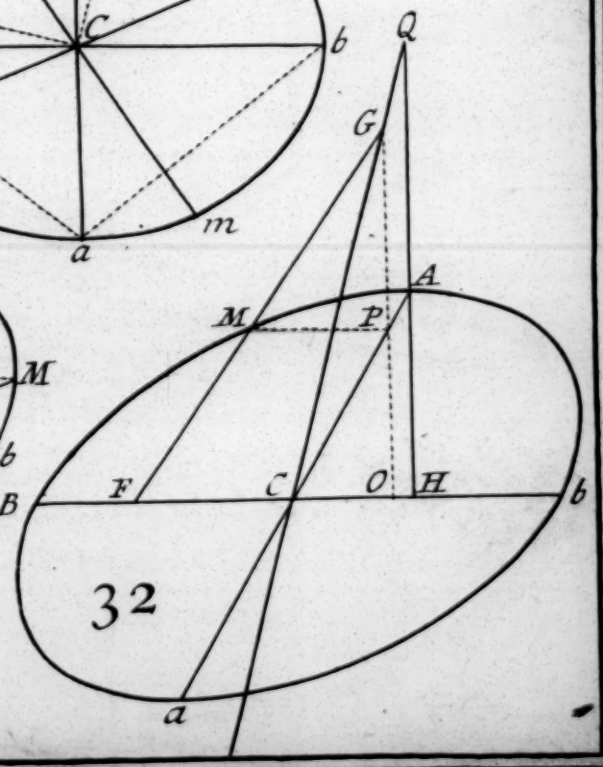

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Becauſe the right Lines *A a*, *B b*, cut one another at right Angles in the Point *C*, it is manifeſt, that in whatſoever ſituation the right Line *G F* is found, during the Motion of its Ends along theſe Lines, the Circle that ſhould have that Line for a Diameter, would always paſs through the Point *C*: And ſo the Line *C D* paſſing through *D* the middle of the Line *F G*, will be always equal to *D F*, becauſe the Lines *C D*, *D F*, *D G*, will always be Radii of that Circle. Whence ariſes the following Deſcription.

Let the right Lines *C D*, *D F*, be each equal to the half of *C Q*, the Sum or Difference of the two Semi-Axes *C B*, *C A*, and faſten them ſo together at their common End *D*, that they may move about the ſame, like the two Legs of a pair of Compaſſes about the Head. This being done, faſten *C* the Extremity of the Line *C D* in the Centre of the Ellipſis, and move *F* the End of the other Line *F D*, along the Axis *B b*, ſo that it cauſes the Side *C D* to move about the fixed Point *C*. Then it is evident that the Point *M* taken in *F D* (produced, if neceſſary) ſo that *F M* be equal to *C A*, will by this Motion deſcribe the Ellipſis ſought.

## PROPOSITION XIV.

### Theorem.

70. *TWO Conjugate Diameters* (A a, B b,) *of an Ellipſis being given; to deſcribe the ſame through ſeveral Points.*

Draw the indefinite right Line *D A D*, through *A* one End of the given Diameter *A a*, parallel to *B b* the Conjugate Diameter, and draw *A O* perpendicular to *A D*, and equal to (*C B*) half the Diameter *B b*, and join *O C*, and about the Centre *O* with the Radius *O A*, deſcribe a Circle. This being done on both Sides of *C A*, draw at pleaſure any Number of Lines *C D*, *C D*, &c. from the Centre *C*, and then draw the Lines *O D*, *O D*, &c. from the Centre *O* to the Points *D*, *D*, &c. cutting the Arc of the Circle in the Points *N*, *N*, &c. and draw the right Lines *N M*, *N M*, &c. parallel to *C O*, and meeting the correſpondent right Lines *C D*, *C D*, &c. in the Points *M*, *M*, &c. then if the Points *m*, *m*, &c. are mark'd in the right Lines *C M*, *C M*, &c. continued, equally diſtant from *C*; it is manifeſt, * that the Curve Line paſſing through all the Points *M*, *M*, &c. *m*, *m*, &c. thus found, will have the right Lines *A a*, *B b*, for two Conjugate Diameters. FIG. 34.

* *Art.* 63.

This may be done otherwiſe thus; divide *C B*, one of the Semidiameters into as many equal Parts, *C E*, *E E*, &c. as poſſible, and draw the

the Perpendiculars $ED$, $ED$, &c. meeting the Arc of a Circle deſcrib'd about the Centre $C$, with the Radius $CB$, in the Points $D$, $D$, &c. Join $AB$, and draw the Line $EP$ through $E$, one of the aforeſaid Points (that is nigheſt to the Centre $C$,) parallel to $AB$, meeting $CA$ in $P$. Then if in the Diameter $Aa$, be taken the equal Parts $PP$, $PP$, &c. on both Sides of the Centre $C$, each equal to $CP$, and through the Points $P$, $P$, &c. be drawn the Lines $PM$, $PM$, &c. parallel to the Diameter $Bb$, (on both Sides of $A$) each equal to its Correſpondent $ED$: I ſay the Curve Line paſſing through all the Points $M$, $M$, &c. will be in the Ellipſis ſought.

For call the given Quantities $CA$, $t$; $CB$ or $CD$, $c$; and the indeterminate one $CP$, $x$; $PM$, $y$; then becauſe the Triangles $CAB$, $CPE$, are ſimilar, we have this Proportion, $CA(t) : CB(c) :: CP(x) : CE = \frac{cx}{t}$. And becauſe the Triangle $CED$ is Right angl'd at $E$, the Square $\overline{ED}^2$ or $\overline{PM}^2 (yy) = \overline{CD}^2 (cc) - \overline{CE}^2 \left(\frac{ccxx}{tt}\right)$. Therefore the Line $PM$ will * be an Ordinate to the Diameter $Aa$; and ſince this Demonſtration extends to all the right Lines $PM$; becauſe every $CP$ is to its Correſpondent $CE$, in the Ratio of $CA$ to $CB$; therefore the Curve paſſing through all the Points $M$, $M$, &c. found as above, will be the Ellipſis ſought.

* *Art.* 41, 55.

*The End of the Second Book.*

BOOK

# BOOK III.

## *Of the Hyperbola.*

### DEFINITIONS.

I.

IF one End of a long Rule $f M O$ be faſten'd in the Point $f$, taken on a Plane, in ſuch a manner, that it may turn freely about that fix'd Point $f$, as a Centre; and if one End of the Thread $F M O$, (being in Length leſs than the ſaid Rule) be fixed to $O$, the other End of the Rule, and the other End of the Thread be fix'd in the Point $F$ taken on the Plane. Then if the Rule $f M O$ be turn'd about the fix'd Point $f$; and at the ſame time you keep the Thread $O M F$ always in an equal Tenſion, and its Part $M O$ cloſe to the Side of the Rule, by means of the Pin $M$: The Curve Line $A X$ deſcrib'd by the Motion of the Pin $M$, is one Part of an *Hyperbola*. FIG. 36.

And if the Rule be turn'd about, and move on the other Side of the fixed Point $F$, the other Part $A Z$ of the ſame Hyperbola may be deſcrib'd after the ſame manner.

But if the End of the Rule be faſten'd in $F$, and that of the Thread in $f$, (the Rule and Thread keeping the ſame Lengths) you may deſcribe another Curve Line $x a z$ after the ſaid manner, which will be oppoſite to $X A Z$, and is called likewiſe an Hyperbola; and both theſe two Curves together are called *oppoſite Hyperbola's*, [or oppoſite Sections.]

2.

The two fixed Points $F, f$, are called the *Foci*.

3.

The Line $A a$, which paſſes thro' the two Foci $F, f$, and terminating both ways in the oppoſite Hyperbola's, is called the *firſt Axis* [or *principal Axis*.]

4.

The Point $C$, dividing the firſt Axis $A a$ in the middle, is called *the Centre*.



5. If

5.

If the indefinite right Line *B b*, be drawn through the Centre *C*, perpendicular to the firſt Axis *A a*; and if about the Point *A*, as a Centre, with the Diſtance *C F*, an Arc of a Circle be deſcrib'd cutting *B b*, in the Points *B*, *b*: Then the Part *B b* of that perpendicular, is called the *Second Axis*, [or the *Conjugate Axis*.]

6.

The two Axes *A a*, *B b*, are together call'd *Conjugate Axes*; ſo that the firſt Axis *A a* is ſaid to be a Conjugate to the Second *B b*; and contrariwiſe the Second *B b*, a Conjugate to the firſt *A a*.

7.

The Lines *M P*, *M K*, drawn from Points (*M*) of the oppoſite Hyperbola's parallel to one of the Conjugate Axes, and terminating in the other, are call'd *Ordinates* to that Axis: So *M P* is an Ordinate to the firſt Axis *A a*, and *M K* one to the ſecond *B b*.

8.

A third Proportional to the two Axes, is call'd the *Parameter* of that which is the firſt Term of the Proportion. So if you make as the firſt Axis *A a* is to the ſecond *B b*, ſo the ſecond *B b* to a third Proportional *p*; then the Line *p* will be the Parameter of the firſt Axis *A a*.

9.

All Lines paſſing through the Centre *C*, are called *Diameters*: Thoſe which meet the oppoſite Hyperbola's, being *Firſt Diameters*, [or principal Diameters] and thoſe which being infinitely produced, do not meet them, *Second Diameters*, [or *Conjugate Diameters*.]

10.

A right Line which meets an Hyperbola but in one Point, and being both ways continued, falls without the Hyperbola, is called a *Tangent* to it in that Point.

## SCHOLIUM.

FIG. 37. 11. THE reaſon why we have ſaid in the firſt Definition, that the Length of the Thread *F M O* muſt be leſs or greater than the Length of the Rule *f M O*, is, becauſe if the Thread was equal to it, the Pin *M* by its motion, would deſcribe a Line, having all the Points (*M*) thereof equally diſtant from the two Points *F*, *f*; becauſe if *M O*, the common Part of the Thread, were taken from the Thread and Rule, the Parts *M F*, *M f*, remaining, would always be equal between

between themſelves. Whence it is manifeſt, that the Line deſcrib'd by the Pin *M* would, in that Caſe, be the indefinite right Line *Bb*, drawn through C the Middle of *Ff*, perpendicular to *Ff*.

## COROLLARY I.

72. IT follows from the firſt Definition, that if the right Lines *MF*, *Mf*, be drawn from any Point (*M*) of one of the oppoſite Hyperbola's to the two Foci *F*, *f*; their Difference *MF*—*Mf* will be always the ſame: for it will be always equal to the Difference between the Lengths of the Rule and Thread. FIG. 36.

## COROLLARY II.

73. WHEN the Point *M* falls in *A*, it is evident, that *MF* will become *Af*; and further, when the Point *M* falls in *a*, in deſcribing the oppoſite Hyperbola *xaz*; it is manifeſt, that *MF* will become *aF*, and *Mf* will become *af*. Whence becauſe the Difference (*MF*—*Mf*) of *MF* and *Mf*, is always the ſame, we ſhall have *Af* — *AF*, or *Ff* — 2 *AF* = *aF* — *af*, or *Ff* — 2 *af*: and therefore *AF* = *af*. Whence it follows,

1. That the focal Diſtance *Ff* is divided into two equal Parts by the Centre C: becauſe *CA* + *AF*, or *CF* = *Ca* + *af* or *Cf*.

2. That the Difference of the two right Lines *MF*, *Mf*, is always equal to the firſt Axis *Aa*; becauſe, in the Hyperbola *XAZ*, we have always *Mf* — *MF* = *Af* — *AF*, or *Af* — *af*; and in the oppoſite Hyperbola we have likewiſe always *MF*—*Mf* = *aF* — *af*, or *aF* — *AF*.

## COROLLARY III.

74. IT follows from the fifth Definition,

1. That the ſecond Axis *Bb* is divided into two equal Parts by the Centre *C*; for the right-angled Triangles *ACB*, *ACb*, will be equal; becauſe the Hypothenuſes *AB*, *Ab* are equal, and the Side *AC* is common.

2. If *CE* be taken in the ſecond Axis *Bb*, equal to *CA* the half of the firſt Axis, and the Hypothenuſe *AE* be drawn; then the ſecond Axis *Bb* will be greater, equal to, or leſs than the firſt *Aa*; according as the right Line *CF* is greater, equal to, or leſs than the Hypothenuſe *AE*; becauſe the Hypothenuſe *Ab* being taken equal to *CF*, will then be found likewiſe greater, equal to, or leſs than the Hypothenuſe *AE*.

3. If $CF, Cf$, be taken in the first Axis $Aa$, (on both Sides the Centre $C$) each equal to $AB$, the Hypothenuse of the right-angled Triangle $CAB$, formed by the two Semi-axes $CA, CB$; then the Points $F, f$, will be the two Foci.

## COROLLARY IV.

75. THE same Things being premised, if you call the given Quantities $CF$ or $AB$, $m$; $CA$ or $Ca$, $t$; then the right-angled Triangle $ACB$, will give $\overline{BC}^2 = mm - tt$. But $AF = m - t$, and $Fa = m + t$; and therefore $AF \times Fa = mm - tt$. Whence it is manifest, that the Square of $CB$, the half of the second Axis $Bb$, is equal to the Rectangle under $AF$, and $Fa$, the Parts of the first Axis $Aa$, comprehended between one of the Foci $F$, and $A, a$, the two Ends of that Axis.

## COROLLARY V.

76. IT will not be difficult now to describe the opposite Hyperbola's, having the two Axes $Aa, Bb$ given, supposing it to be known that $Aa$ is the first Axis. For if the two Foci $F, f$, be * found in the first Axis $Aa$, and one End of a Thread $FMO$ be fixed in the Point $F$; and if then you fix $O$, the other End of that Thread, to the End of a long Rule $OMf$, (whose Length must * be less or greater than the Length of the Thread $OMF$, by the Length of the Line $Aa$.) And if the other End of that Rule be fasten'd in the Focus $f$, so as to move about the same: Then may you describe the two opposite Hyperbolas $XAZ$, $xaz$, as is directed in *Def.* 1. and it is evident, that the Line $Aa$ will be their first Axis, and the Line $Bb$ the second.

* *Art.* 74.

* *Art.* 71.

*Note*, The longer the Rule $OMf$ is, the greater will the Parts of the opposite Hyperbola's described, by means thereof, be; so that they may be augmented at pleasure, by equally augmenting the Length of the Rule and Thread.

## PROPOSITION I.

### Theorem.

77. *IF an Ordinate* MP *be drawn to the first Axis* Aa, *and* AD *be taken in that Axis (produced) equal to* MF, *from* A *towards the Focus* F, *when the Point* M *is in the Hyperbola* XAZ, *and towards the Focus* f, *when that Point does fall in the opposite Hyperbola* xaz. *I say, as* CA : CF :: CP : CD.

Call

Call (as before) the given Quantities $CA$ or $Ca$, $t$; $CF$, or $Cf$, $m$; and moreover, the indeterminate Quantities $CP$, $x$; $PM$, $y$; and the unknown Quantity $CD$, $z$; in the firſt Caſe, we ſhall have $AD$ or $ME = z - t$, $aD$ or $Mf = z + t$, $FP = x - m$, or $m - x$ (according as the Point $P$ falls below or above the Focus $F$), $Pf = x + m$: And in the ſecond Caſe, $AD$ or $MF = z + t$, $aD$ or $Mf = z - t$, $FP = x + m$, $Pf = x - m$ or $m - x$, according as the Point $P$ falls above or below the Focus $f$.

Now the right-angled Triangle $MPF$ will give $zz \mp 2tz + tt = yy + xx \mp 2mx + mm$; *viz.* $-$ in the firſt, and $+$ in the ſecond Caſe; and the other right-angled Triangle $MPF$ will give $zz \pm 2tz + tt = yy + xx \pm 2mx + mm$; *viz.* $+$ in the firſt, and $-$ in the ſecond Caſe.

Then if each Member of the firſt Equation, in the firſt Caſe, be orderly taken from thoſe of the ſecond Equation; and contrariwiſe, (in the ſecond Caſe) each Member of the ſecond Equation from thoſe of the firſt, there will be had $4tz = 4mx$: Whence $CD$ $(z)$ will be $= \frac{mx}{t}$. Therefore $CA\ (t) : CF\ (m) :: CP\ (x) : CD\ (z)$. *W. W. D.*

## COROLLARY.

78. HENCE if you call the given Quantities $CA$ or $Ca$, $t$; $CF$, or $Cf$, $m$; and the indeterminate Quantity $CP$, $x$; it is evident, we ſhall have always $MF = \frac{mx}{t} - t$, and $Mf = \frac{mx}{t} + t$, when the Point $M$ happens in the Hyperbola $XAZ$, whoſe Focus is the Point $F$: and contrariwiſe, $MF = \frac{mx}{t} + t$, and $Mf = \frac{mx}{t} - t$, when the Point $M$ falls in the oppoſite Hyperbola $xaz$, whoſe Focus is the Point $f$.

## PROPOSITION II.

### Theorem.

79. *THE Square of any Ordinate* (PM) *to the firſt Axis* Aa, *is to the Rectangle under* AP, Pa, *the Parts of the Axis produced, as the Square of the Conjugate Axis* Bb, *to the Square of the firſt Axis* Aa.

*We are to prove, that* $\overline{PM}^2 : AP \times Pa :: Bb^2 : Aa^2$.

The ſame Things being premiſed as in the laſt Propoſition, if $\frac{mx}{t}$ be put for its Value $z$, in the Equation $zz \mp 2tz + tt = yy + xx \mp 2mx + mm$, found

found by means * of the right-angled Triangle $MPF$, we shall have this Equation always formed, *viz.* $tty\text{=}mmxx\text{—}mmtt\text{—}ttxx\text{+}t^4$, which being reduced to a Proportion, and then $\overline{PM}^2(yy) : AP \times Pa\,(xx\text{—}tt)$

* *Art.* 77. $:: \overline{BC}^2$ * $(mm\text{—}tt) : \overline{CA}^2(tt) :: \overline{Bb}^2 : \overline{Aa}^2$. *W. W. D.*

### COROLLARY I.

20. IF an Ordinate $(MK)$ be drawn to the ſecond Axis $Bb$, which call $2c$; then it is manifeſt, that $MK = CP\,(x)$, and $CK = PM\,(y)$. But $\overline{PM}^2(yy) : AP \times Pa\,(xx\text{—}tt) :: \overline{Bb}^2(4cc) : \overline{Aa}^2(4tt)$. And therefore $4ccxx = 4cctt + 4ttyy$; from whence we get this Proportion, $\overline{MK}^2(xx) : \overline{CK}^2 + \overline{CB}^2(yy+cc) :: \overline{Aa}^2(4tt) : \overline{Bb}^2(4cc)$.

That is the Square of any Ordinate $(MK)$ to the ſecond Axis $Bb$, is to the Square of $CK$ *plus* the Square of $CB$, the half of the ſecond Axis, as the Square of the Conjugate Axis $Aa$, to the Square of the ſecond Axis $Bb$.

### *A Fundamental* COROLLARY. I.

FIG. 38, *and* 39. 81. IF the firſt Axis $Aa$ be called $2t$; the ſecond Axis $Bb$, $2c$; the Parameter $p$; each of the Ordinates $PM$, $y$; and each of the correſpondent Parts $(CP)$, contain'd between the Centre, and the Points of Concurrence of the Ordinates, $x$; we ſhall have * always

* *Art.* 79, *and* 80. $\overline{PM}^2(yy) : \overline{CP}^2 \mp \overline{CA}^2(xx \mp tt) :: \overline{Bb}^2(4cc) : \overline{Aa}^2(4tt) :: p : Aa\,(2t)$. Becauſe, by the Definition of a Parameter, $Aa\,(2t) : Bb\,(2c) :: Bb\,(2c) : p = \frac{4cc}{2t}$. Where you muſt obſerve, that it is $xx\text{—}tt$, when $Aa$ is the firſt Axis, and then the Rectangle $AP \times Pa$ may be ſubſtituted for $\overline{CP}^2 - \overline{CA}^2$; and on the contrary, it is $xx+tt$, when $Aa$ is the ſecond Axis. Hence multiplying the Extremes, and Means of the firſt Proportion $yy : xx \mp tt : 4cc : 4tt$. And then thoſe of the other $yy : xx \mp tt : p : 2t$. there will ariſe $yy = \frac{ccxx}{tt} \mp cc$, and $yy = \frac{pxx}{2t} \mp \frac{1}{2}pt$. And ſince this Property agrees equally to all the Points of the oppoſite Hyperbola's, and determines their Poſition with regard to the Axes; therefore the Equation $yy = \frac{ccxx}{tt} \mp cc$, or $yy = \frac{pxx}{2t} \mp \frac{1}{2}pt$, entirely expreſſes the Nature of the Hyperbola with regard to the Axes.

COROL-

### COROLLARY III.

82. IF any two Ordinates ($MP$, $N\mathcal{Q}$) be drawn to the Axis $Aa$, it is manifeſt that $\overline{MP}^2 : \overline{\mathcal{Q}N}^2 :: \overline{CP}^2 \mp \overline{CA}^2 : \overline{C\mathcal{Q}}^2 \mp \overline{CA}^2$. For $\overline{PM}^2 : \overline{CP}^2 + \overline{CA}^2 :: \overline{Bb}^2 : \overline{Aa}^2 :: \overline{\mathcal{Q}N}^2 : \overline{C\mathcal{Q}}^2 + \overline{CA}^2$. Whence, &c.

It is neceſſary here to take notice, that the Rectangles $AP \times Pa$, $A\mathcal{Q} \times \mathcal{Q}a$ may be ſubſtituted for $\overline{CP}^2 - \overline{CA}^2$, and $\overline{C\mathcal{Q}}^2 - \overline{CA}^2$, as being equal to them; which I would have hereafter always obſerved.

### COROLLARY IV.

83. IF any right Line $MPM$ be drawn through any Point ($P$) of either Axis, as $Aa$ (produc'd if it be the firſt Axis) parallel to $Bb$ the Conjugate Axis to $Aa$; then that Line will meet one or both the oppoſite Hyperbola's in only two Points $M$, $M$, equally diſtant from the Point $P$. For in order that the Points $M$, $M$, be in one or both the Hyperbola's, it is neceſſary * that the Squares of both the $PM$'s ($y$) on each Side the Axis $Aa$, be each equal to the ſame Quantity $\frac{ccxx}{ss} \mp cc$. * *Art.* 81.

### COROLLARY V.

84. HENCE it follows, that ſince $yy$ is $= \frac{ccxx}{ss} \mp cc$, the more $CP$ ($x$) (taken on either Side the Centre $C$) increaſes, the more will the correſpondent Ordinates ($PM = y$), on each Side of the Axis ($Aa$), likewiſe increaſe, even infinitely: And contrariwiſe, the more $CP$ ($x$) diminiſhes, the more will $PM$ ($y$) likewiſe diminiſh; ſo that (*Fig.* 38.) $CP$ ($x$) being equal to $CA$ or $Ca$ ($t$) when $Aa$ is the firſt Axis; $PM$ ($y$) will then be equal to nothing, and (*Fig.* 39.) $CP$ ($x$) being equal to nothing, when $Aa$ is the ſecond Axis, both the $PM$'s ($y$) which then will become $CB$ or $Cb$ ($c$), are leſs than any of the Ordinates ($PM = y$) taken on both Sides the Centre. Whence it is manifeſt; FIG. 38, *and* 39.

1. If Parallels be drawn (*Fig.* 39.) to the ſecond Axis $Aa$, thro' $B$, $b$, the Ends of the firſt Axis $Bb$; then theſe Parallels will touch the oppoſite Hyperbola's in the Points $B$, $b$.

2. The oppoſite Hyperbola's recede more and more from their Conjugate Axes, even infinitely, beginning from the Extremes of the firſt Axis; but yet with this Difference, that the firſt Axis meets each of the oppoſite Hyperbola's in one Point, and being continued, will be

be ever after within them; whereas the ſecond Axis falls quite without the Hyperbola's, and being infinitely produced, will never meet either of them.

### COROLLARY VI.

85. IT follows, becauſe $yy$ is $= \frac{ccxx}{tt} \mp cc$, that if the Points $P$, $P$, be taken on both Sides the Centre ($C$) equally diſtant from the ſame; then the Ordinates $PM$, $PM$, will be equal. Whence it is manifeſt, that if a right Line $MM$, terminating in one Hyperbola, or in the oppoſite ones, be cut into two equal Parts by an Axis $Bb$, in the Point $K$ not being the Centre, that Line will be parallel to the Conjugate Diameter $Aa$. For if $MP$, $MP$ be drawn parallel to the Axis $Bb$, then the Line $PP$, will be biſected in $C$, becauſe $MM$ is ſo in $K$; and therefore the Ordinates $PM$, $PM$, will be equal, and the right Line $MM$ parallel to the Axis $Aa$.

### COROLLARY VII.

86. IF the two Parts of the Plane, whereon the oppoſite Hyperbola's are drawn, on each Side of the Axis $Aa$, be conceiv'd to be
FIG. 39. folded together, it is manifeſt, that when $Aa$ is the ſecond Axis, the two oppoſite Hyperbola's will exactly agree, or coincide, *viz.* the Points $B$, $M$, &c. with the Points $b$, $M$, &c. becauſe all the Perpen-
* *Art.* 83. diculars $Bb$, $MM$, &c. drawn to that Axis, are * biſected in the Points $C$, $P$, &c.

And for the ſame Reaſon (*Fig.* 38.) when $Aa$ is the firſt Axis, the Parts of the oppoſite Hyperbola's on each Side the Axis will perfectly agree or coincide.

## ADVERTISEMENT.

We have hitherto, in this Book, kept to the ſame Method, as in the Ellipſis, and might have continued it on to the End; but becauſe the other Properties of the Hyperbola may be eaſier demonſtrated from certain particular Lines appertaining to the ſame, which muſt neceſſarily be ſpoken of; therefore I here deviate from that Method, and firſt lay down the following Definitions of theſe Lines, and afterwards demonſtrate the remaining Properties by means of them.

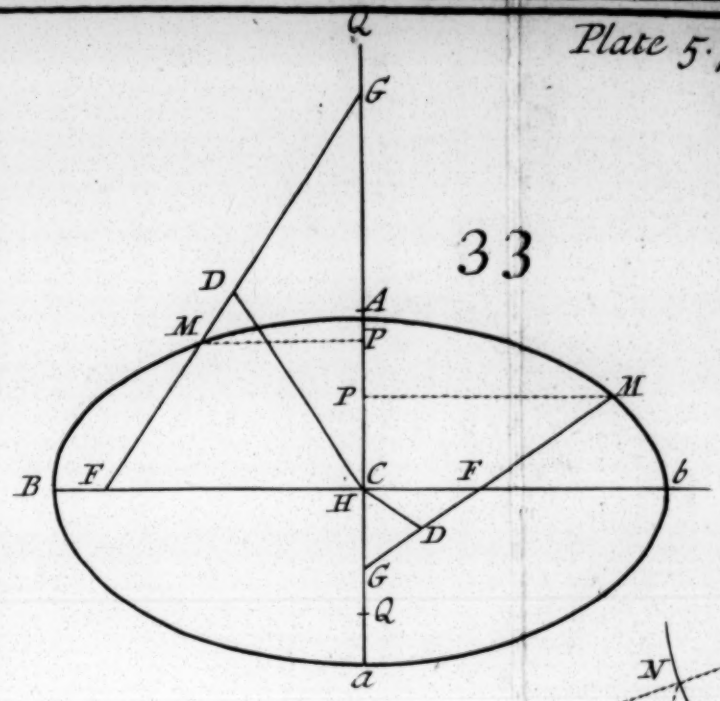
### DEFINITIONS.

II.

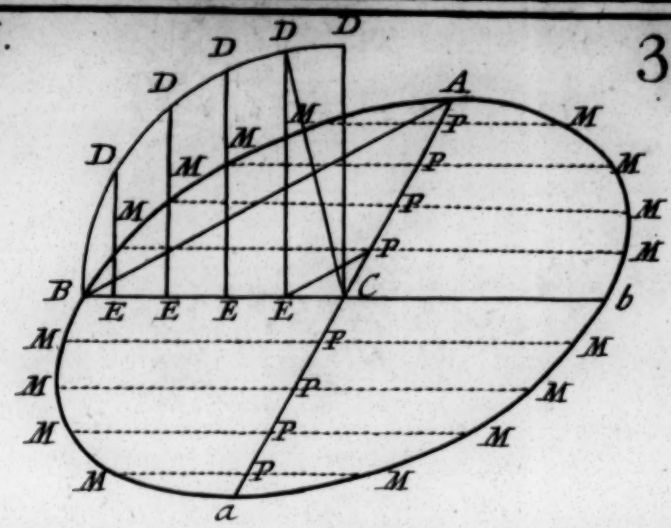
FIG. 40. If from the Centre $C$ the indefinite right Lines $CG$, $Cg$, be drawn parallel to the Lines $Ab$, $AB$, drawn from the End $A$ of the firſt Axis

Plate 5. page 46.

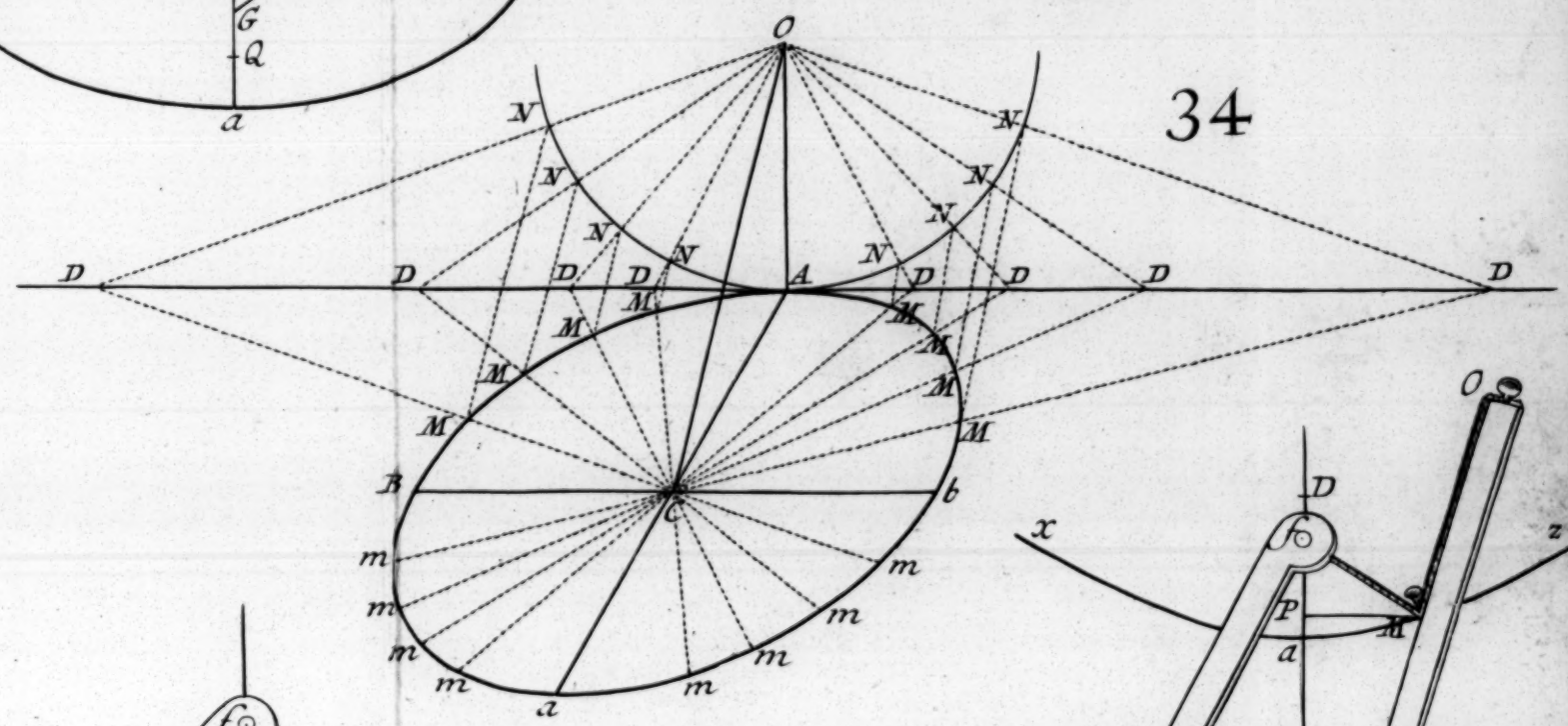
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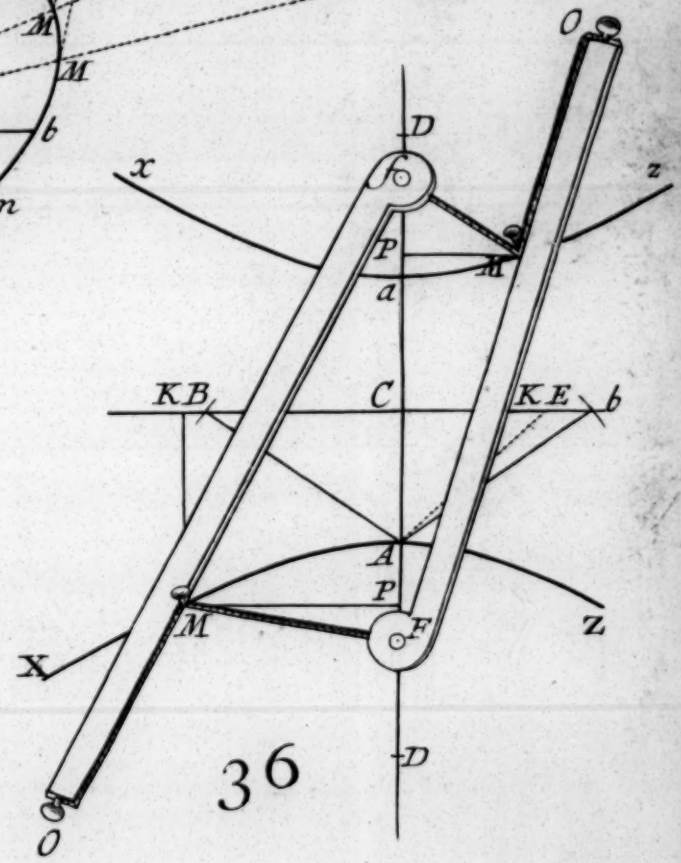



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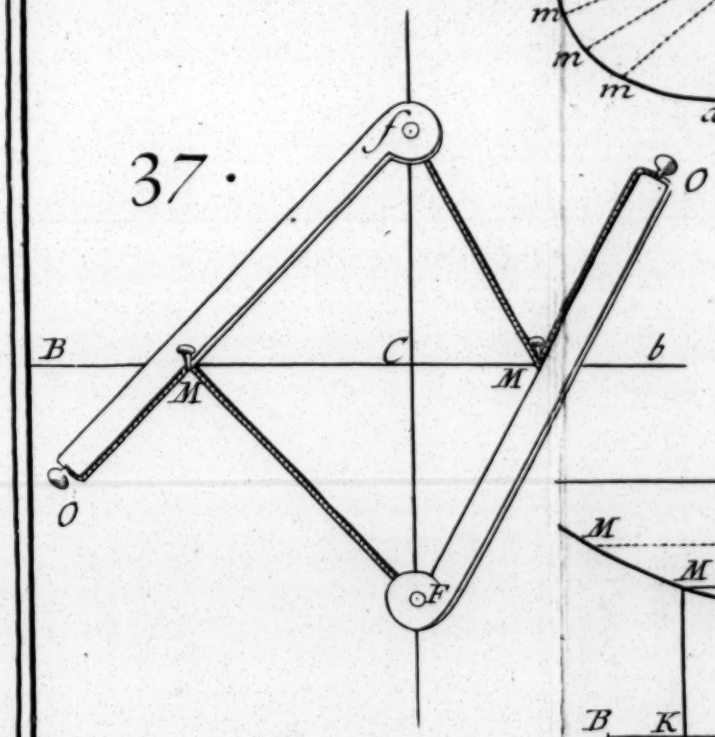

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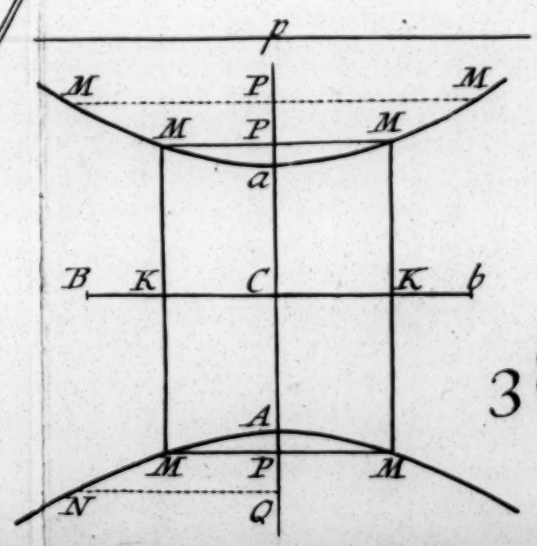



36

37.




38

Axis $Aa$, to the two Ends $B$, $b$, of the ſecond; then the ſaid right Lines $CG$, $Cg$, are called the *Aſymptotes* of the Hyperbola $MAM$; and being infinitely produced on the other Side of the Centre, they are called the *Aſymptotes* of the oppoſite Hyperbola $MaM$.

12.

The Square of $CG$ or $Cg$, that Part of an Aſymptote contain'd between the Centre $C$, and the Point of Concurrence of the Line $AB$, or $Ab$, drawn from $A$, the End of the firſt Axis, to $B$, or $b$, the Extremity of the ſecond Axis, is called the *Power* of the Hyperbola $MAM$, or $MaM$.

## COROLLARY I.

87. HENCE it is evident, that the Angle $GCg$, (or its equal $BAb$) form'd by the Aſymptotes of an Hyperbola, is leſſer, equal to, or greater than a right Angle; according as the ſecond Axis $Bb$ is leſs, equal to, or greater than the firſt $Aa$. For when the firſt Axis $Aa$ exceeds the ſecond $Bb$; then ($CA$) the half thereof, will exceed $CB$, the half of the ſecond Axis: and conſequently $CAB$, the Angle of the right-angled Triangle $CAB$, is leſs than half a right Angle; therefore the two equal Angles $CAB$, $CAb$, which together make up the Angle $BAb$, will be leſs than a right Angle. And after the ſame manner may the two other Caſes be demonſtrated.

## COROLLARY. II.

88. BEcauſe the Triangles $BAb$, $BGC$, are ſimilar, it is plain that the Line $AB$ is divided by the Aſymptote $CG$ into two equal Parts, in the Point $G$, and alſo $CG$ is equal to half of $Ab$; becauſe $BC$ is the half of $Bb$. After the ſame manner we prove, that $Ab$ is divided by the Aſymptote $Cg$, in two equal Parts in the Point $g$, and $Cg$ is the half of $AB$. Therefore all the Lines $CG$, $GA$, $GB$, $Cg$, $gA$, $gb$, are equal to each other: Becauſe every of them is equal to the half of $AB$, or $Ab$, which by *Def.* 5. are equal to one another.

## COROLLARY. III.

89. THE Power of an Hyperbola is equal to one fourth Part of the Sum of the Squares of the two Semiaxes. For if you call $CA$, $t$; $CB$, $c$; $CG$, $m$; we ſhall have * $BA = 2m$. And becauſe $ACB$ is a right-angled Triangle, the Square $\overline{AB}^2$ ($4mm$) will be $= tt + cc$. And conſequently $\overline{CG}^2$ ($mm$) $= \frac{tt+cc}{4}$. * *Art.* 88.

PRO-

## PROPOSITION III.

### Theorem.

FIG. 40. 90. *IF from any Point* M *in either of the oppoſite Hyperbola's, be drawn a right Line* R r *perpendicular to the firſt Axis* A a, *(meeting the ſame in* P,*) and terminating in the Aſymptotes in* R *and* r: *I ſay, the Rectangle under* RM, Mr, *is equal to the Square of* BC, *the half of the ſecond Axis* B b.

*We are to prove, that* $RM \times Mr = \overline{BC}^2$.

Call the known Quantities $CA, t$; $CB, c$; and the indeterminate ones, $CP, x$; $PM, y$; then becauſe the Triangles $ACB$, $CPr$, and $ACb$, $CPR$, are ſimilar; therefore $CA\,(t) : CB$ or $Cb\,(c) :: CP\,(x) : Pr$, or $PR = \frac{cx}{t}$. Whence $RM$ or $PR \pm PM = \frac{cx}{t} \pm y$; and $Mr$, or $Pr + PM = \frac{cx}{t} \mp y$. And conſequently $RM \times Mr = \frac{ccxx}{tt}$ $-yy = \overline{BC}^2\,(cc)$ in ſubſtituting $yy$ for its Value * $\frac{ccxx}{tt} - cc$. *W.W.D.*

* *Art.* 81.

### COROLLARY. I.

91. IT is manifeſt, that $\overline{PM}^2\left(\frac{ccxx}{tt} - cc\right)$ is always leſs than $\overline{PR}^2$ or $\overline{Pr}^2\left(\frac{ccxx}{tt}\right)$. And conſequently all the Points of the oppoſite Hyperbola's fall in the Angles formed by their Aſymptotes; ſo that none of them can fall in the adjoining Angles.

### COROLLARY II.

92. IF from any two Points *M*, *N*, in one Hyperbola, or in the oppoſite Hyperbola's, there be drawn two right Lines *Rr*, *Kk*, perpendicular to the firſt Axis, and terminating in the Aſymptotes: then it is manifeſt, that the Rectangles $RM \times Mr$, $KN \times Nk$, will be always equal to one another; becauſe each of them is equal to the Square of *BC*, the half of the ſecond Axis *Bb*. Whence it appears, that $RM : KN :: Nk : Mr$.

PRO-

## PROPOSITION IV.

### Theorem.

93. IF *the right Lines* Hh, Ll *be drawn from any two Points* (M, N) *of an Hyperbola, or the oppoſite Hyperbola's, parallel to one another, and terminating in the Aſymptotes: I ſay, the Rectangles* HM × Mh, LN × NI, *will be equal between themſelves.*

*We are to prove, that* HM × Mh = LN × NI.

The right Lines *Rr*, *Kk*, being drawn perpendicular to the firſt Axis *Aa*; it is then manifeſt, that the Triangles *MRH*, *NKL*, and *Mrh*, *Nkl* will be ſimilar; becauſe they are formed by Parallels: And therefore we have *RM* : *KN* :: *HM* : *LN*. And *Nk* : *Mr* :: *Nl* : *Mh*. But * *RM* : *KN* :: *Nk* : *Mr*. Therefore *HM* : *LN* :: *Nl* : *Mb*. And conſequently *HM* × *Mb* = *LN* × *Nl*. *W. W. D.* * *Art.* 92.

### COROLLARY I.

94. IF the Line *NL*, parallel to *MH*, be ſuppos'd to paſs through the Centre *C*; or, which is all one, be ſuppos'd to become the Line *CE*: Then it is evident, that the two Points *L*, *l*, will coincide in the Centre *C*; and ſo the Rectangle *LN* × *Nl*, will become the Square $\overline{EC}^2$. From whence it appears, that if the right Line *CE* be drawn from any Point *E* of one of the oppoſite Hyperbola's to the Centre *C*, and then likewiſe another Line *MHb* be drawn thro' any Point *M* of either of thoſe Hyperbola's, parallel to *CE*, and meeting the Aſymptotes in *H* and *b*; the Square of *CE* will be equal to the Rectangle under *HM* and *Mb*.

### COROLLARY II.

95. IF thro' any Point (*N*) of one of the oppoſite Hyperbola's be drawn a right Line *Ll*, terminating in the Aſymptotes, and meeting either of thoſe Hyperbola's in ſome other Point *n*; then the Parts *LN*, *ln*, of that right Line taken between the Points of the Hyperbola, and the Point of Concurrence of the Aſymptotes will be equal to one another. For if *LN*, be called *a*; *Nn*, *b*; *nl*, *c*; we ſhall have *LN* × *Nl* (*ab* ∓ *ac*) = *HM* × *Mb* = *Ln* × *nl* (*bc* ∓ *ac*) and ſo we get *LN* (*a*) = *ln* (*c*).

### COROLLARY III.

96. IF it be ſuppos'd in the laſt Corollary, that the Line $Nn$, terminating in the oppoſite Hyperbola's, paſſes thro' the Centre $C$; that is, if it be ſuppos'd to become the firſt Diameter $ED$; then it is manifeſt, that the two Points $L, l$, will coincide in the Centre $C$; and ſo $NL$, will become $EC$, and $nl$, $CD$. From whence it appears, that every firſt Diameter $DE$, is divided into two equal Parts by the Centre $C$.

### COROLLARY IV.

97. IF two right Lines $Mm$, $Nn$, being parallel to one another, be terminated by one Hyperbola, or the oppoſite Hyperbola's, and meet an Aſymptote in the Points $H, L$; I ſay, the Rectangles $MH \times Hm$, $NL \times Ln$, will be equal to one another: for if thoſe two Lines be produced (if neceſſary) until they meet one of the Aſymptotes in the Points $h, l$; then the Parts $MH$, $mh$, and $NL$, $nl$, will be * equal to one another: And therefore, ſince $HM \times Mh = LN \times Nl$, it follows, that $MH \times Hm = NL \times Ln$.

* *Art.* 95.

## PROPOSITION V.

### Theorem.

FIG. 41. 98. *IF thro' any two Points* M, N, *of an Hyperbola, or the oppoſite Hyperbola's, be drawn two right Lines* MH, NL, *parallel to each other, and terminating in one Aſymptote; as likewiſe two other right Lines* Mh, NI, *parallel to one another, and terminating in the other Aſymptote; I ſay, the Rectangles* H M × M h, N L × N I, *are equal to one another.*

This Propoſition is proved in the ſame manner as the laſt, the Demonſtration being the very ſame.

### COROLLARY I.

FIG. 42. 99. IF the right Lines $MH$, $Mh$, and $NL$, $Nl$, be parallel to the two Aſymptotes; then it is manifeſt, that the Parallelograms $MHCh$, $NLCl$, as likewiſe the Triangles $CHM$, $CLN$, being the Halves of them, are equal to one another; becauſe the Sides of the ſaid Parallelograms about the equal Angles $HMh$, $LNl$, are reciprocally proportional.

COROL-

## COROLLARY II.

100. THE ſame Things being premiſed as in the foregoing Corollary, it is manifeſt, that $CH \times HM = CL \times LN$; becauſe in this Suppoſition $Mb = CH$, and $Nl = CL$: that is, if two right Lines $MH$, $NL$, be drawn thro' any two Points $M$, $N$; in one, or the oppoſite Hyperbola's, parallel to one of the Aſymptotes, and terminating in the other; then the Rectangles $CH \times HM$, $CL \times LN$, will be always equal to one another; and ſo $CH : CL :: LN : MH$.

## COROLLARY III.

101. BEcauſe the End ($A$) of the firſt Axis, is one Point of the Hyperbola; and the Line $AB$, which cuts one of the Aſymptotes $CG$ in $G$, is parallel to the other Aſymptote $Cg$; therefore * the Rectangle $CH \times HM$ will always be equal to the ſame Rectangle $CG \times GA$, or to the Square * $\overline{CG}^2$, that is, (according to *Def.* 12.) equal to the Power of the Hyperbola. Then if you call the given Quantity $CG$, $m$; and the indeterminate ones, $CH$, $x$; $HM$, $y$; we ſhall have always $CH \times HM\ (xy) = \overline{CG}^2\ (mm)$. But becauſe this Property equally extends to all Points of the oppoſite Hyperbola's, and determines their Poſition with regard to the Aſymptotes; it is evident, that this Equation $xy = mm$ entirely expreſſes the Nature of the Hyperbola with regard to the Aſymptotes.

* *Art.* 100.

* *Art.* 88.

## COROLLARY IV.

102. BEcauſe $HM\ (y)$ is $= \frac{mm}{x}$, it follows, that the more $CH\ (x)$ increaſes, the more doth $HM\ (y)$ diminiſh; ſo that when $CH\ (x)$ becomes infinitely great, $HM\ (y)$ will then be infinitely ſmall; that is, equal to nothing. From whence it appears, that the Hyperbola $AM$, and its Aſymptote $CH$ (being both produced) will accede nearer and nearer to one another; ſo that at laſt their Diſtance will become leſs than any given Quantity; and yet they will never meet, unleſs it be at an infinite Diſtance, to which they can never be produced. The ſame is to be underſtood of the other Aſymptote $Cg$.

## COROLLARY V.

103. AMong all the Lines that paſs through the Centre $C$, (1.) Thoſe, (as $Aa$) that fall in thoſe Angles of the Aſymptotes next to the Hyperbola's, meet each of the oppoſite Hyperbola's in only

one Point *A*, or *a*; and being produced, will ever after be within the Hyperbola's: for becauſe of the Angles *GCA*, *gCA*, and thoſe vertical to them, it is manifeſt, that the Line *Aa*, recedes more and more from both the Aſymptotes; whereas the oppoſite Hyperbola's approach nearer * and nearer to them. (2.) Thoſe Lines (as *Bb*) which fall in the adjoining Angles, alſo formed by the ſaid Aſymptotes, will never meet the oppoſite Hyperbola's altho' infinitely produced; becauſe none of the Points of the Hyperbola's can fall * in theſe latter Angles.

* *Art.* 102.

* *Art.* 91.

Whence it appears, * that all firſt Diameters fall in the Angle, form'd by the Aſymptotes, next to the Curve, and the ſecond Diameters in the Angles adjoining to them.

* *Def.* 9.

## COROLLARY VI.

FIG. 43. 104. IF the Line *HM* be drawn through any Point *H*, in one of the Aſymptotes *CE*, parallel to the other Aſymptote *Ce*, then that Line *HM* will meet the Hyperbola in the Point *M* only; and being continued, will be ever after within the ſame: for the Diſtance from *HM* to *Ce*, remains every where the ſame; but the Hyperbola continually comes * nearer and nearer to *Ce*.

* *Art.* 102.

## COROLLARY. VII.

105. HENCE if two indefinite right Lines *MH*, *Mb*, be drawn thro' any Point *M* of an Hyperbola, parallel to the Aſymptotes *Ce*, *CE*,

1. All the Points of the oppoſite Hyperbola will fall in the Angle *HMb*; becauſe they all fall * in the Angle formed by the Aſymptotes, which is included in the Angle *HMb*.

* *Art.* 91.

2. The two Parts of the Hyperbola *MN* will fall in the Angles, on each Side *HMb*: ſo that no Point thereof will fall in the Angle vertical to *HMb*.

3. All Lines, as *MF*, which fall in the Angle *HMb*, and being continu'd towards *F*, do meet the oppoſite Hyperbola in one Point *N*, and fall within the Curve: becauſe they recede more and more from the right Lines *MH*, *Mb*, and conſeqnently from the Aſymptotes which are parallel to them: But being produced on the other Side of the Point *M*, they will fall within the Hyperbola paſſing thro' that Point *M*, and will never after meet the ſame.

4. All Lines, as *Ee*, which fall in the Angles adjoining to *HMb*, do meet the two Aſymptotes of the Hyperbola paſſing thro' the Point *M*; ſo when thoſe Lines fall within one Part of the Hyperbola, they muſt needs meet the ſame in ſome Point (*N*), ſince they go on to meet the Aſymptote falling without that Part. Co-

### COROLLARY VIII.

106. 1. IF a right Line $Ff$, be drawn through any Point $M$ of an Hyperbola, meeting one of its Aſymptotes in the Point $F$, and one of the Aſymptotes of the oppoſite Hyperbola in the Point $f$; and if the ſaid Line be prolonged to $N$, ſo that $fN$ be equal to $FM$: I ſay, the Point $N$ will be in the oppoſite Hyperbola. For the Line $Ff$, falls in the Angle $HMh$, and conſequently meets the oppoſite Hyperbola in ſome Point $N$, as we have demonſtrated in the laſt Corollary. Whence *, *&c.*

* *Art.* 95.

2. If from any Point $M$ of an Hyperbola, there be drawn a right Line $Ee$, terminating in the Aſymptotes, and if you take in it the Part $eN$ equal to $EM$: I ſay, the Point $N$ will yet be in that Hyperbola. For drawing $MH$ parallel to one Aſymptote $Ce$, and terminated by the other in $H$; then if $CL$ be taken in that Aſymptote, equal to $HE$, and the Line $LN$ be drawn parallel to $HM$; we have demonſtrated in *Art.* 104. that the Line $LN$ will meet the Hyperbola in one Point $N$; and in *Art.* 100. that that Point will be ſuch that $CL$ or $HE : HM :: CH$ or $EL : LN$. From whence it appears, that the Line $LN$, meets the Hyperbola in the ſame Point as it meets the right Line $Ee$. But becauſe $HM$, $LN$, are parallel, it is manifeſt that $eN = EM$, ſince $CL = HE$. Therefore, *&c.*

## PROPOSITION VI.

### Problem.

107. *FROM a given Point* M, *in an Hyperbola, whoſe Aſymptotes* CE, Ce, *are given, to draw the Tangent* DMd; *and demonſtrate, that there can be drawn but one only to that Point.* FIG. 43.

Draw the right Line $MH$ from the given Point $M$ parallel to one of the Aſymptotes $Ce$, and terminating in the other ($CE$) in the Point $H$; aſſume the Line $HD$ in $CE$ equal to $HC$, and draw the right Line $DM$ through the given Point $M$, meeting the Aſymptote $Ce$ in the Point $d$. I ſay in the firſt place, that the Line $DMd$, will touch the Hyperbola in the Point $M$.

For becauſe the Triangles $CDd$, $HMD$ are ſimilar, the Line $Dd$, terminated by the Aſymptotes, is divided by the Point $M$ into two equal Parts, like as $CD$ is in $H$. And if it were poſſible for $DMd$ to meet the Hyperbola in ſome other Point $O$, then it is manifeſt, that $Od$ would be * equal to $MD$, and conſequently to $Md$, that is, the Part equal to the whole; which is impoſſible: therefore the Line

* *Art.* 95.

$DMd$

*D M d* cannot meet the Hyperbola in any other Point but *M*. Moreover, if the ſaid Line ſhould fall within the Hyperbola, as the Line *Ee*, it is evident, that it would meet the Curve in ſome other Point
* *Art.* 91. *N*; becauſe it would meet * the Aſymptote falling without the Curve, in the Point *e*. Therefore it is plain, that the Line *D d*, meets the Hyperbola only in the Point *M*, and doth not fall within the ſame, that is, the ſaid Line touches the Hyperbola in the Point *M*.

2. I ſay, there is no Line but *D M d* only can touch the Hyperbola in the Point *M*: For if *H E* be taken in one of the Aſymptotes *C E*, either greater or leſs than *H D*, and if the right Line *E M* be drawn from the Point *M*, meeting the other Aſymptote *C e* in the Point *e*: Then becauſe *M H*, *C e* are parallel, it is manifeſt, that *M E* will be greater or leſs than *M e*; ſince *H E* was aſſum'd greater or leſs than *H D* or *H C*. Now this being premis'd, if the Point *N* be taken in the greater Part *M e*, ſo that *N e* be equal to *M E*; then it is evident,
* *Art.* 106. that the ſaid Point *N* will be * in the Curve, and ſo the Line *E e* will not touch the ſame in the Point *M*. Which was the ſecond Thing to be demonſtrated.

## SCHOLIUM.

108. IT has been demonſtrated in *Art.* 102. that the more *C H* increaſes, the more doth *H M* diminiſh; ſo that when *C H* becomes infinitely great, *H M* will become infinitely ſmall, or nothing. But when *C H* is infinitely great, then *H D* (being equal thereto) will be ſo likewiſe; and conſequently the Lines *M D*, *H D*, meeting one another at an infinite Diſtance, being taken as Parallels, will fall in each other; becauſe the Points *M* and *H* will then coincide: that is, the Aſymptote *C E* being infinitely produced, (as alſo the Hyperbola) may be taken for a Tangent to the Hyperbola, in the Extremity thereof. The ſame may be ſaid of the other Aſymptote *C e*, which may be eſteem'd as touching the ſame Hyperbola in the other Extremity thereof.

Hence it appears, that the two Aſymptotes may be taken as infinite Tangents touching the oppoſite Hyperbola's in the Extremities thereof.

## COROLLARY I.

109. SINCE there is but one Line *D M d*, only which terminating in the Aſymptotes, is divided into two equal Parts in the Point *M*; it follows, if any right Line *D M d*, terminating in the Aſymptotes of an Hyperbola, meets the ſame in the Point *M*, divi-

ding that right Line into two equal Parts; then that Line *D M d*, will touch the Hyperbola in the Point *M*. And contrariwiſe, if a right Line *D M d*, terminating in the Aſymptotes of an Hyperbola, touches the ſame in *M*; then will that Line be biſected in the Point *M*.

## COROLLARY. II.

110. IF any firſt Diameter *MCm* be drawn thro' *M* the Point of Contact of any Tangent *D M d*, terminating in the Aſymptotes *CL*, *Cl*, of an Hyperbola; and if *E e* be drawn thro' the Point *m*, wherein *MCm* meets the oppoſite Hyperbola, parallel to the Tangent *D d*, and terminating in the Aſymptotes in the Points *E*, *e*; then will the Line *E e* be a Tangent to the Hyperbola in the Point *m*. For the Triangles *CMD*, *CmE*, will be ſimilar and equal, becauſe * *CM* is equal to *Cm*: therefore the Line *mE* will be equal to *MD*. We prove after the ſame manner (becauſe the Triangles *CMd*, *Cme* are ſimilar and equal) that *me* is equal to *Md*: therefore the Line *Ee* is divided in the Point *m* into two equal Parts; becauſe *D d* is ſo divided in the Point *M*; and conſequently the Line *E e*, touches * the Curve in the Point *m*. FIG. 44. * *Art.* 96. * *Art.* 109.

Hence it appears, that the Tangents *D d*, *E e*, paſſing thro' the Extremities of any firſt Diameter *Mm*, are parallel to one another; and alſo equal, when they are terminated by the Aſymptotes.

## DEFINITIONS.

13.

If there are two Diameters *Mm*, *Ss*, whereof one, as *Ss*, is parallel to the Tangents paſſing through the Extremities of the other *Mm*; and terminated beſides in *S*, *s*, by the right Lines *MS*, *Ms*, drawn through the Point *M*, one End of the Diameter *Mm*, parallel to the Aſymptotes; the ſaid two Diameters *Mm*, *Ss* are call'd together *Conjugate Diameters*. FIG. 44.

14.

Right Lines drawn from Points of the oppoſite Hyperbola's parallel to one of the Conjugate Diameters, and terminating in the other, are called *Ordinates* to that Diameter. So *NO* is an Ordinate to the Diameter *Mm*.

15.

If a third Proportional be taken to the two Conjugate Diameters, then the ſame will be the *Parameter* of that Diameter, which is the firſt Term of the Proportion.

Co-

### COROLLARY. I.

111. THE thirteenth *Definition* hath Relation to the two Axes; becauſe, according to *Art.* 84. the ſecond Axis is parallel to the Tangents paſſing thro' the Ends of the firſt; and moreover (by *Def.* 11.) is terminated by two right Lines drawn from one End of the firſt Axis parallel to the Aſymptotes: Whence it appears, that the two Axes may be taken as two Conjugate Diameters, being at right Angles with one another.

### COROLLARY II.

112. BEcauſe the Diameter $SCs$, is parallel to the Tangent $DMd$, paſſing through $M$, one End of the Diameter $MCm$; and ſince that Tangent meets the two Aſymptotes ($CD$, $Cd$) of the Hyperbola, paſſing through the Point $M$: therefore the Diameter $SCs$ falls in the Angles adjacent to the Angle $DCd$, form'd by the Aſymptotes of the Hyperbola; and ſo it will be a ſecond Diameter.

Hence it appears, that among any two Conjugate Diameters, as $MCm$, $SCs$, there is always a firſt Diameter $Mm$, and a ſecond Diameter $Ss$.

### COROLLARY. III.

113. THE ſecond Diameter $SCs$, is divided into two equal Parts by the Centre $C$, and is alſo equal to the Tangent $DMd$, which paſſing thro' $M$, one End of the firſt Diameter $Mm$, being a Conjugate to $SCs$, does terminate in the Aſymptotes. For ſince $MS$, $Cd$, and $Ms$, $CD$, are parallel; it is manifeſt, that $CS$ is equal to $Md$, and $Cs$ to $MD$. But $DMd$, is divided * into two equal Parts in $M$ the Point of Contact. Therefore, *&c.*

**Art.* 109.

### COROLLARY IV.

114. IF two Conjugate Diameters $Mm$, $Ss$ be given, and it is known which of them is the firſt Diameter; then you may have the Aſymptotes $CD$, $Cd$, in drawing right Lines parallel to the two right Lines $MS$, $Ms$, (drawn from $M$ the Extremity of the firſt Diameter $Mm$, to $S$, $s$, the two Ends of the ſecond.)

And contrariwiſe, if the two Aſymptotes $CD$, $Cd$, of an Hyperbola be given, together with ſome Point $M$ thereof; you may find the two Conjugate Diameters $MCm$, $SCs$, by drawing the Line $MH$ parallel to one of the Aſymptotes ($Cd$), meeting the other Aſymptote

$CD$

$CD$ in $H$, and producing the ſame to $S$, ſo that $HS$ be equal to $HM$; and then drawing the right Lines $CM$, $CS$; for if $MD$ be drawn parallel to $CS$, it is evident, (ſince the Triangles $CHS$, $MHD$, are ſimilar,) that $HD$ is equal to $HC$, becauſe $MH$ is equal to $HS$; and ſo $MD$ touches * the Curve in $M$: Therefore, by *Def.* 13. the Lines $CM$, $CS$, are two Semi-Conjugate Diameters. * *Art.* 107.

Hence, if two Conjugate Diameters $Mm$, $Ss$ be given in Poſition and Magnitude, and if it be known which of them is the firſt Diameter, then the two Aſymptotes $CD$, $Cd$, together with the Point $M$, one Point of the oppoſite Hyperbola's, is had.

And contrariwiſe, the Aſymptotes $CD$, $Cd$ of an Hyperbola being given, together with one Point $M$ of the ſame; we have the two Conjugate Diameters $Mm$, $Ss$ thereof given both in Poſition and Magnitude; as likewiſe we know which of them is the firſt Diameter, being that paſſing thro' the given Point $M$.

## COROLLARY IV.

115. ANY ſecond Diameter $SCs$, being given in Poſition, if the Magnitude thereof be requir'd, as alſo the firſt Diameter $Mm$, the Conjugate to $SCs$, you muſt draw the right Line $Ll$, any where in the Angle form'd by the Aſymptotes, parallel to the ſecond Diameter, and terminated by the Aſymptotes in the Points $L$, $l$; and then through $O$ the middle of $Ll$, you muſt draw the firſt Diameter $CO$, meeting the Hyperbola in one Point $M$. This being done, if the right Lines $MS$, $Ms$, be drawn from the Point $M$ parallel to the Aſymptotes: It is manifeſt, by *Def.* 13. that the Points $S$, $s$, wherein thoſe Parallels meet the ſecond Diameter $SCs$ given in Poſition, do determine the Magnitude thereof; as alſo that the firſt Diameter $MCm$ is the Conjugate thereto. For if the Line $Dd$ be drawn thro' the Point $M$, parallel to $Ll$, and terminating in the Aſymptotes; then the ſaid Line will be biſected in the Point $M$, becauſe $Ll$ is ſo in the Point $O$; and therefore $Ll$ will touch * the Curve in the Point $M$. * *Art.* 109.

Hence it is evident, that any ſecond Diameter $SCs$ being given in Poſition, the Magnitude thereof is ſo determin'd, as that it cannot vary; as likewiſe the Magnitude and Poſition of the firſt Diameter $Mm$, which is a Conjugate thereto.

## COROLLARY V.

116. ANY ſecond Diameter $SCs$, being given in Poſition and Magnitude, together with the Parameter, and the Poſition of the Ordinates to it; then it will not be difficult to find the Poſition

and Magnitude of the first Diameter $M\,C\,m$, being the Conjugate to $S\,C\,s$, as also its Parameter. For through the Centre $C$ draw an indefinite right Line parallel to the Ordinates to the Diameter $S\,s$, and denote two Points $M, m$ in this Line equally distant each way from the Centre $C$; so that $M\,m$ be a mean Proportional between the second Diameter $S\,s$, and the Parameter thereof. Then if a third Proportional to the two Lines $M\,m$, $S\,s$ be found, it is manifest, by *Def.* 14, and 15. that $M\,m$ will be the first Diameter, being the Conjugate to $S\,s$, and the Parameter thereof will be that third Proportional.

## PROPOSITION VII.

### Theorem.

FIG. 44. 117. *THE Square of any Ordinate* (O N) *to the first Diameter* M m, *is to the Rectangle under* M O, O m, *the Parts of that Diameter produced; as the Square of its Conjugate Diameter* S s, *to the Square of that first Diameter* M m.

*We are to prove, that* $\overline{O\,N}^2 : M\,O \times O\,m :: \overline{S\,s}^2 : \overline{M\,m}^2$.

If the right Line $D\,d$, be drawn through one End ($M$) of the first Diameter $M\,m$, parallel to the second Diameter $S\,s$, and terminating in the Asymptotes; then, (by *Def.* 13.) that Parallel will touch the
*Art. 109. Curve in the Point $M$, and so will be * bisected by that Point: therefore if the Ordinate $O\,N$ (which by *Def.* 13. is parallel to the Diameter $S\,s$) be produced both ways from the Diameter $M\,m$, the same will meet the Asymptotes in two Points $L$, $l$, each way equally distant from the Point $O$. This being premis'd, call the given Quan-
*Art. 113. tities $C\,M$, or $C\,m$, $t$; $C\,S$, or $C\,s$, or * $M\,D$, or $M\,d$, $c$; and the indeterminate Quantities $C\,O$, $x$; $O\,N$, $y$; then because the Triangles $C\,M\,D$, $C\,O\,L$, are similar, we have this Proportion, $C\,M\,(t) : M\,D\,(c) :: C\,O\,(x) : O\,L$ or $O\,l = \frac{cx}{t}$. Whence $L\,N$ or $L\,O \pm O\,N = \frac{cx}{t} \pm y$, and $N\,l$ or $O\,l \mp N\,O = \frac{cx}{t} \mp y$; and therefore $L\,N \times N\,l = \frac{ccxx}{tt} - yy$
*Art. 90. $= * D\,M \times M\,d = cc$. Whence it follows, that $\overline{O\,N}^2\,(yy) : M\,O \times O\,m\,(xx - tt) :: \overline{S\,s}^2\,(4cc) : \overline{M\,m}^2\,(4tt)$. Because by multiplying the Means and Extremes, we have $4ttyy = 4ccxx - 4cctt$; that is (by dividing by $4tt$, and transposing) the same Equation $\frac{ccxx}{tt} - yy = cc$, as at first. *W. W. D.*

A Se-

### *A General* COROLLARY.

118. HENCE it is manifeſt, that what has been demonſtrated * in *Prop.* 2. with regard to the two Axes *A a*, *B b*, extends it ſelf, by means of this Propoſition, to any two Conjugate Diameters *M m*, *S s*. And becauſe the 80th, 81ſt, 82d, 83d, 84th and 85th Articles ariſe from the ſecond Propoſition, and are of equal Force, whether the Angle *ACB* be a right one or not; therefore it follows, that if the Lines *Aa*, *Bb*, inſtead of the two Axes, be ſuppoſed in theſe Articles to be any two Conjugate Diameters; the ſaid Articles will yet be true according to this Suppoſition, for their Demonſtration remains always the ſame; and there is nothing more requir'd to make this appear, but reading them over again, and uſing the Word Diameter for Axis. * *Art.* 79.

## PROPOSITION VIII.

### Theorem.

119. IF DE, FG *be any two Tangents to the Hyperbola* MA, *terminating in the Aſymptotes, and cutting one another in the Point* O; *I ſay, the Sides of the Triangles* CDE, CFG, *about the common Angle* C, *are reciprocal proportional.* FIG. 45.

*We are to prove, that* CD : CF :: CG : CE.

Draw the Lines *MH*, *AL* through the Points of Contact *M*, *A*, parallel to the Aſymptote *CG*; then it is manifeſt, (becauſe the Triangles *CDE*, *HDM* are ſimilar) that *CD* is the Double of *CH*, and *CE* the Double of *HM*; ſince *DE* is * the Double of *DM*. And becauſe the Triangles *CFG*, *LFA* are ſimilar, *CF* is the Double of *CL*, and *CG* the Double of *LA*, becauſe *FG* is the Double of *FA*. But * *CH* : *CL* :: *LA* : *HM*. And therefore, if the Double of each Term be taken, we ſhall have 2*CH* or *CD* : 2*CL* or *CF* :: 2*LA* or *CG* : 2*HM* or *CE*. *W. W. D.* * *Art.* 109. * *Art.* 100.

### COROLLARY.

120. IT follows from this Propoſition, that the right Lines *DG*, *FE*, are parallel to one another: Whence it is manifeſt,

1. The Triangles *CDE*, *CFG*, are equal: For the Triangles *FDE*, *FGE*, having the ſame Baſe *FE*, and being between the ſame Parallels *DG*, *FE*, are equal; and therefore, if the ſame Triangle *CFE* be added to both the Triangles *CDE*, *CFG*, there will be form'd the Triangles *CDE*, *CFG*, which ſhall be equal to one another.

2. The Line $DE$ is divided in the ſame Proportion in the Points $M$, $O$, as the Line $FG$ is in the Points $A$ and $O$. For if the right Line $MA$ be drawn thro' the Points of Contact, then it is manifeſt, that this Line will be parallel to the two right Lines $DG$, $FE$; becauſe it biſects the right Lines $DE$, $FG$, included between thoſe Parallels.

## PROPOSITION IX.

### Theorem.

FIG. 46, and 47. 121. *IF through any Point* M *in an Hyperbola, be drawn an Ordinate* (MP) *to any one of its Diameters* Aa, *and if the Tangent* MT *be alſo drawn meeting that Ordinate in* T; *I ſay,* CP : CA :: CA : CT. *Obſerving that the Points* P, T *fall on the ſame Side the Centre* C, *when the Line* Aa *is a firſt Diameter; and on both Sides, when it is a ſecond Diameter.*

FIG. 46. *Caſe* 1. When the Line $Aa$ is a firſt Diameter, produce the Tangent $MT$, meeting the Aſymptotes $CD$, $CG$, in the Points $D$, $E$; and produce the Ordinate P$M$, meeting the Aſymptote $CD$ in the Point $N$; alſo draw the Line $AK$ through the Point $A$ parallel to $DE$, meeting the Aſymptote $CG$ in the Point $K$; and likewiſe draw the Tangent $FG$, terminating in the Aſymptotes (which will be * parallel to $PM$) and meeting the other Tangent $DE$ in the Point $O$.

* *Def.* 14.

This being laid down, $AP$ is to $AC$, or $FN$ to $FC$, in a Ratio compounded of $FN$ to $FD$, or of $OM$ to $OD$, or * of $OA$ to $OG$, or of $EK$ to $EG$, and of $FD$ to $FC$, or * of $EG$ to $EC$. But $AT$ is to $TC$, or $KE$ to $EC$, in the Ratio compounded of $EK$ to $EG$, and of $EG$ to $EC$: therefore $AP : AC :: AT : TC$. Becauſe the Ratio's compounded of thoſe two Ratio's are the ſame; and conſequently $AP + AC$, or $CP : CA :: AT + TC$, or $CA : CT$. *Which was in the firſt place to be demonſtrated.*

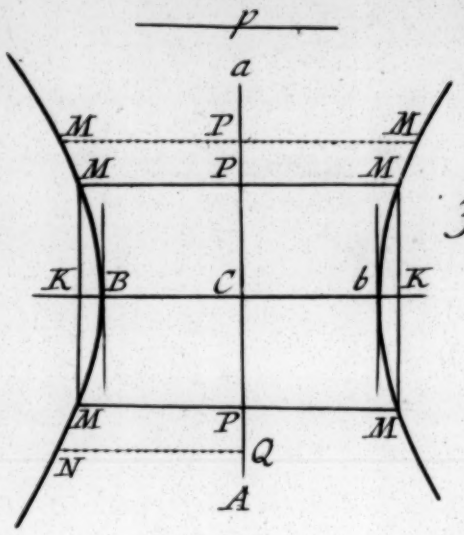
* *Art.* 120.

* *Art.* 120.

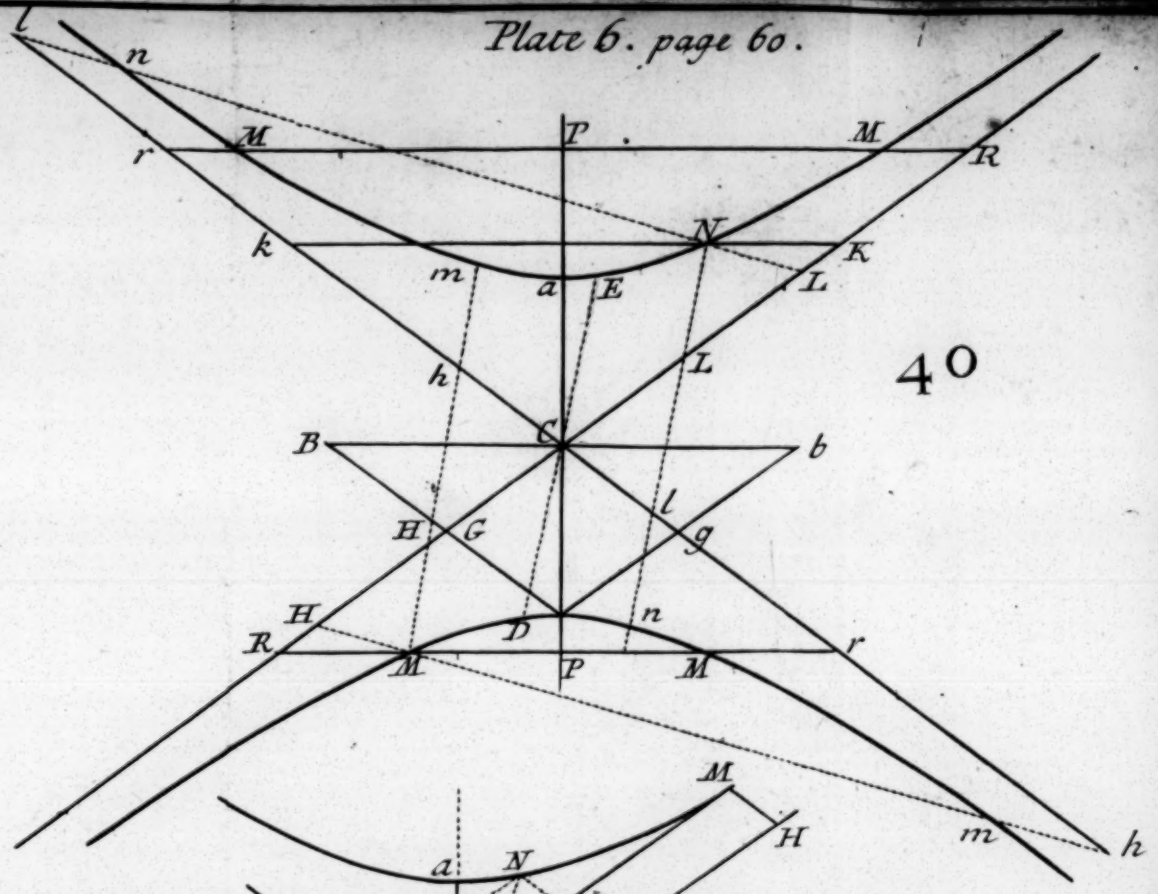
FIG. 47. *Caſe* 2. When the Line $Aa$ is a ſecond Diameter, draw the Line $CK$ through the Centre C parallel to the Ordinate $PM$, meeting the Hyperbola in the Point $B$, and the Tangent $MT$, in the Point $R$, and draw the Line $MK$ through the Point of Contact $M$ parallel to $Aa$; then it is manifeſt, that $CB$ will be a firſt Semi-Diameter, the Conjugate to the Second $Aa$; and ſo $MK$ will be an Ordinate to that Diameter.

This being premiſed, if the given Quantities $CA$, or $Ca$ be called $t$; $CB$, $c$; and the indeterminate Quantities $CP$ or $MK$, $x$; $PM$ or $CK$, $y$; then from what has been demonſtrated in *Caſe* 1. we have $CR = \frac{cc}{y}$; and therefore $RK$ or $CK - CR = \frac{yy - cc}{y}$. But the ſimilar Triangles $KRM$, $CRT$, give this Proportion $KR\left(\frac{yy-cc}{y}\right) : RC\left(\frac{cc}{y}\right) :: MK$

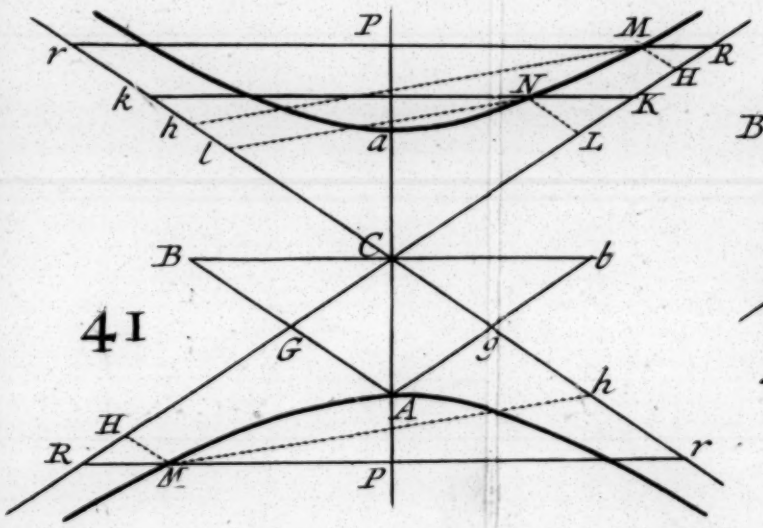
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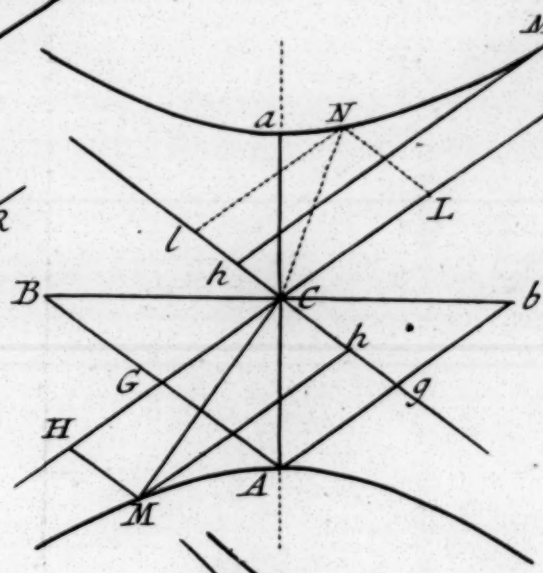

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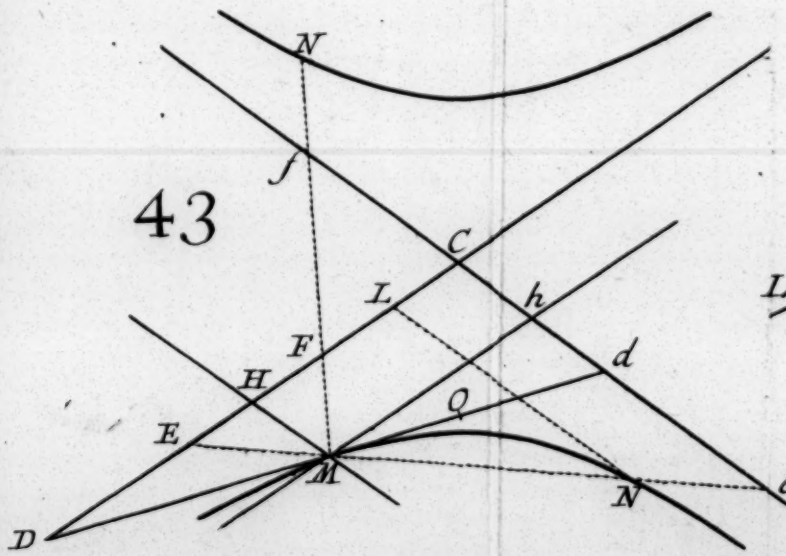

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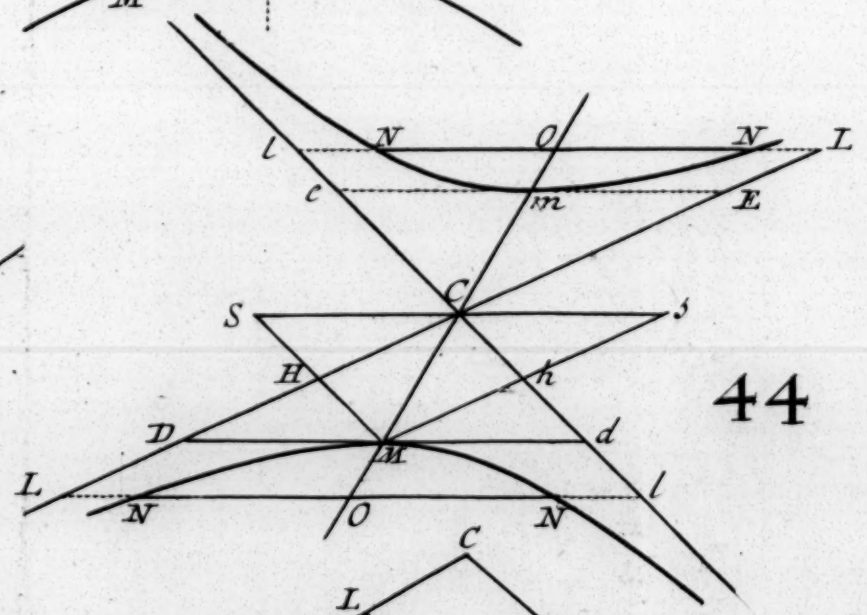

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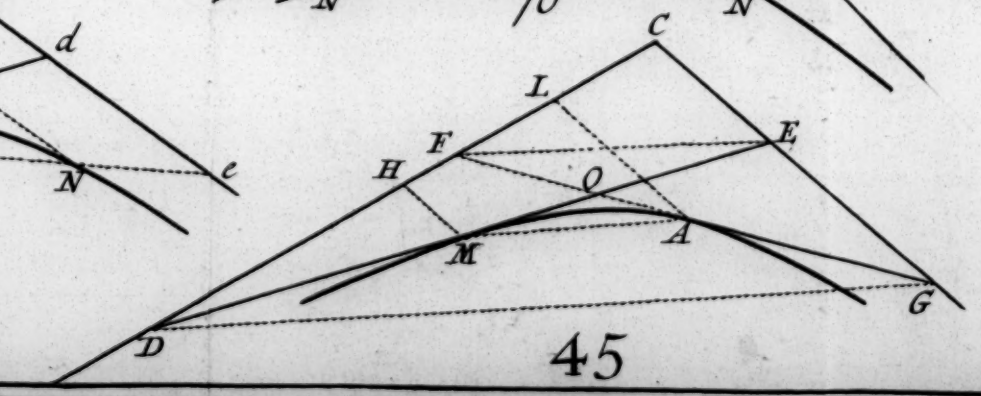

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:: $MK(x) : CT = \frac{cxx}{yy-cc} = \frac{tt}{x}$, (by ſubſtituting $\frac{ccxx}{tt}$ for its Value $yy-cc$ (becauſe $yy = {}^{*}\frac{ccxx}{tt} + cc$.) That is, $CP : CA :: CA : CT$. *W. W. D.* * *Art.* 80, *and* 118.

## PROPOSITION X.

### Theorem.

122. *IF through any Point* M *in an Hyperbola, whoſe Centre is* C, *there be drawn an Ordinate* MP *to one of the Axes* Aa, *as alſo the Perpendicular* MG *to the Tangent* MT *paſſing through* M: *I ſay,* CP *will always be to* PG, *in the given Ratio of the Axis* Aa *to the Parameter thereof.* FIG. 48, *and* 49.

For call the Semi-Axis $CA$ or $Ca$, $t$; and the indeterminate Quantities $CP, x$; $PM, y$; then we ſhall have * $CT = \frac{tt}{x}$; and therefore * *Art.* 121.

$PT = \frac{xx \mp tt}{x}$, according as $Aa$, is the firſt or ſecond Axis. But the right-angled ſimilar Triangles $TPM$, $MPG$, give this Proportion, $TP\left(\frac{xx \mp tt}{x}\right) : PM(y) :: PM(y) : PG = \frac{xyy}{xx \mp tt}$. Whence we get this Proportion, *viz.* $CP(x) : PG\left(\frac{xyy}{xx \mp tt}\right) :: \overline{CP}^2 \mp \overline{CA}^2 \ (xx \mp tt) : \overline{PM}^2 (yy)$. Since by multiplying the Means and Extremes, the ſame Product $xyy$ ariſes. But $\overline{CP}^2 \mp \overline{CA}^2$ is to $\overline{PM}^2$, as * the Axis $Aa$ to the Parameter thereof. Therefore $CP$ is to $PG$ likewiſe in the ſame Ratio. *W. W. D.* * *Art.* 81.

## PROPOSITION XI.

### Theorem.

123. *IF the right Lines* MF, Mf, *be drawn from any Point* M, *in an Hyperbola, to the two Foci* F, f: *I ſay, the Tangent* MT, *paſſing through that Point* M, *does divide the Angle* FMf *into two equal Parts.* FIG. 50.

For draw the Lines $FD, fd$, perpendicular to the Tangent $MT$; likewiſe draw the firſt Axis $Aa$, paſſing through the two Foci $F, f$, and meeting the Tangent in $T$; as likewiſe the Ordinate $MP$ to that Axis: Then call the given Quantities $CA$ or $Ca, t$; $CF$ or $Cf, m$; and the indeterminate Quantity $CP, x$. This being done, we have $MF$ * * *Art.* 78.

$\left(\frac{mx}{t} - t\right) : Mf\left(\frac{mx}{t} + t\right) :: TF$ or $CF(m) - CT$ * $\left(\frac{tt}{x}\right) : Tf$ or $Cf$ * *Art.* 121.

(m)

$(m) + CT\left(\frac{tt}{x}\right)$. ſince by multiplying the Means and Extremes, the ſame Product ariſes. But the right-angled ſimilar Triangles $TFD$, $Tfd$, give this Proportion, $TF : Tf :: FD : fd$; therefore the Hypothenuſe $(MF)$ of the right-angled Triangle $MDF$, will be to the Hypothenuſe $Mf$, of the right-angled Triangle $Mdf$, as the Side $DF$ is to the Side $df$; and conſequently theſe two Triangles will be ſimilar. Therefore the Angles $FMD, fMd$, which are oppoſite to the Homologous Sides $DF, df$, will be equal to one another. *W. W. D.*

### COROLLARY.

124. HENCE it is manifeſt, that the Tangent $MT$ being infinitely produced both ways from $(M)$ the Point of Contact, leaves the Hyperbola $AM$ entirely next to its Focus $F$. And becauſe this every where happens, let the Point $M$ be taken where it will in the Curve: Therefore it is manifeſt, that the Hyperbola being extended never ſo much, is Concave next to its Focus $F$.

## PROPOSITION XII.

### Theorem.

FIG. 51. 125. *THE Difference of the Squares of any two Conjugate Diameters* M m, S s, *is equal to the Difference of the Squares of the two Axes* A a, B b.

*We are to prove, that* $\overline{CS}^2 - \overline{CM}^2 = \overline{CB}^2 - \overline{CA}^2$, *or* $\overline{CM}^2 - \overline{CS}^2 = \overline{CA}^2 - \overline{CB}^2$.

If the right Lines $MS$, $AB$ are drawn, they will be * parallel to one of the Aſymptotes $Cg$; as alſo cut into equal Parts by the other Aſymptote $CG$ in the Points $H, G$; becauſe * the Lines $Ms$, $Ab$, are parallel to the Aſymptote $CG$, and the ſecond Diameters $Ss$, $Bb$, are * divided by the Centre $C$ into two equal Parts; therefore, if the right Lines $AF$, $BE$, $ML$, $SK$, be drawn perpendicular to the Aſymptote $CG$, the Triangles $GAF$, $GBE$, and $HML$, $HSK$, will be formed, which ſhall be ſimilar and equal. This being premis'd, call the given Quantities $CG$ or * $GA$, $m$; $GE$ or $GF$, $a$; $AF$ or $BE$, $b$; and the indeterminate Quantities $CH$, $x$; $HM$, $y$; then we have $CE = m + a$, $CF = m - a$; $\overline{CE}^2 + \overline{EB}^2$, or $\overline{CB}^2 = mm + 2am + aa + bb$, $\overline{CF}^2 + \overline{FA}^2$, or $\overline{CA}^2 = mm - 2am + aa + bb$: And therefore $\overline{CB}^2 - \overline{CA}^2 = 4am$. But the ſimilar Triangles $GAF$, $HML$, give.

* *Def.* 11, *and* 13.
* *Def.* 11, *and* 13.
* *Art.* 113.
* *Art.* 88.

give this Proportion, $GA\,(m) : AF\,(b) :: HM\,(y) : ML$ or $KS = \frac{by}{m}$. And $GA\,(m) : GF\,(a) :: HM\,(y) : HL$ or $HK = \frac{ay}{m}$. Therefore $CK = x + \frac{ay}{m}$, $CL = x - \frac{ay}{m}$; $\overline{CK}^2 + \overline{KS}^2$ or $\overline{CS}^2 = xx + \frac{2axy}{m} + \frac{aayy}{mm} + \frac{bbyy}{mm}$, $\overline{CL}^2 + \overline{LM}^2$ or $\overline{CM}^2 = xx - \frac{2axy}{m} + \frac{aayy}{mm} + \frac{bbyy}{mm}$. And therefore, $\overline{CS}^2 - \overline{CM}^2 = \frac{4axy}{m} = 4am$, by putting * $mm$ for $xy$. And consequently $\overline{CS}^2 - \overline{CM}^2 = \overline{CB}^2 - \overline{CA}^2$. *W. W. D.*

* *Art.* 101.

If the Angle $GCg$, formed by the Aſymptotes, ſhould be acute; whereas in this Figure, and the Reaſoning appropriated thereto, it is obtuſe; then $CF$ would be greater than $CE$; and it would be proved after the ſame manner, that $\overline{CM}^2 - \overline{CS}^2 = \overline{CA}^2 - \overline{CB}^2$. But if the Angle $GCg$, form'd by the Aſymptotes, was a right Angle; then it is manifeſt, that the Lines $AB$, $MS$, would be perpendicular to the Aſymptote $CG$; and ſo the two Semi-Conjugate Diameters $CM$, $CS$, would be equal to one another, like as the two Semi-Axes $CA$, $CB$. But becauſe the Difference of the two Conjugate Diameters $Mm$, $Ss$, is nothing; as likewiſe the Difference of the two Axes $Aa$, $Bb$; therefore it follows, that this Propoſition is true in all its Caſes.

## COROLLARY.

126. HENCE it is manifeſt, that any firſt Diameter $Mm$, is leſs, greater than, or equal to the ſecond Diameter $Ss$, being the Conjugate thereto; according as the Angle $GCg$, formed by the Aſymptotes, is obtuſe, acute or right.

## DEFINITION.

16.

Two oppoſite Hyperbolas are called *Equilateral*, when any two of their Conjugate Diameters are equal to one another; or elſe when the Angle form'd by their Aſymptotes is a right Angle.

## COROLLARY.

127. IF from any Point $M$ in an equilateral Hyperbola, there be drawn any Ordinate $(MP)$ to either of the Diameters, as $Aa$, FIG. 52.

then we ſhall have * $\overline{MP}^2 = \overline{CP}^2 \mp \overline{CA}^2$: *viz.* —, when $Aa$ is a firſt Diameter and +, when it is a ſecond Diameter. For the Conjugate Diameter to $Aa$, will be * always equal to it. * *Art.* 81, *and* 118. * *Art.* 126.

PRO-

## PROPOSITION XIII.

### Problem.

FIG. 53, 54, and 55. 128. *GIVEN two Conjugate Diameters being given, and knowing which of them is the first Diameter; or, which comes* * *to the same thing, the Asymptotes* CD, CF *of an Hyperbola being given, together with any Point* (M) *of the Curve: to draw two Conjugate Diameters* Aa, Bb, *that shall make an Angle with each other, equal to an Angle given.*

* *Art.* 114.

In any Circle whose Centre is *o*, draw the Chord *df*, so that the Angle in the Segment *dcf* be equal to the Angle *DCF* form'd by the Asymptotes; and draw the Line *ec* through *e* the Middle of the Chord *df*, making the Angle *dec*, or *fec* with that Chord equal to the given Angle; and thro' the Point *c*, wherein the Line *ec* meets the Arc *dcf*, draw the right Lines *cd*, *cf*. This being done, assume *CD*, *CF*, in the Asymptotes equal to the Chords *cd*, *cf*; then if *DF* be drawn, and the *second Diameter Bb* be drawn parallel to the same; and the first Diameter *Aa*, through *E* the middle Point; I say, the two Diameters *Aa*, *Bb*, make an Angle with one another equal to the given Angle, and they are Conjugate Diameters.

For by Construction, the Angle *dcf*, is equal to the Angle *DCF* form'd by the Asymptotes; and consequently the Triangles *DCF*, *dcf*, and *DCE*, *dce*, are equal and similar; therefore the Angle *BCa* made by the Diameters *Aa*, *Bb*, will be equal to the Angle *DEC* or *dec*, which was made equal to the given Angle. Moreover, if through the Point *A*, one End of the first Diameter *Aa*, there be drawn a Parallel to *DF*, it is manifest, that this Parallel will be divided into two equal Parts by the Point *A*, because *DF* is so divided in the Point *E*; and so * that Parallel will touch the Curve in *A*; therefore * *Aa*, *Bb*, are Conjugate Diameters.

* *Art.* 109.

* *Def.* 13.

Now to determine the Magnitudes of the said two Conjugate Diameters, draw the Line *MKL* thro' the given Point *M* parallel to the Diameter *Aa*, meeting one of the Asymptotes (*CD*) in the Point *K*, and the other Asymptote *CF*, (produced beyond the Centre *C*) in the Point *L*: This being done, if *CA* be taken a mean Proportional between *KM*, *ML*; then it is * manifest, that the Point *A* will be one End of the first Diameter *Aa*; and so if the Lines *AB*, *Ab*, are drawn parallel to the Asymptotes *CF*, *CD*, those * Parallels will determine the Magnitude of the second Diameter *Bb*, by their Points of Concurrence *B*, *b*.

* *Art.* 94.

* *Def.* 13.

Because there can be drawn two different Lines *ec*, *ec*, making the Angles *dec*, *fec* each way with the Chord *df*, equal to the given Angle, if it be not a right Angle; therefore we can find always two diffe-

different Pair of Conjugate Diameters (*A a*, *B b*) which will anſwer the Problem, as they be ſeen in *Fig.* 54, and 55. But it muſt be noted, that the Conjugate Diameters *A a*, *B b*, of *Fig.* 55. have the ſame Poſition with reſpect to the Aſymptote *CF*, as thoſe of *Fig.* 54. have to the other Aſymptote *CD*; and their Magnitudes will remain the ſame in theſe two different Poſitions. For,

1. If the Line *o e* be drawn from the Centre *o* to *e*, the Middle of the Chord *d f*; this Line will be perpendicular to that Chord: and conſequently the Angles *o e c*, *o e c*, will be equal; therefore drawing the Radii *o c*, *o c*, the Triangles *o e c*, *o e c*, which have the Side *o e* common, the Angles *o e c*, *o e c*, and the Sides *o c*, *o c*, equal to one another, will have alſo their third Sides *e c*, *e c*, equal: therefore the Triangles *f e c*, *d e c*, which have the Sides *e f*, *e d*, and *e c*, *e c*, as alſo the Angles *f e c*, *d e c*, equal, will be equal and ſimilar: And ſo it appears, that the Angle *ecf*, or *ECF*, of *Fig.* 55. is equal to the Angle *ecd*, or *ECD*, of *Fig.* 54. and conſequently the Poſition of the Diameter *Aa*, in *Fig.* 55. with regard to the Aſymptote *CF*, is the ſame as the Poſition of the Diameter *A a*, in *Fig.* 54. with regard to the other Aſymptote *CD*.

2. If the Line *M l* be drawn, in *Fig.* 55. making the Angle *M l C*, with the Aſymptote *CF*, (produced) equal to the Angle *M L C*, or *ECF* of *Fig.* 54. then it is plain, that the Lines *M l*, *Mk*, in *Fig.* 55. will be equal to the Lines *M L*, *MK*, of *Fig.* 54. becauſe the Poſition of the Point *M*, in reſpect of the Aſymptotes, is ſuppos'd to be the ſame in both Figures. But the Angle *M l L*, being the Complement of the Angle *M l C*, in *Fig.* 55. or of *ECF*, in *Fig.* 54. is equal to the Angle *M K k*, being the Complement of the Angle *ECD*, in *Fig.* 55. or of *ECF*, in *Fig.* 54. and conſequently (in *Fig.* 55.) the two Triangles *L M l*, *k M K*, having the Angle at *M* common, and the Angles at the Point *l*, *K*, equal, will be ſimilar; and ſo *L M* : *M l* :: *k M* : *MK*. Therefore *L M* × *MK* = *l M* × *Mk*, or *L M* × *MK*, in *Fig.* 54. Hence it appears, * that *C A*, *C A* the Halves of the firſt Diameters, in *Fig.* 54 and 55. are equal. The ſame may be ſaid of the Diameter *B b*; becauſe the Poſition and Magnitude thereof depends on the Poſition and Magnitude of the firſt Diameter *A a*, to which it is the Conjugate. * *Art.* 94.

Becauſe there can be drawn but one Line *e c*, making an Angle either way with the Chord *d f*, equal to the given Angle, when it is a right one; therefore there can be but two Conjugate Diameters *A a*, *B b*, making right Angles with one another that will anſwer the Problem, and theſe * will be the Axes. But becauſe the Triangle *d cf*, or *DCF*, is then Iſoſceles, the firſt Axis *Aa* will biſect the Angle *D CF* formed by the Aſymptotes; and ſo there is nothing more requir'd for finding the Poſition of the two Axes, but drawing the two right Lines *A a*, *B b*, FIG. 55, *and* 57. * *Art.* 111.

perpendicular to one another; one of which, as $Aa$, biſects the Angle $DCF$, formed by the Aſymptotes; for afterwards their Magnitude may be determined, as is directed for finding the Magnitudes of the Conjugate Diameters.

The two Axes may be found otherwiſe thus: Draw $MH$ through the Point $M$, parallel to $CF$ one of the Aſymptotes, and terminating in the Point $H$ by the other Aſymptote $CD$. And in the Aſymptote $CD$, aſſume $CG$, a mean Proportional between $CH, HM$; and draw $AB$ thro' the Point $G$ parallel to $CF$, ſo that each of its Parts $GA, GB$, be equal to $CG$. Then it is manifeſt, that the Lines $CA$, $CB$, will be * the two Semi-Axes both in Poſition and Magnitude.

**Art.* 101, *and* 88.

## COROLLARY.

129. IT is now evident, 1. That there are but two Conjugate Diameters that cut each other at right Angles; and ſo there can be but two Axes. 2. There can be but two different Pair of Conjugate Diameters making an Angle with each other equal to a given Angle, when this Angle is not a right one; and the two firſt Diameters of theſe two Pair have the ſame Poſition to one Aſymptote, as the two others have to the other Aſymptote; and ſo they are alike ſituate on both Sides of the two Axes, ſince the two Axes biſect the Angles form'd by the Aſymptotes: And finally, their Magnitudes remain the ſame in both thoſe different Poſitions.

## PROPOSITION XIV.

### Problem.

130. *ANY two Conjugate Diameters being given, and which of the two is the* firſt Diameter *being known; or, which* * *is the ſame thing, the Aſymptotes of two oppoſite Hyperbola's being given, together with any one of their Points, to deſcribe the ſaid Hyperbola's by a continued Motion.*

**Art.* 114.

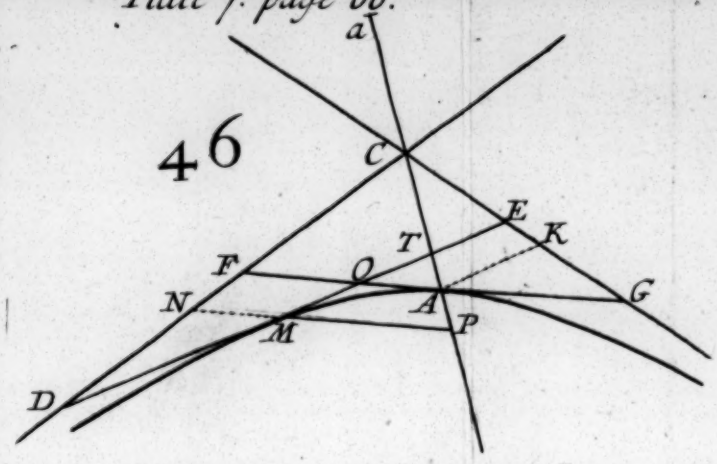
*The Firſt Way.*

Find the two Axes, as is directed in the laſt Propoſition, and then deſcribe the oppoſite Hyperbola's by Article 76.

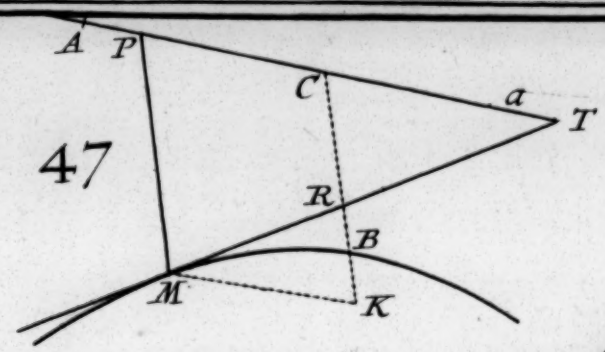
*Second Way.*

FIG. 58. Let $Aa, Bb$, be any two given Conjugate Diameters, whereof $Aa$ is the firſt; or elſe let $CG, CG$, be two Aſymptotes given, together with the Point $A$, through which one of the oppoſite Hyperbola's paſſes.

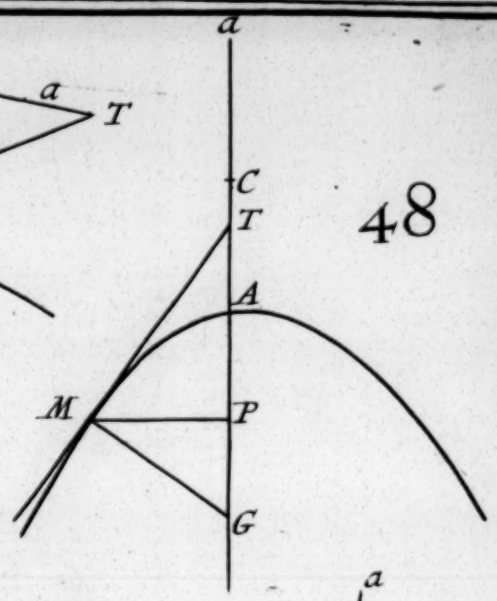
Plate 7. page 66.



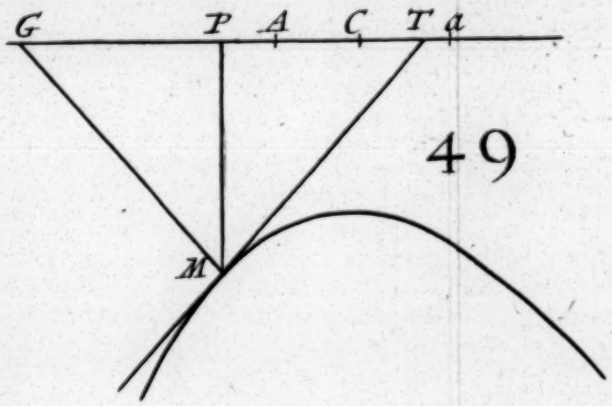
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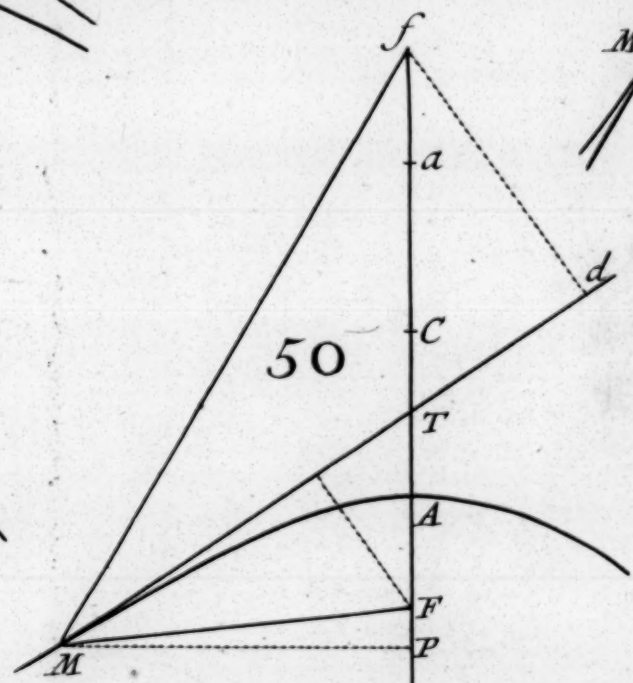
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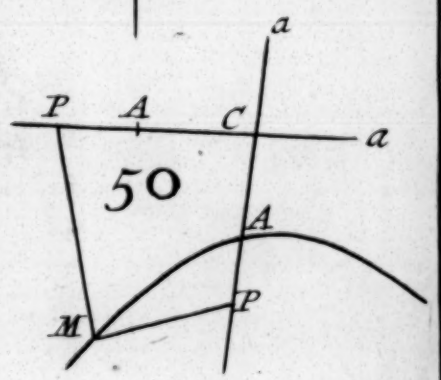
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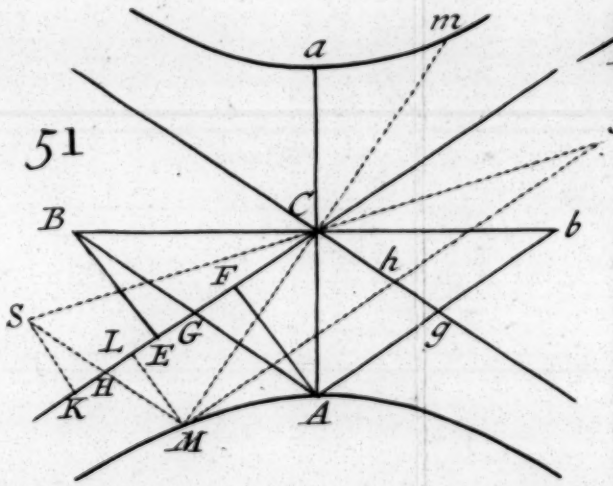
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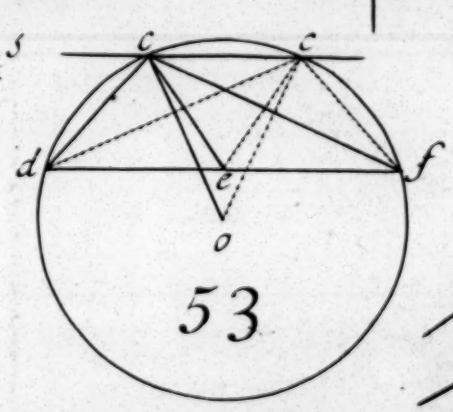
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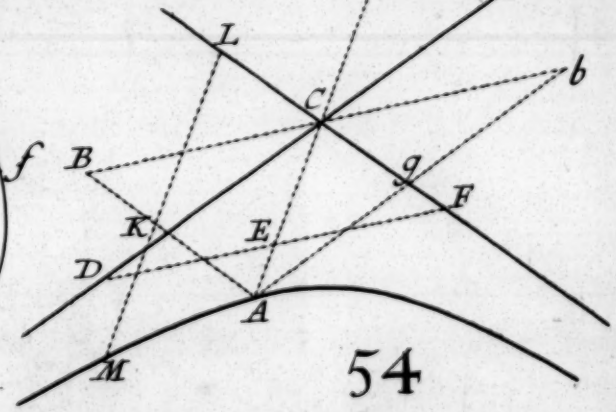
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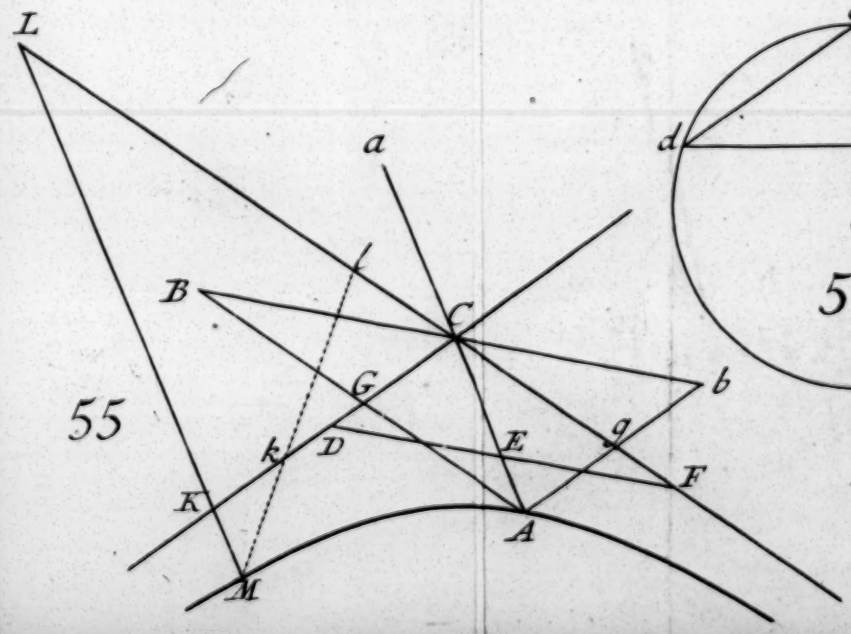
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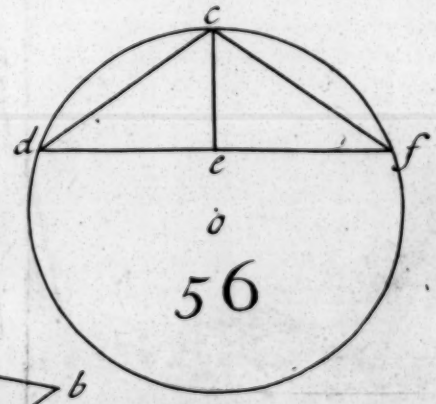
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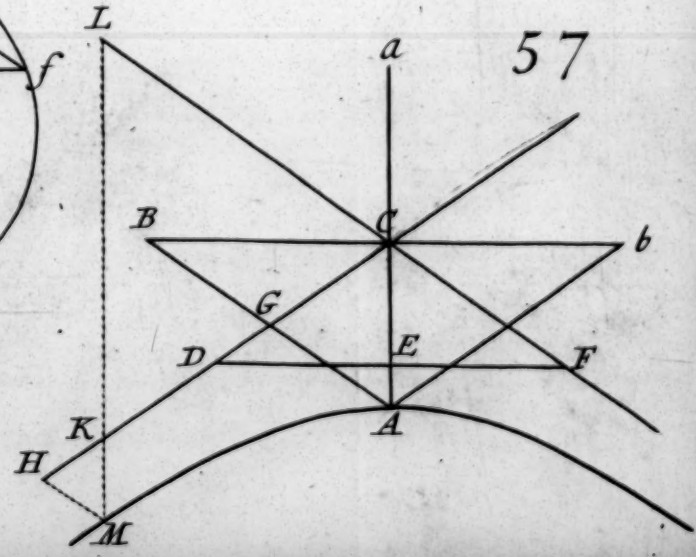
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57

paſſes. Draw $AG$ through the given Point $A$, parallel to one of the Aſymptotes $Cg$, and terminating in the other in $G$; then move the right Line $HK$, (equal to $CG$) along the Aſymptote $CG$, both ways indefinitely produced; ſo that one End $H$ thereof carries along with it the Line $HM$ parallel to the Aſymptote $Cg$, and the other End $K$, the right Line $KA$, moveable about the fixed Point $A$. Then I ſay, the Point ($M$), being the continual Interſection of the right Lines $AK$, $HM$, will by this Motion deſcribe the two oppoſite Hyperbola's ſought.

For becauſe the Triangles $KHM$, $KGA$, are ſimilar, we have always this Proportion, $KH$ or $CG : HM :: KG$ or $CH : GA$. And therefore $CH \times HM = CG \times GA$. Therefore the Point $M$ will be * in the Hyperbola paſſing through the given Point $A$, or in the oppoſite Hyperbola, and whoſe Aſymptotes are the given right Lines $CG$, $Cg$. * *Art.* 101.

## PROPOSITION XV.

### Problem.

131. *THE ſame Things being given, as in the laſt Propoſition, to deſcribe the oppoſite Hyperbola's by finding many Points thereof.*

#### *Firſt Way.*

Let $CD$, $CE$, be the given Aſymptotes, and $A$ the given Point. Thro' this Point $A$ draw any Number of Lines $DE$, $DE$, $DE$, &c. terminating in the Aſymptotes; and in them aſſume $EM$, $EM$, $EM$, &c. each equal to its Correſpondent $AD$, $AD$, $AD$, &c. Then it is * manifeſt, 1. That the Points $M$, $M$, $M$, &c. will be in the Hyperbola paſſing thro' the Point $A$, when the Points $E$, $E$, $E$, &c. fall below the Centre. 2. That the right Lines $CD$, $CE$, are the Aſymptotes of the Hyperbola's: Therefore, if two Curves be drawn through all the Points $M$, $M$, $M$, &c. falling in the oppoſite vertical Angles; thoſe two Curves will be the oppoſite Hyperbola's ſought. FIG. 59. * *Art.* 106.

#### *Second Way.*

Let the Lines $Aa$, $Bb$, be two given Conjugate Diameters, whereof $Aa$ is the 2*d Diameter*. Aſſume any Number of little equal Parts $CE$, $CE$, $CE$, &c. of any Magnitude at Pleaſure, in the Semidiameter $CB$ indefinitely produced towards $B$; and through that Point $E$, which is neareſt to the Centre $C$, draw the Line $EP$ parallel to $BA$. This being done, in the ſecond Diameter $Aa$, both ways continued, aſſume FIG. 60.

the ſmall Parts $CP$, $PP$, &c. each equal to $CP$, as many in Number as the Parts $CE$, $EE$, $EE$, &c. and draw $CD$ perpendicular and equal to $CB$; then if the Lines $MPM$, $MPM$, $MPM$, &c. be drawn parallel to the firſt Diameter $Bb$, and in each of them you aſſume, both ways from the Point $P$, the Parts $PM$, $PM$, each equal to its Correſpondent $ED$. I ſay, the two Curve Lines paſſing thro' all the Points $M$, $M$, $M$, &c. thus found will be the two oppoſite Hyperbola's ſought.

For call the given Quantities $CA$, $t$; $CB$ or $CD$, $c$; and the indeterminate Quantities $CP$, $x$; $PM$, $y$; then the ſimilar Triangles $CAB$, $CPE$ give this Proportion, $CA\ (t) : CB\ (c) : : CP\ (x) : CE = \frac{cx}{t}$. And ſince the Triangle $ECD$ is right-angled at $C$ (ſuppoſing every Hypothenuſe $ED$ to be drawn, which, for avoiding Confuſion, we have omitted in the Figure) the Square $\overline{ED}^2$ or $\overline{PM}^2$ $(yy)$ is $= \overline{CE}^2 \left(\frac{ccxx}{tt}\right) + \overline{CD}^2\ (cc)$. Therefore the Line $PM$ will be * an Ordinate to the ſecond Diameter $Aa$, having the firſt Diameter $Bb$, the Conjugate thereto. And becauſe the ſame Demonſtration extends to every of the Lines $PM$, ſince $CP$ is always to its Correſpondent $FE$, in the Ratio of $CA$ to $CB$: Therefore, &c.

* *Art.* 81, *and* 118.

FIG. 61. When the Conjugate Diameters $Aa$, $Bb$, are equal to one another, * *Def.* 16. that is, * when the Hyperbola's ſought are equilateral, then the Conſtruction will become much more eaſy. For if $CD$ be drawn perpendicular and equal to $CA$, and if $MPM$ be drawn through any Point $P$ in the Diameter $Aa$, parallel to the firſt Diameter $Bb$; then you need but take in that Line (both ways produced) the Parts $PM$, $PM$, &c. each equal to $PD$, and you will have two Points thro' which the oppoſite Hyperbola's muſt paſs. For becauſe the Triangle $PCD$, is right-angled at $C$ (ſuppoſing an Hypothenuſe $CD$ to each of them) we ſhall have always $\overline{PD}^2$ or $\overline{PM}^2 = \overline{CP}^2 + \overline{CD}^2$ or $\overline{CA}^2$; and therefore the Line $PM$ will be * an Ordinate to the ſecond Diameter $Aa$, whoſe Conjugate Diameter $Bb$ is equal to it.

* *Art.* 127.

## DEFINITION.

### 17.

FIG. 62. Let there be two oppoſite Hyperbola's $AM$, $am$, whoſe firſt Axis is the Line $Aa$, and ſecond the Line $Bb$; and let there be two other oppoſite Hyperbola's, whoſe firſt Axis is the Line $Bb$, and ſecond the Line $Aa$; then the two laſt Hyperbola's $BS$, $bs$, are ſaid to be *Conjugates* to the two former ones $AM$, $am$; and all the four together are call'd *Conjugate* Hyperbola's.

COROL-

### COROLLARY.

132. IT is manifeſt, that the Lines $Ba$, $Ab$, are parallel; becauſe * the Lines $Aa$, $Bb$, terminated by them, do biſect each other in the Point $C$. Whence it follows, by *Def.* 11. that the Hyperbola $BS$, being the Conjugate to $AM$, hath $CG$ the Aſymptote of the Hyperbola $AM$ for one of its Aſymptotes, and $Cg$ the other Aſymptote of the ſame Hyperbola, for the other Aſymptote; becauſe thoſe two Lines paſs thro' the Centre $C$, and are parallel to the two right Lines $Ba$, $BA$, drawn from ($B$) the End of the firſt Axis ($Bb$) of the Hyperbola $BS$, to the two Ends $A$, $a$, of the ſecond. Therefore it is manifeſt, that the two right Lines $CG$, $Cg$, parallel to $Ab$, $AB$, both ways infinitely produced, are not only the Aſymptotes of the oppoſite Hyperbola's $AM$, $am$; but likewiſe are the Aſymptotes of the two other Hyperbola's $BS$, $bs$, which are Conjugates to them. * *Def.* 4. *and* 5.

## PROPOSITION XVI.

### Theorem.

133. *IF through any Point* H *in the Aſymptote* CG, *which is common to both the Hyperbola's* AM, BS, *there be drawn a Parallel* (MS) *to the other Aſymptote* Cg; *I ſay, that Parallel will meet the ſaid two Hyperbola's in the Points* M, S, *equally diſtant from the Point* H.

For 1. The Line $MS$ will meet * each of the oppoſite Hyperbola's $AM$, $BS$, in one Point. 2. From the Nature of the Hyperbola $AM$, the * Rectangle $CH \times HM$ is $= CG \times GA$; and from the Nature of the Hyperbola $BS$, the Rectangle $CH \times HS$ is $= CG \times GB$. Whence (ſince * $GB = GA$,) $CH \times HS$ will be $= CH \times HM$; and ſo $HS$ is $= HM$. *W. W. D.* * *Art.* 104. * *Art.* 101. * *Art.* 88.

### COROLLARY I.

134. IF from the Points $M$, $S$, in the two Hyperbola's $AM$, $BS$, be drawn the Diameters $MCm$, $SCs$, terminating in the two other oppoſite Hyperbola's $am$, $bs$; then it is manifeſt, * that the Diameter $Ss$ will be the ſecond Diameter, which is the Conjugate to the firſt Diameter ($Mm$) of the two oppoſite Hyperbola's $AM$, $am$; and contrariwiſe; the Diameter $Mm$, will be the ſecond Diameter, which is the Conjugate to the firſt Diameter $Ss$, of the oppoſite Hyperbola's $BS$, $bs$. Whence it appears, that any two Conjugate Diameters ($Mm$, $Ss$) of the two oppoſite Hyperbola's $AM$, $am$, are alſo * *Art.* 114.

alſo two Conjugate Diameters of the two other Hyperbola's *B S*, *b s*, Conjugates to the former; but yet with this Difference, *viz.* that the firſt Diameter *M m* becomes a ſecond Diameter; and contrariwiſe, the ſecond Diameter *S s* a firſt.

### COROLLARY II.

135. HENCE it is manifeſt, that the Hyperbola's *B S*, *b s*, the Conjugates to the Hyperbola's *A M*, *a m*, paſs through the Ends *S*, *s*, of all the ſecond Diameters (*SCs*) of thoſe Hyperbola's: And contrariwiſe, the Hyperbola's *A M*, *a m*, paſs through the Ends *M*, *m*, of all the ſecond Diameters (*MCm*) of the two Hyperbola's *B S*, *b s*, which are the Conjugates to them.

## *The End of the Third Book.*



BOOK

# BOOK IV.

## *Of the Three Conick Sections.*

### DEFINITION.

BY the Term *Conick Section* in general, we underſtand each of the Curves treated of in the foregoing Books, *viz.* the *Parabola*, *the Ellipſis*, *the Hyperbola*, or *the oppoſite Hyperbola's*.

### PROPOSITION I.

### Theorem.

136. *IF through the End* (A) *of any Diameter* (Aa,) *of an Ellipſis, or of any firſt Diameter of an Hyperbola, the Line* AG *be drawn parallel to the Ordinates* (PM) *thereof, and equal to its Parameter; and if the right Line* aG *be drawn from* a *the other End of that Diameter, cutting any Ordinate* PM *(produced if neceſſary) in the Point* O. *I ſay, the Square of the Ordinate* PM, *is equal to the Rectangle under* AP *and* PO. FIG. 63, and 64.

*We are to prove, that* $\overline{PM}^2 = AP \times PO$.

By the 41ſt and 55th Articles of the ſecond Book, and the 81ſt and 118th of the third, we have this Proportion, $Aa : AG :: AP \times Pa : \overline{PM}^2$. But becauſe the Triangles $aAG$, $aPO$, are ſimilar; therefore $Aa : AG :: Pa : PO :: AP \times Pa : AP \times PO$. And ſo $\overline{PM}^2 = AP \times PO$. *W. W. D.*

### COROLLARY.

137. HENCE it is evident, that the Square of any Ordinate ($PM$) to a Diameter $Aa$, is always leſs in the Ellipſis, and greater in the Hyperbola, than the Rectangle under the Parameter $AG$, and the Part ($AP$) of that Diameter between its End $A$, and the Point of Concurrence ($P$) of the Ordinate; whereas * in the Parabola they are equal. And from this Property was it, that *Apollonius*, otherwiſe * *Art.* 7, *and* 20. FIG. 65.

wiſe call'd the great Geometrician, gave thoſe Names, which we have made uſe of, to the Conick Sections: For by the Word *Parabola* is underſtood the Exactneſs or Equality, by the Word *Ellipſis* the Deficiency, and by the Word Hyperbola the Exceſs, which is found in the Compariſon of the Squares of the Ordinates *PM*, with the Correſpondent Rectangles $AP \times AG$.

## PROPOSITION II.

### Theorem.

FIG. 66, and 67. 138. *IN the Ellipſis, every Diameter* A a, *and in the oppoſite Hyperbola's, every firſt Diameter* A a, *is divided by the Centre* C *into two equal Parts, and meets the Section but in two Points.*

This Propoſition has been demonſtrated in the 50th Article of the ſecond Book; and in the 96th and 103d Articles of the third Book.

## PROPOSITION III.

### Theorem.

139. *THERE can be but one Tangent* (L A L) *paſſing through a given Point* (A) *in a Conick Section.*

This Propoſition is found demonſtrated in the 21ſt Article of the firſt Book, the 56th of the ſecond, and the 107th of the third Book.

## PROPOSITION IV.

### Theorem.

140. *THE Tangents* L A L, l a l, *paſſing through the Ends* A, a, *of any Diameter of an Ellipſis, or two oppoſite Hyperbola's, are parallel to each other.*

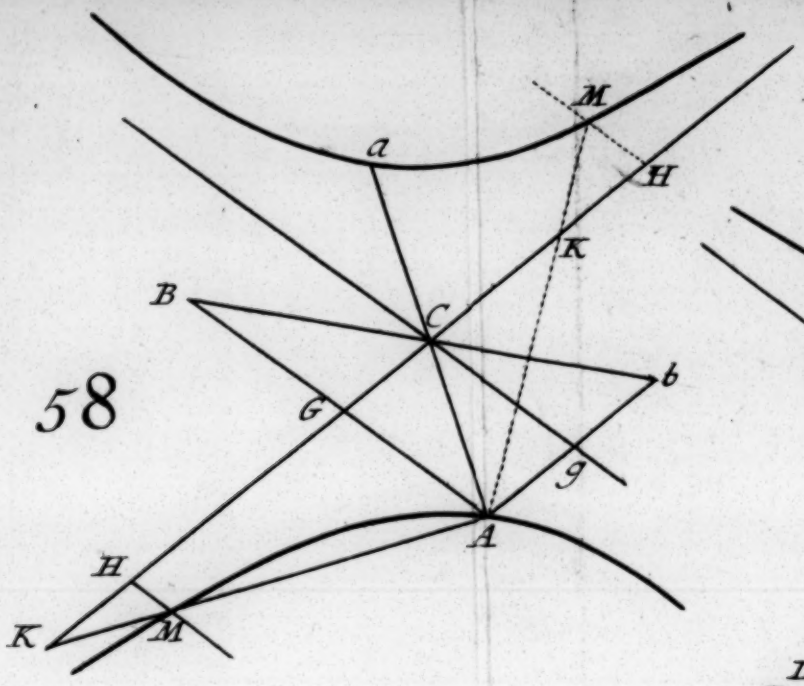
This has been prov'd in the 44th and 45th Articles of the ſecond Book, and the 110th of the third.

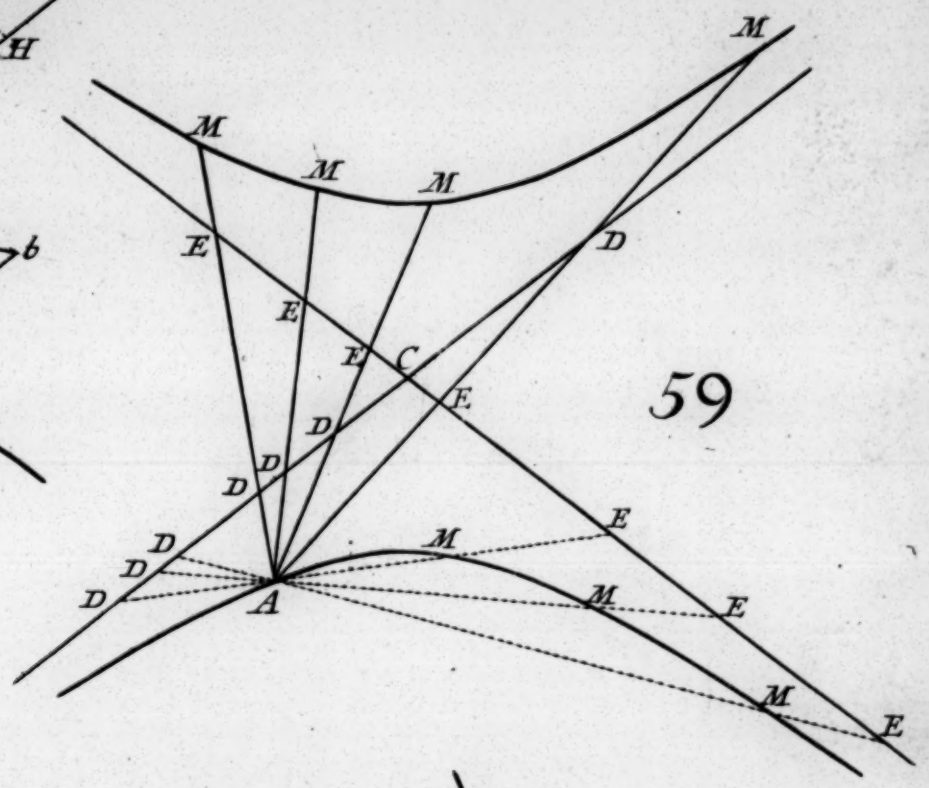
## PROPOSITION V.

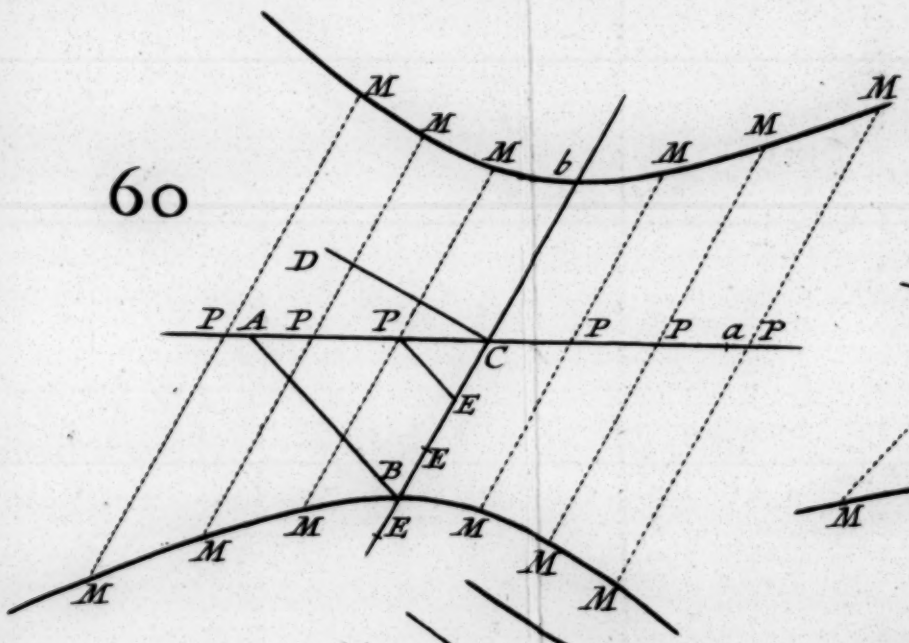
### Theorem.

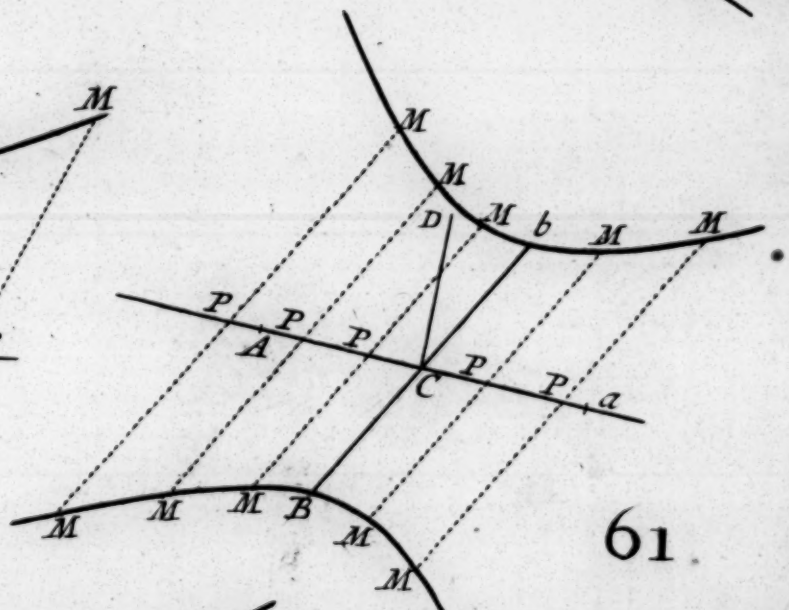
141. *ANY Diameter of an Ellipſis, or the oppoſite Hyperbola's being given: I ſay, the Poſition of the Diameter, which is the Conjugate to it, is ſo determin'd, as that there can be but one only.*

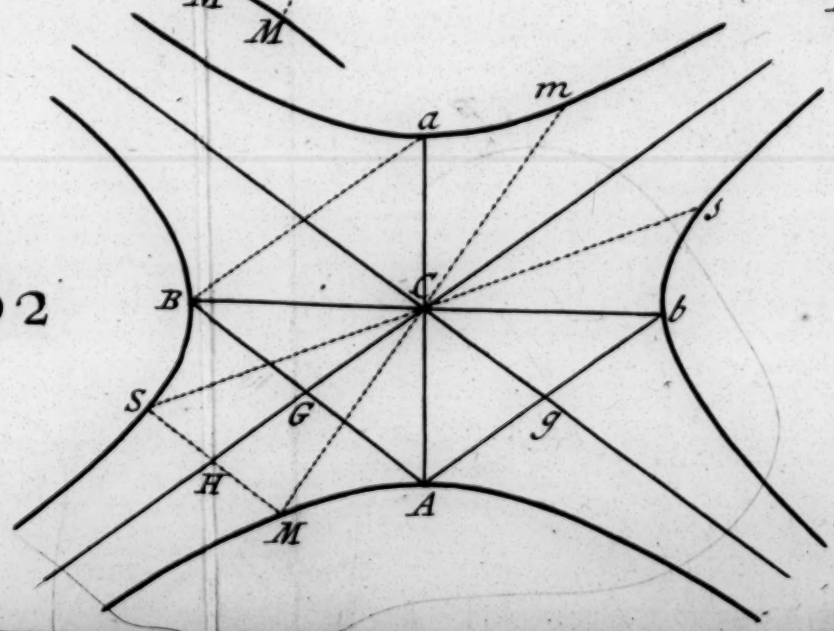
For 1. If the Section be an Ellipſis, or the oppoſite Hyperbola's, and if $Aa$ be a firſt Diameter; then it is manifeſt, by the 56th


58
M
a
H
K
B
C
b
G
g
A
H
K
M


59
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M
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M
E
D
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E
C
F
D
D
D
D
E
M
M
E
A


60
M
M
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b
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a
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C
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B
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M
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M
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M
M


61
M
M
D
M
b
M
M
P
P
P
A
P
C
P
a
M
M
M
B
M
M


62
m
a
s
C
B
b
S
G
g
H
A
M

56th Article of the ſecond Book, and the 13th Definition of the third Book, that the Conjugate Diameter (*Bb*) to *Aa* will be parallel to the Tangent *LAL*, paſſing thro' the Point *A*. Therefore, * *&c.* *Art. 139.

2. When *Bb* is a ſecond Diameter of two oppoſite Hyperbola's: This has been demonſtrated in the 115th Article of the third Book.

### COROLLARY.

142. HENCE it appears, that a Conick Section being given, together with one of its Diameters; the Poſition of the Ordinates to that Diameter is ſo determined, that every of them can have but one Diameter, and all of them are parallel to one another. For in the Parabola, the Ordinates muſt be * parallel to the Tangent paſſing through the Origin of the given Diameter, and in the other Sections parallel to that Diameter, which is the Conjugate to the Diameter given. * *Art.* 21. * * *Def.* 12, II. *and* 14, III.

## PROPOSITION VI.

### Theorem.

143. *IN an Ellipſis, every Diameter* Aa, *as alſo in the oppoſite Sections, every firſt Diameter* A a, *divides the Section into two Parts or Portions,* A M, a m, *which being taken in contrary Poſitions on each Side that Diameter, are ſimilar and equal to one another.*

For take *CP*, *Cp*, equal to one another, in the Diameter *Aa* (produced in the oppoſite Sections) on each Side the Centre *C*; and draw the Ordinates *PM*, *Pm*; then it is manifeſt, * that thoſe Ordinates are equal to one another, and the Angles * *CPM*, *Cpm*, are equal. Therefore, if the Plane *Cpm* be ſuppos'd to be laid upon the Plane *CPM* on the other Side of the Diameter *Aa* in a contrary Poſition, ſo that the right Line *Cp* coincides with *CP*, and *pm* with *PM*; then it is manifeſt, that the Point *a* will * coincide with *A*, and the Point *m* with the Point *M*. And ſince this always happens, let the equal Parts *CP*, *Cp* be of what Magnitude ſoever: It follows, that all the Points (*m*) of the Part *am* will coincide with all the Points (*M*) of the Part *AM*; and conſequently the ſaid two Parts or Portions will coincide and be equal. *W. W. D.* * *Art.* 45, 55, 85, *and* 118. *Art. 142. *Art. 138.

## PROPOSITION VII.

### Theorem.

144. *IF the right Line* MPM, *be drawn through any Point* P, *taken in* A a, *the Diameter of a Conick Section (produced in the Hyperbola, when the ſame is a principal Diameter) parallel to the Ordinates to* FIG. 68, 69, 70, 71.

*that Diameter: I ſay, the Line* MPM *will meet the Section in only two Points* M, M, *equally diſtant each way from* P. *And contrariwiſe, if any right Line* MM, *terminating in a Conick Section, be cut into two equal Parts by the Diameter* Aa *in the Point* P, *not being the Centre; then the ſaid Line* MM *will be parallel to the Ordinates to that Diameter.*

This has been demonſtrated in the 9th, 11th, and 20th Articles of the firſt Book; the 43d, 45th, and 55th of the ſecond Book; and the 83d, 85th, and 118th of the third.

### COROLLARY.

145. HENCE if any right Line *M M*, terminating in a Conick Section, be cut into two equal Parts by the Diameter *A a*, in the Point *P*, not being the Centre; then all the Parallels to that Line terminating in the Section, will be biſected likewiſe by that Diameter.

## PROPOSITION VIII.

### Problem.

146. *A Conick Section being given, to find a Diameter thereof.*

A Draw the right Lines *M M*, *N N*, parallel to one another, and terminating in the Section; then if they be biſected in the Points *P*, *Q*, the right Line *A a* drawn thro' *P*, *Q*, will be a Diameter.

* *Art.* 145. For * the Diameter paſſing through *P* the Middle of *M M*, muſt paſs likewiſe through *Q* the middle of *N N*.

### COROLLARY. I.

147. IF any other Diameter *D d* be drawn, as above; then it is manifeſt, that the Conick Section will be * a Parabola, when *D d* is parallel to *A a*; an Ellipſis, * when *D d* meets *A a* within the Section; and laſtly, an Hyperbola, * or the oppoſite Sections, when the Lines *D d*, *A a*, meet each other in the Point *C*, without the Section; in which two laſt Caſes, the Point of Concurrence *C* is the Centre. And ſo you have here a Succeſſion of the Definitions of the Diameters of the Parabola, Ellipſis, and Hyperbola.

* *Def.* 7. I.
* *Def.* 9. II.
* *Def.* 9. III.

When the whole Ellipſis is given, you need only draw a Diameter *A a*, for having the Centre thereof: For becauſe the Magnitude of the Diameter is * determin'd by the Ellipſis, you need only biſect that Diameter in *C*. The ſame is to be done * when oppoſite Hyperbola's are given.

* *Art.* 50.
* *Art.* 96.

COROL-

### COROLLARY. II.

148. HENCE, a Conick Section being given, together with ſome Point *O*, in the ſame Plane; we may always draw ſome Diameter *D d*, through that Point: For in the Parabola, you need only draw *D d* through the given Point *O*, parallel to any Diameter *A a*; and in the Ellipſis, Hyperbola, or oppoſite Sections, the right Line *D d*, paſſing through the given Point *O*, and the Centre *C*, found by the laſt Corollary.

### COROLLARY. III.

149. HENCE it is evident, that a right Line *M M* can meet a Conick Section in only two Points *M*, *M*. For if a Diameter *A a* be drawn through *P*, the Middle of *M M*; it is manifeſt, by *Art.* 144. that the ſame will be parallel to the Ordinates to that Diameter; and ſo by the ſame Article, the Line *M M* can meet the Section in only two Points *M*, *M*.

*Note*, If the Line ſhould paſs through the Centre *C*, then Recourſe muſt be had to *Art.* 138. where this has been prov'd already.

### COROLLARY IV.

150. ANY Ellipſis or Hyperbola being given, to find two Conjugate Diameters (*A a*, *B b*) thereof; and moreover, to draw the Aſymptotes *CG*, *Cg*, when the given Section is an Hyperbola. FIG. 69, 70.

Find a Diameter *A a*, by means of the Parallels *M M*, *N N*, and through *C* draw *B b* parallel to thoſe two Lines: Then it is evident, * that *A a*, *B b* will be Conjugate Diameters; becauſe the Lines *M M*, *N N* being divided by the Diameter *A a* into two equal Parts in the Points *P*, *Q*, will be * Ordinates both ways to that Diameter. * *Def.* 12, II. *and* 14, III. * *Art.* 144.

But now to draw (*Fig.* 70.) the Aſymptotes *CG*, *Cg*; make as $\overline{AP \times Pa} : \overline{PM}^2 :: \overline{CA}^2 : \overline{CB}^2$ or $\overline{Cb}^2$. or (which is the ſame thing) make as a mean Proportional between *AP* and *Pa* to *PM*, ſo is *CA* to *CB* or *Cb*. This being done, if the right Lines *AB*, *Ab* are drawn, then the indefinite right Lines *Cg*, *CG*, drawn thro' the Centre *C* parallel to *AB*, *Ab*, will be the Aſymptotes ſought. For it is manifeſt, that *Bb* will be * the Conjugate Diameter to the firſt Diameter *A a*; and what remains, is manifeſt by *Def.* 13 and 14. *Book* 3. * *Art.* 81, *and* 118.

## PROPOSITION IX.

### Problem.

151. *ANY Conick Section being given, together with one of its Diametees* A a; *to find the Poſition of the Ordinates* (P M) *to that Diameter.* FIG. 68, 69, 70, 71.

Draw two Parallels to the given Diameter *A a*, equally diſtant each way from the ſame, and meeting the Section in the Points *M*, *M*; then I ſay, the Line *M M*, which cuts the given Diameter in the Point *P*, is an Ordinate both ways to the Diameter *A a*, if the Point *P* does not fall in the Centre.

For, by Conſtruction, the Line *M M* will be biſected by the Dia-
*Art. 144. meter *A a* in the Point *P*; and conſequently the Line *M M* will be * an Ordinate both ways to that Diameter.

After this manner may be found always the Poſition of any Ordinate (*P M*) to the given Diameter *A a*. For 1. in the Parabola, and Hyperbola (*Fig.* 68 and 70.) when the given Diameter *A a* is a firſt Diameter, it is manifeſt, that at whatever Diſtance the two Parallels be drawn from the Diameter *A a*, they will each meet the Sectio-
* *Art.* 10, 20, 84, *and* 118. on in one Point *M*; becauſe the Section * infinitely recedes more and more from the Diameter *A a*. 2. In the Ellipſis (*Fig.* 69.) and the oppoſite Sections (*Fig.* 71.) when the given Diameter *A a* is a ſecond Diameter; then it is manifeſt, that there can be drawn two Parallels on both Sides the Diameter *A a*, each of which will cut the Section in one Point *M*, ſo that the Line *M M* will meet the given Diameter *A a* in the Point *P*, not being the Centre; becauſe in the Ellipſis the
* *Art.* 44, *and* 55. Ordinates to the Diameter *A a* do continually * diminiſh from the Centre *C* to *A*; and contrariwiſe, in the oppoſite Sections, do * in-
* *Art.* 84, *and* 118. creaſe as they go from the Centre *C*.

## COROLLARY I.

FIG. 68, 69, 70. 152. FROM hence ariſes a new Way of drawing a Tangent through a Point (*A*) given in a Conick Section. For draw * a Dia-
*Art. 148. meter (*A a*) through that Point, and find a double Ordinate (*M P M*) to that Diameter. This being done, it is manifeſt, * that a Line
* *Art.* 10, 20, 44, 55, 84, 118, *and Def.* 9, I. 12, II. 7, III. drawn through the Point *A* parallel to *M M*, will touch the Section in *A*.

## COROLLARY II.

FIG. 69, 70, 71. 153. HENCE it farther appears, when an Ellipſis, or two oppoſite Sections are given, together with any one of their Diameters *A a*, how to find the Conjugate Diameter (*B b*) to that. For you need only draw *B b* through the Centre *C*, parallel to the Ordinates to *A a*.

Or elſe, if *B b* be the given Diameter, and the Conjugate to it (*A a*) is requir'd, draw *M M* parallel to *B b*, and terminating in the Section; and then through the Points *P*, *C*, the Middles of *M M* and *B b*, draw the Diameter *A a* ſought.

COROL-

### COROLLARY III.

154. AN Hyperbola *MAM* being given, together with one of the ſecond Diameters *Bb* in Poſition, to determine the Magnitude thereof, and alſo to find the Poſition of the Ordinates to the ſame. FIG. 70.

Firſt ſeek the firſt Diameter *Aa*, to which *Bb* is the Conjugate, by the ſecond Part of the laſt Corollary; and then make $AP \times Pa : \overline{PM}^2 :: \overline{CA}^2 : \overline{CB}^2$ or $\overline{Cb}^2$. This being done, it is * manifeſt, that *Eb* will be the Magnitude of the ſecond Diameter *Bb*, and the Ordinates thereof will be parallel to *Aa*. * *Art.* 81, *and* 118.

## PROPOSITION X.

### Problem.

155. *FROM a given Point* T *without a given Conick Section, to draw two Tangents* TM, TM, *to that Section.* FIG. 72, 73, 74.

*For the Parabola.*

Draw * a Diameter through the given Point *TC* (*Fig.* 72.) meeting the Parabola in *A*, and take *AP* equal to *AT*; moreover, draw * a Parallel to the Ordinates to that Diameter through the Point *P*, which will meet * the Parabola in the two Points *M*, *M*; then if the right Lines *TM*, *TM*, be drawn through the given Point *T*, the ſaid Lines will be * the Tangents ſought. * *Art.* 148. * *Art.* 151. * *Art.* 144. * *Art.* 22, *and* 23.

*For the Ellipſis.*

Draw * the Diameter *Aa* through the given Point *T*, and take *CP* a third Proportional to *CT*, *CA*; likewiſe through *P* draw a Parallel to the Ordinates, which will meet * the Ellipſis in the Points *M*, *M*; then if the right Lines *TM*, *TM*, be drawn from the given Point *T*, thoſe Lines will touch * the Ellipſis. FIG. 73. * *Art.* 148. * *Art.* 144. * *Art.* 57, *and* 58.

*For the Hyperbola and oppoſite Sections.*

Draw * the Diameter *Aa* through the given Point *T*, and determine * the Magnitude thereof, if it be a ſecond Diameter, and take *CP* a third Proportional to *CT*, *CA viz.* on the ſame Side the given Point *T*, with regard to the Centre, when that Point falls in one of the Angles formed by the Aſymptotes; and on the oppoſite Side, when it falls in the adjoining Angles: This being done, through the Point *P* draw a Parallel to the Ordinates, which will meet * the Hy- FIG. 74. * *Art.* 148. * *Art.* 154. * *Art.* 144.

perbola, or oppoſite Sections, in the Points $M, M$; then if the right Lines $TM$, $TM$ be drawn through the Point $T$, they will be * the Tangents ſought. *Art. 121.

If the given Point ſhould fall in the Centre $C$, then would the two * Tangents be the Aſymptotes $CG$, $Cg$; which might be drawn by *Art.* 150. And laſtly, if the given Point ſhould fall in one Aſymptote, as in $S$; then there muſt be drawn through the Point $H$, the Middle of $CS$, the Parallel $HM$ to the other Aſymptote $CG$, which will * meet the Hyperbola in the Point $M$, through which, and the given Point $S$, if the right Line $SM$ be drawn; this Line will be * one of the Tangents ſought, and the other Tangent will be the Aſymptote $Cg$ it ſelf, wherein the given Point $S$ is given. *Art. 108. *Art. 104. *Art. 107.

## COROLLARY I.

156. BEcauſe the Line $MPM$, being parallel to the Ordinates, always meets * the Section in two Points $M, M$, equally diſtant each way from the Point $P$, and not more; it follows, that there can be drawn but two right Lines $TM$, $TM$, from a given Point $T$, without a Conick Section, to touch the ſame. Whence it is evident, that the Diameter drawn throngh $T$, the Point of Concurrence of the Tangents, biſects the Line $MM$, joining the Points of Contact, in the Point $P$; and contrariwiſe, the Diameter biſecting the right Line $MM$, joining the Points of Contact of the two Tangents $MT$, $MT$, in the Point $P$, paſſes through $T$ the Point of Concurrence of the Tangents. *Art. 144.

## COROLLARY II.

157. ANY two Tangents to the Parabola (*Fig.* 72.) will meet each other, being produced if neceſſary. For if any two Points of Contact $M, M$, be joined by a right Line; and then if that Line be biſected in the Point $P$, and the Line $AP$ be taken in the Diameter paſſing through the Point $P$, and meeting the Parabola in $A$, equal to $AP$; it is manifeſt, that the two Tangents $MT$, $MT$, paſſing through the Points $M, M$, will meet one another in the ſaid Point $T$.

## COROLLARY III.

158. IT is farther evident, (*Fig.* 74.) that any two Tangents to an Hyperbola, will meet each other, if produced according to Neceſſity; and that always within the Angle form'd by the Aſymptotes: For if any two Points of Contact $M, M$, be joined by a ſtraight Line, and after having biſected the ſame in $P$, there be taken $CT$ in the

the Diameter paſſing through that Point (and meeting the Hyperbola in *A*,) a third Proportional to *CP*, *CA*; it is then manifeſt, that the Tangents *MT*, *MT* will meet each other in the Point *T*, which will be always * within the Angle form'd by the Aſymptotes; becauſe the Semi-diameter *CA* falls within that Angle. *Art. 103.

## COROLLARY IV.

159. ANY two Tangents to an Ellipſis, or the oppoſite Sections, (*Fig.* 73, 74.) will meet each other, when the Line joining the Points of Contact does not paſs through the Centre; *viz.* the Tangents to the Ellipſis, on the ſame Side the Centre as that Line falls, and thoſe to the oppoſite Sections, on the contrary Side. This is prov'd by means of the Propoſition above, as we have ſhewn in the two laſt Corollaries.

## PROPOSITION XI.

### Problem.

160. *A Conick Section being given, to find a Diameter thereof, which ſhall make Angles either way with the Ordinates equal to an Angle given.*

*For the Parabola.*

Find * *AP* one of its Diameters, and draw the Line *AN* through *A* the Origin thereof, making the Angle *PAN* with *AP*, equal to the given Angle, and meeting the Parabola in the Point *N*; then biſect *AN* in *O*, and draw *OM* parallel to *AP*; and, I ſay, the Line *OM* is the Diameter ſought. *Art. 146. FIG. 75, 76.

For 1. Becauſe all the Diameters of a Parabola muſt be parallel to one another, by *Def.* 7. *Book* I. therefore the Line *MO* will be a Diameter; ſince *AP* is one likewiſe.

2. Becauſe the Line *AN*, terminating in the Parabola, is biſected by the Diameter *MO*; therefore that Line will be * an Ordinate each way to the Diameter *MO*. *Art. 144.

3. Becauſe *MO*, *AP*, are parallel, the Angle *MOA*, made by the Diameter *MO*, and its Ordinate *OA*, will be equal to the Angle *PAN*, which was made equal to the given Angle. Therefore, *&c.*

If the Angle given be a right one, then it is manifeſt, that the Diameter *MO*, found as above, will be * the Axis of the Parabola. * *Art.* 23.

*For the other Sections.*

*Art. 146. FIG. 77, 78, 79, 80. Find * $Aa$ one of the Diameters, and upon the ſame deſcribe the Segment $ANa$ of a Circle, that may contain an Angle equal to the Angle given, or the Complement thereof to two right Angles; then draw the right Lines $NA$, $Na$, from the Point $N$, wherein the Circle cuts the Section, to the Ends $A$, $a$, of the Diameter $Aa$. This being done, if through the Points $O$, $Q$, being the Middles of $NA$, $Na$, and the Centre $C$, you draw the two Diameters $Mm$, $Ss$; I ſay, each of theſe Diameters make Angles with their Ordinates equal to the given Angle.

For becauſe the Line $AN$, terminating in the Section, is biſected *Art. 144. by the Diameter $Mm$ in the Point $O$, the ſaid Line $AN$ will be * an Ordinate both ways to that Diameter; ſince it biſects the Lines $Aa$, $AN$ in the Points $C$, $O$: Therefore the Angle $mOA$ made by the Diameter $Mm$ and Ordinate $AO$, will be equal to the Angle $aNA$, which by Conſtruction is equal to the Angle given, or its Complement to two right Angles. After the ſame manner we prove, that the Diameter $Ss$ makes an Angle with its Ordinate $QN$, equal to the given Angle, or its Complement to two right ones. Whence, *&c.*

* *Def.* 12, II. *and* 14, III. *Art.* 58, *and* 128. It is manifeſt, 1. That the Diameter $Ss$ is * the Conjugate to the Diameter $Mm$; becauſe the ſame is parallel to the Ordinate $ON$. 2. The Conjugate Diameters $Mm$, $Ss$ will become * the two Axes, when the Angle given is a right Angle.

## PROPOSITION XII.

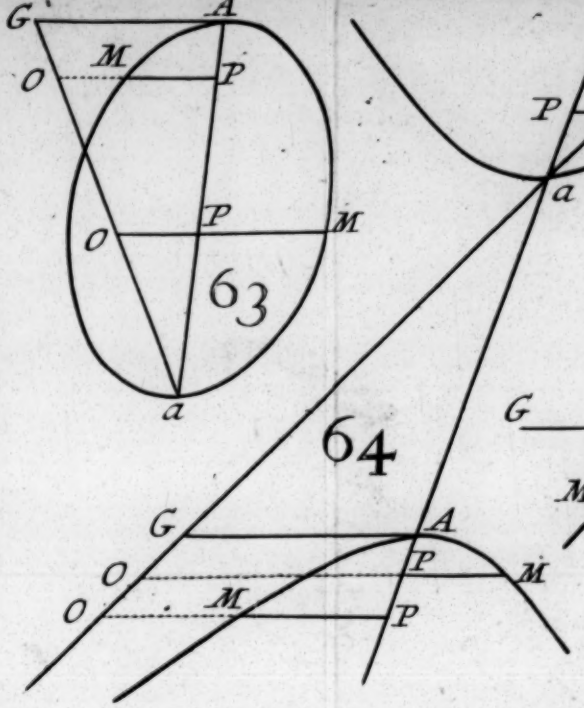
### Problem.

161. *ANY Diameter of a Conick Section being given, together with the Parameter thereof, and the Poſition of its Ordinates; and knowing moreover, whether the ſame be a firſt or ſecond Diameter, when we have to do with the Hyerbola, to deſcribe all the three Sections by one uniform Method.*

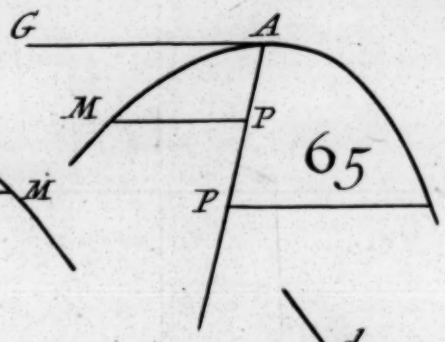
*Firſt Way.*

* *Art.* 27. FIG. 81. *For the Parabola.* Find * the Axis $AP$, its Origin $A$, and Parameter $AG$, aſſum'd in the Axis continued out beyond $A$, and draw an indefinite right Line $DD$ through the Point $G$ perpendicular to $PG$. This being done, if the indefinite right Line $DM$ be mov'd along $DG$, always parallel to $AG$; ſo that the End $D$ thereof carries along with it the Side $DA$, of the right Angle $DAM$, whoſe Vertex $A$ is moveable about $A$ the Origin of the Axis: I ſay, the continual Interſection

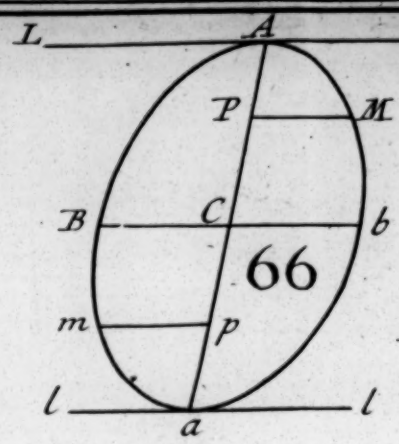
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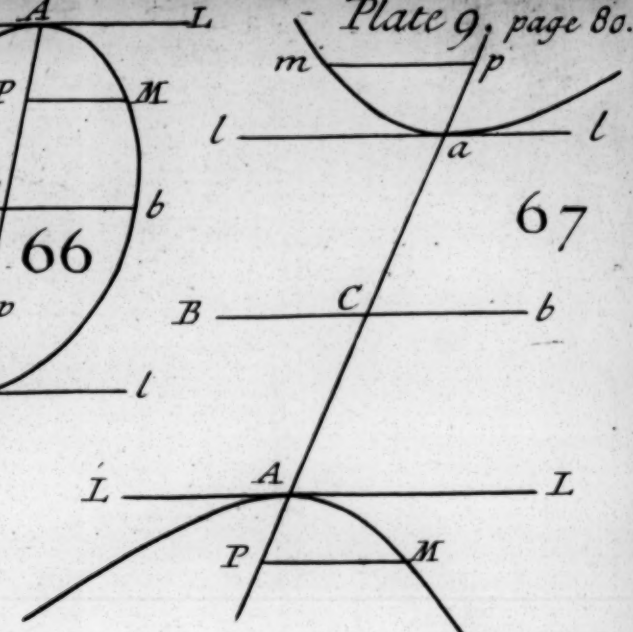

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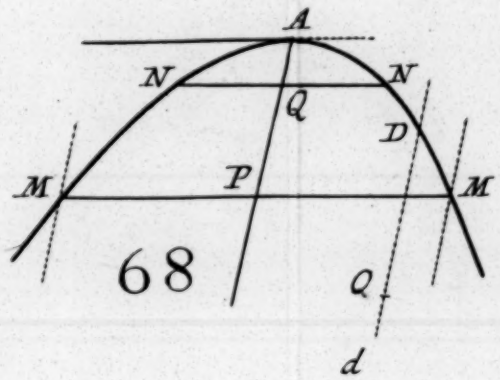



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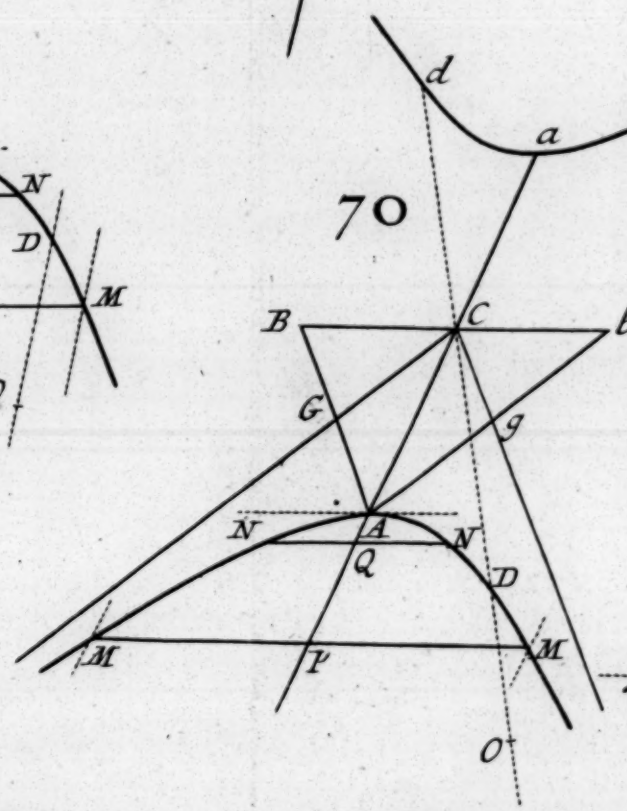



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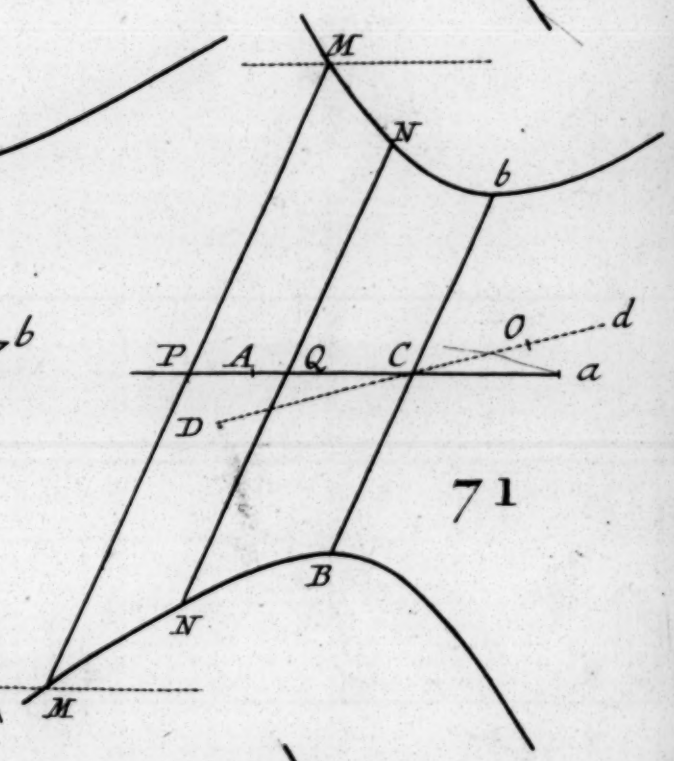


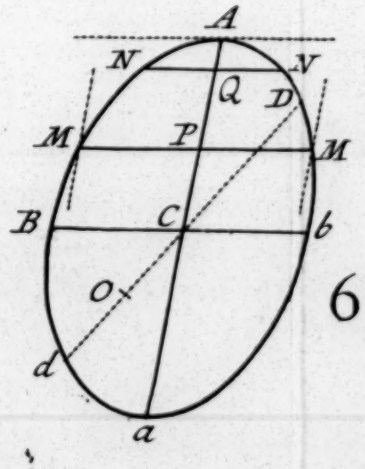

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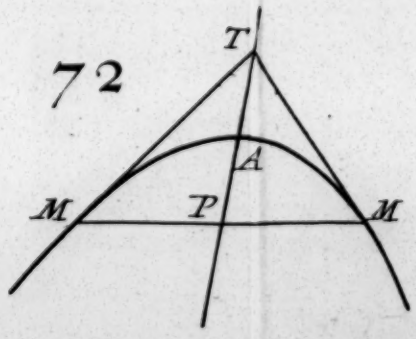

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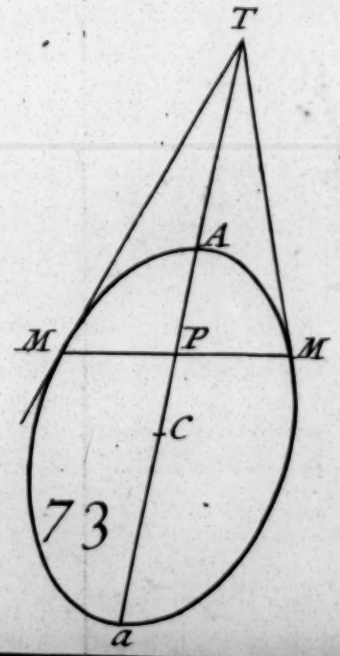

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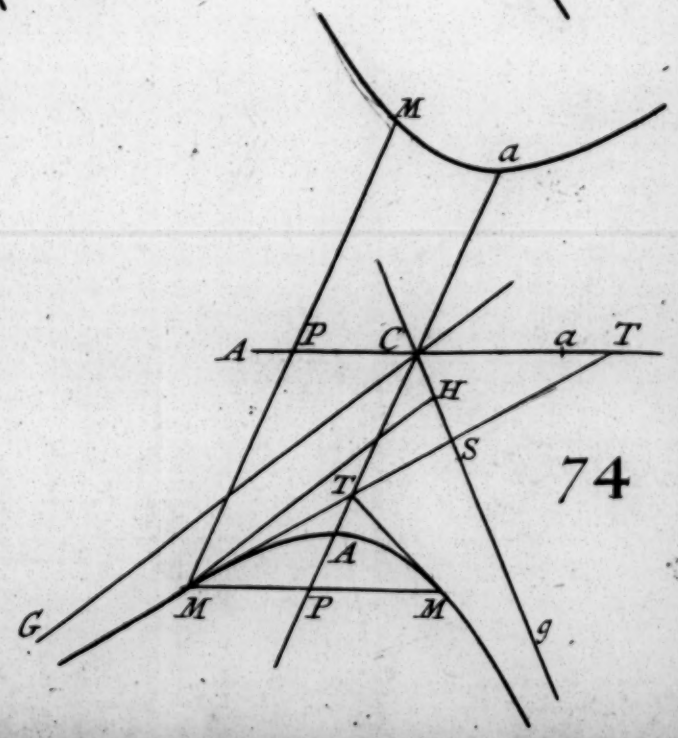

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terſection (*M*) of the Line *D M*, and the Side *A M*, will by this Motion deſcribe the Parabola ſought.

For if *M P* be drawn perpendicular to the Axis, then the right-angled Triangles *A G D*, *M P A*, will be ſimilar; becauſe each of the Angles *G A D*, *P M A*, being added to the Angle *P A M*, makes up a right Angle. Therefore we have this Proportion, *viz.* *A G* : *G D*, or *P M* : : *P M* : *A P*. And ſo $\overline{PM}^2$ will be $= G A \times A P$; and conſequently *P M* will be * an Ordinate to the Axis *A P*. * *Art.* 7.

*Note*, This Conſtruction has been already laid down in Book I. Art. 29. ſo as to agree to all Diameters: And the Reaſon why we have here only confin'd the ſame to the Axis, is to ſhow its Affinity with the Conſtruction we are going to give for the other Sections.

*For the other Sections.* Find a mean Proportional between the given Diameter and its Parameter, and place the ſame ſo that it be parallel to the Ordinates, and divided into two equal Parts by the Centre; then it is manifeſt, * that we ſhall have two Conjugate Diameters; by means of which find * the two Axes, and the Parameter of either of them in the Ellipſis, and that of the firſt Axis in the Hyperbola. This being done, * *Def.* 13, II. *and* 15, III. * *Art.* 64, *and* 128.

Produce the Axis of the Ellipſis to *G*, and divide the Axis (*A a*) of the Hyperbola in *G*; ſo that *a G* be to *G A*, as the Axis *A a* is to its Parameter; and draw the indefinite right Line *D D* through the Point *G* perpendicular to *A a*; then if the Point *D* moves along this Line, and at the ſame time this Point carries with it the right Line *D a*, moveable about the End *a* of the Axis (*A a*); as alſo the Side *D A*, of the right Angle *D A M*, moveable about its Vertex *A*, plac'd on the other End (*A*) of the ſaid Axis *A a*: I ſay, the continual Interſection (*M*) of the right Lines *A M*, *a D*, will by this Motion deſcribe the Section requir'd. FIG. 82, 83.

For if *M P* be drawn perpendicular to the Axis *A a*, the ſimilar Triangles *a P M*, *a G D*, will give this Proportion, *viz.* *a P* : *P M* : : *a G* : *G D*. But the right-angled Triangles *A G D*, *M P A*, are ſimilar; becauſe each of the Angles *G A D*, *P M A*, being added to the Angle *P A M*, makes a right Angle: And therefore *A P* : *P M* : : *G D* : *G A*. And by multiplying the Antecedents and Conſequents of the two firſt Ratio's, by the Antecedents and Conſequents of the two laſt, there will ariſe $aP \times PA : \overline{PM}^2 :: aG \times GD : GD \times GA :: aG : GA$; that is, as the Axis *A a* to its Parameter. Therefore, * *&c.* * *Art.* 41, *and* 81.

*Note*, The farther the Point *D* is diſtant from *G*, the greater will the Angle *P a M* be; and contrariwiſe, the leſs the Angle *P A M*; ſo that when the Lines *a M*, *A M*, becoming parallel in the Hyperbola, and afterwards cutting one another on the other Side the Line *D D*,

their continual Interſection will deſcribe the oppoſite Hyperbola.

In the Ellipſis and Hyperbola, if the Point *a* be conceiv'd to be infinitely diſtant from *A*, or (which is the ſame thing) if the Axis *Aa* be ſuppos'd infinitely great; then the Lines *GA*, *Da*, which will meet one another at an infinite Diſtance, may be taken for Parallels. And ſo this laſt Conſtruction falls again into the Caſe of the foregoing Conſtruction. Therefore the Ellipſis or Hyperbola will then become a Parabola, whoſe Parameter is the Line *AG*; and conſequently a Parabola may be eſteem'd as an Ellipſis or Hyperbola, whoſe Axis is infinite, *viz.* the firſt Axis in the Hyperbola, and either of the two in the Ellipſis.

*Second Way.*

FIG. 84. *For the Parabola.* Let *HAL* be an Iſoſcelles Triangle, one of whoſe Sides *AH* is ſituate in the given Diameter *AP* indefinitely produced both ways from its Origin *A*, and the other Side *AL* in the indefinite Tangent *LAL* paſſing through the Point *A*. Now if *HL*, the Baſe of this Triangle, be ſuppos'd to move always parallel to it ſelf; ſo that one End *L* thereof carries along with it the indefinite right Line *LM* parallel to *AP*, and the other End *H*, the Line *FH*, parallel to *AL*, and equal to the given Parameter of the Diameter *AP*; and if the Extremity *F* of the Line *FH*, likewiſe carries along with it the right Line *FA*, moveable about the fix'd Point *A*: Then, I ſay, the continual Interſection (*M*) of the two right Lines *FA*, *LM*, during the Motion of the Line *HL* in the Angle *HAL*, and that vertical thereto, will deſcribe the Parabola *MAM* requir'd.

For if the Ordinate *MP* be drawn to the Diameter *AP*, the ſimilar Triangles *AHF*, *APM*, will give *AH*, or *AL*, or *PM* : *HF* :: *AP* : *PM*, and therefore $\overline{PM}^2 = AP \times HF$. Whence, * *&c.*

* *Art.* 7, *and* 20.

It muſt be obſerv'd here, that the Point *H* falls beyond *A*, the Origin of the Diameter *AP*, when the Points *F*, *L* fall on both Sides that Diameter.

FIG. 85, 86. *For the other Sections.* Here the Conſtruction is the ſame as that of the Parabola, only the Line *LM* muſt move about *a*, the other End of the given Diameter *Aa*; whereas in the Parabola, *LM* muſt be parallel to *Aa*. *Note*, In the Hyperbola, we ſuppoſe that the given Diameter is a firſt Diameter; for if it be a ſecond Diameter, the firſt Diameter being the Conjugate to it may be found, as alſo its Parameter by *Art.* 115.

For if the Ordinate *MP* be drawn to the Dinmeter *Aa*; then the ſimilar Triangles *aPM*, *aAL*, and *APM*, *AHF*, will give theſe Proportions, *aP* : *PM* :: *aA* : *AL* or *AH*. And *AP* : *PM* :: *AH* : *HF*.

*HF*. And therefore, if the Antecedents and Conſequents of the two firſt Ratio's be multiply'd by thoſe of the ſecond, there will ariſe $aP \times PA : \overline{PM}^2 :: aA \times AH : AH \times HF :: aA : HF$. Whence, * *&c.* * *Art.* 41, 55, 81, *and* 118.

It muſt be obſerv'd, that the Points *H*, *a*, ought to fall on both Sides the Point *A* in the Ellipſis, and on one Side in the Hyperbola, when the Points *F*, *L*, fall on both Sides the Diameter *Aa*.

## COROLLARY I.

162. HENCE, if any Diameter *Aa*, together with one of its Ordinates *PM* be given; the Parameter *HF* thereof may be found. For 1. In the Parabola, aſſume *AH*, in the Diameter *AP*, equal to *PM*; and then if the Line *HF* be drawn parallel to *PM*, and terminated in *F* by the Line *AM*, drawn through *A* the Origin of the Diameter, and *M* the Extremity of the Ordinate: I ſay, it is manifeſt, that the Line *HF* will be the Parameter of the Diameter *AP*. FIG. 84.

2. In the other Sections, draw the Line *aM* through *a*, one End of the given Diameter *Aa*, meeting the Tangent *AL*, paſſing thro' the other End, in the Point *L*; then aſſume *AH*, in the Diameter *Aa*, equal to *AL*, and draw *HF* parallel to *PM*, meeting the Line *AM* in *F*, and the Line *HF* will be the Parameter of the Diameter *Aa*. FIG. 85, 86.

## COROLLARY II.

163. HENCE ariſes a very exact and uniform practical Method for deſcribing a Conick Section thro' ſeveral Points: An Example of which I ſhall give in the Ellipſis, which will ſerve as a Rule for deſcribing the other Sections.

Aſſume *AG*, in the Tangent *AL*, paſſing through (*A*) one End of the given Diameter *Aa*, equal to the Parameter of that Diameter, and draw the indefinite right Line *GF* parallel to *Aa*; and through the Point *A* draw any Number of right Lines *AF*, *AF*, *&c.* at pleaſure. This being done, aſſume *AL*, *AL*, *&c.* in the indefinite Tangent each equal to its Correſpondent *GF*, *GF*, &c. and draw the right Lines *aL*, *aL*, &c. then, I ſay, the Interſections *M*, *M*, &c. of the correſpondent right Lines *FA*, *La*, *FA*, *La*, &c. will be Points of the Ellipſis, whoſe Diameter is the Line *Aa*, Tangent the Line *AL*, and the Parameter of the Diameter *Aa*, the Line *AG*. This is manifeſt by drawing *FH* parallel to *AG*, and alſo the Line *HL* through the Point *L*, anſwering to *F*. For the Triangle *HAL* will be Iſoſcelles, becauſe * *AL* is equal to *GF*, or *AH*, and *HF* will be * *Hyp.* FIG. 87.

the Parameter of the Diameter $Aa$; and ſo this Conſtruction falls again into that of the ſecond of the two aboveſaid ways.

Becauſe the Lines $GF$, $AL$, become very great, when the Points $(M)$ to be found are near to the Point $a$; theſe Points may be found, by uſing the Tangent $al$ paſſing through the other End $a$ of the Diameter $Aa$, together with the Line $gf$ Parallel to $Aa$, as may be ſeen in the Figure.

If the Ordinates $MP$, $MP$, &c. be drawn parallel to the Tangent $AL$, and if they be produced on the other Side of the Diameter $Aa$, to the Points $M$, $M$, &c. ſo that each of them be cut into two equal Parts by that Diameter; then it is * evident, that the new Points $M$, $M$, &c. will be moreover in the ſame Ellipſis. * *Art.* 43.

The ſame Extent of your Compaſſes, *viz.* $GF$, or $AL$, will ſerve for denoting any Number of Points $F$, $F$, &c. $L$, $L$, &c. at pleaſure, in the Lines $GF$, $AL$. For by that means, all thoſe ſmall Parts being equal to one another, every $GF$ will be equal to its Correſpondent $AL$: Which is the Foundation of the Demonſtration.

## PROPOSITION XIII.

### Theorem.

Fig. 88, 89, 90, 91. 164. *IF there be two right Lines* MN, AR, *terminating in a Conick Section, meeting each other in* P, *and if they be parallel to two right Lines given in Poſition: I ſay, the Rectangle* MP × PN *will be always to the Rectangle* AP × PR, *in a given Ratio, in whatſoever Part of the Section the ſaid right Lines* MN, AR *fall.*

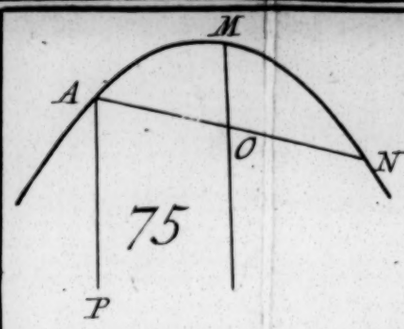
*For the Parabola.*

Fig. 88. Let the Tangents $CB$, $EB$, meeting one another in the Point $B$, be parallel to the right Lines $MN$, $AR$: I ſay, $MP \times PN : AP \times PR :: \overline{CB}^2 : \overline{EB}^2$.

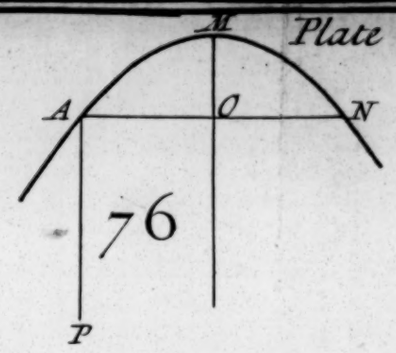
* *Art.* 148. For if the Diameter $CG$ be drawn * through $G$, the Middle of $MN$; and if $CB$ be drawn through $C$, the Origin thereof, parallel to $MN$; then it is * manifeſt, that $CB$ will touch the Section in $C$. And after the ſame manner muſt the Tangent $EB$ be drawn parallel to $AR$, which produce till it meets the Diameter $CG$ in the Point $K$; and if the Ordinate $EL$ be drawn through the Point of Contact $E$, we ſhall have * $KC = CL$; and ſo $KB = BE$. Again, draw the Ordinate $AD$, and the Line $AE$ parallel to the Diameter $CG$, and call the given Quantities $KB$ or $BE$, $m$; $BC$, $n$; $CK$, $e$; the Parameter $(CH)$ of the Diameter $CG$, $p$; and the indeterminate Quantities $AP$, $x$; $PM$, $y$; $AD$, $r$; $CD$, $s$. * *Art.* 10, and 21. * *Art.* 22, and 23.

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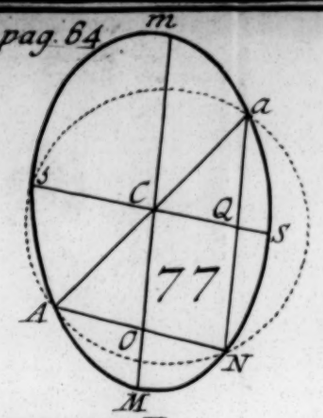
Plate 10. pag. 64



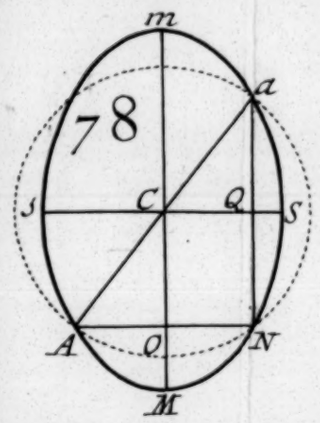
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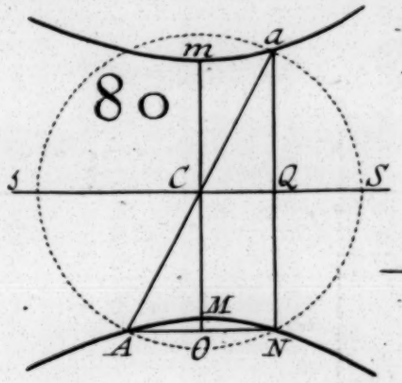
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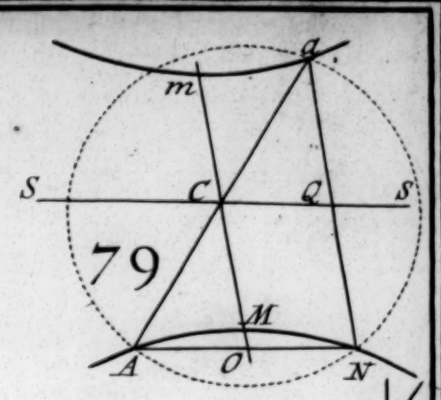
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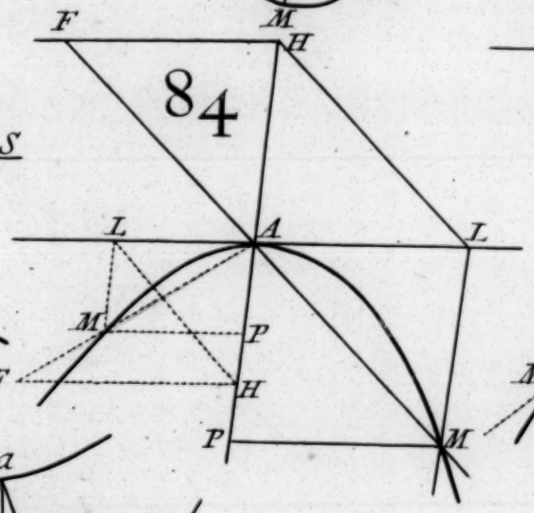
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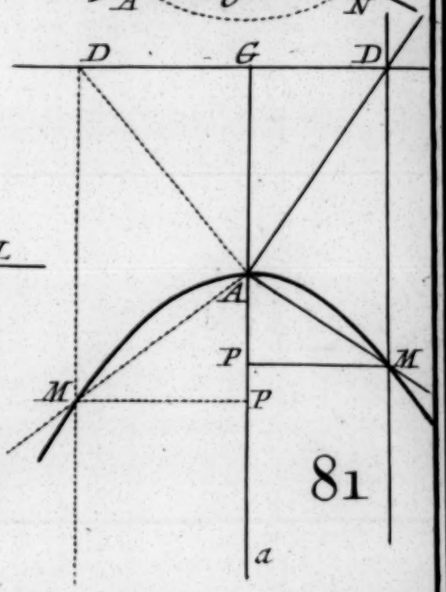
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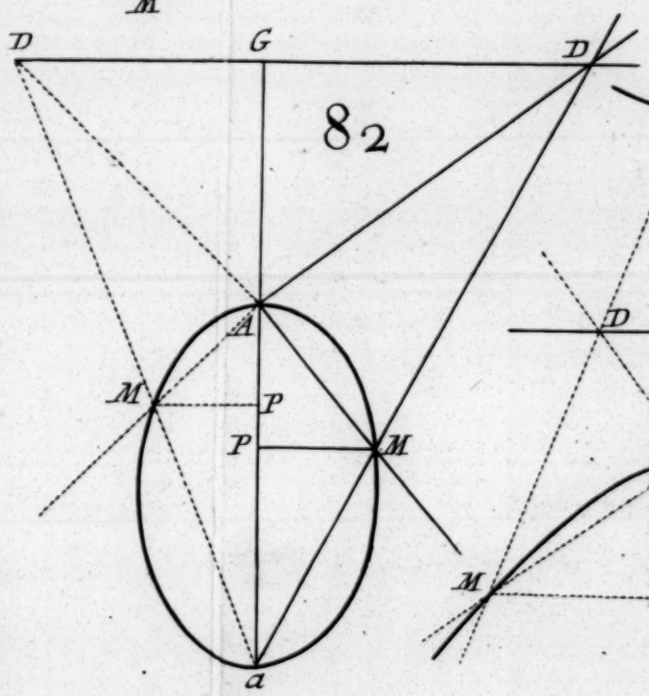
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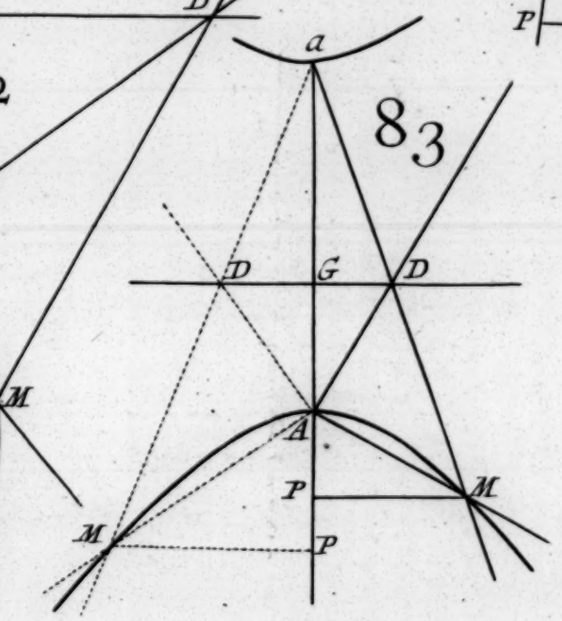
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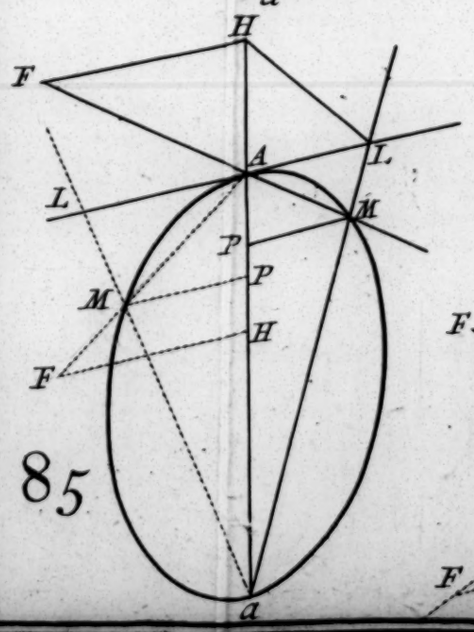
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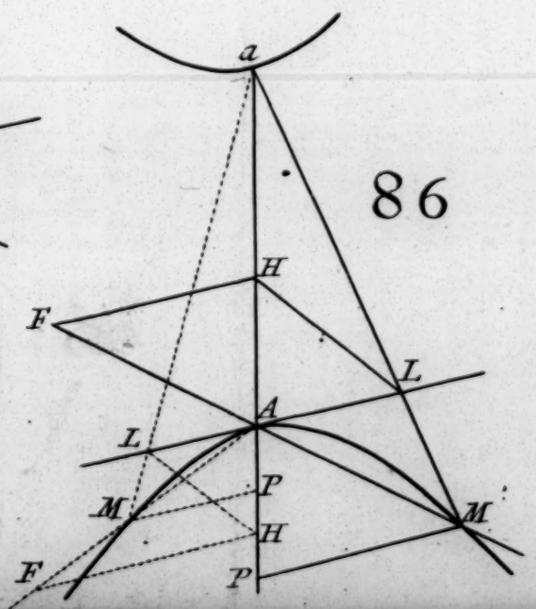
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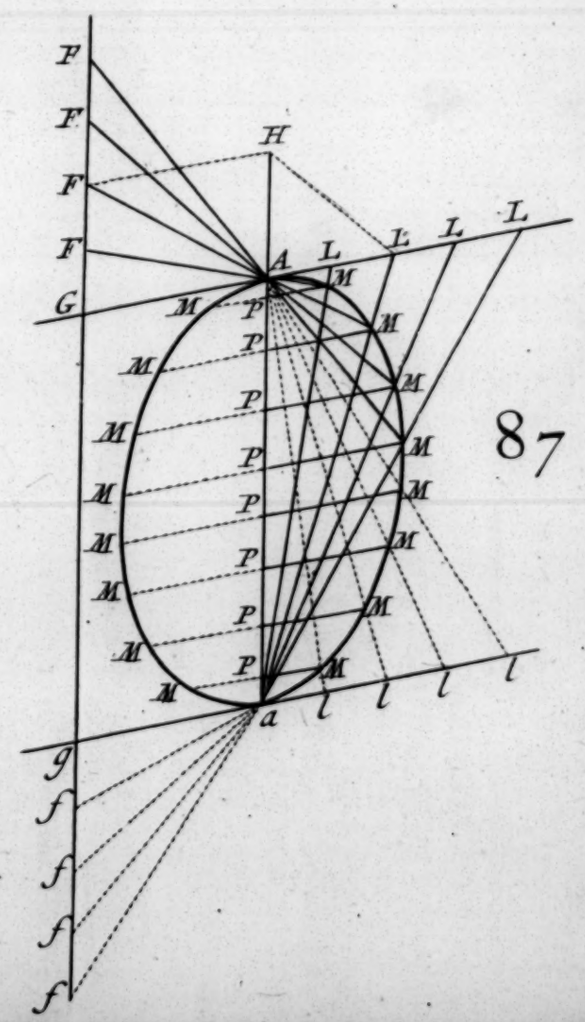
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This being laid down, becauſe the Triangles $KBC$, $APF$, are ſimilar, therefore $PF$ is $=\frac{nx}{m}$, $AF$ or $DG=\frac{ex}{m}$: And conſequently $CG=\frac{ex}{m}+s$, $GM$ or $GN=y+\frac{nx}{m}+r$, $PN$ or $GN+GP=y+\frac{2nx}{m}+2r$; $MP\times PN=yy+\frac{2nx}{m}y+2ry$, $\overline{GM}^2=yy+\frac{2nx}{m}y+2ry+\frac{nn}{mm}xx+\frac{2nr}{m}x+rr$. But * $CD(s):CG\left(\frac{ex}{m}+s\right)::\overline{AD}^2(rr):\overline{GM}^2=rr+\frac{err}{ms}x=rr+\frac{ep}{m}x$, becauſe $\overline{AD}^2(rr)=CD\times CH(ps)$. And comparing the two Values of $\overline{GM}^2$ together, there will be form'd the following Equation $yy+\frac{2nx}{m}y+2ry+\frac{nn}{mm}xx+\frac{2nr}{m}x-\frac{ep}{m}x=o$, which agrees to all Points of the Parabola, when the Line $AR$ falls above the Diameter $CG$, and the Point of Interſection $P$ falls between the Points $A$, $R$.

* *Art.* 8, *and* 20.

Now in the aforeſaid Equation if you make $y=o$, and ſtrike out all the Terms affected with $y$, we ſhall have $\frac{nn}{mm}xx+\frac{2nr}{m}x-\frac{ep}{m}x=o$. From hence we get $x=\frac{emp}{nn}-\frac{2mr}{n}=AR$; becauſe when $PM(y)$ is equal to nothing it is manifeſt that $AP(x)$ will be equal to $AR$. Therefore $AP\times PR=\frac{emp}{nn}x-\frac{2mr}{n}x-xx$; and conſequently $MP\times PN\left(yy+\frac{2nx}{m}y+2ry\right):AP\times PR\left(\frac{emp}{nn}x-\frac{2mr}{n}x-xx\right)::\overline{CB}^2(nn):\overline{EB}^2(mm)$: becauſe by multiplying the Means and Extremes, the preceeding Equation will come out again. And ſince the Tangents $CB$, $BE$ remain always the ſame, let their Parallels $MN$, $AR$, fall in any Part of the Parabola; therefore, &c.

Here it muſt be obſerv'd that there may happen different Caſes, according to the different Poſitions of the right Lines $MN$, $AR$; but ſince the Demonſtration is every where the ſame, excepting only in the Alteration of ſome Lines, or Terms that vaniſh; therefore I ſhall not explain all the Caſes ſeparately; which I would have obſerv'd alſo in the other Sections.

*For the other Sections.*

If the two Semi-diameters $CO$, $CB$, be drawn Parallel to the right Lines $MN$, $AR$: I ſay $MP\times PN:AP\times PK::\overline{CO}^2:\overline{CB}^2$. FIG. 89, 90, 91.

Draw

Draw the Diameter $CG$, having the Line $MN$ for a double Ordinate, and draw $BE$, $AD$, parallel to $MN$; alſo draw $AF$ parallel to $CG$, and call the given Quantities $CB$, $m$; $BE$, $n$; $CE$, $e$; and the Semi-diameter $CK$, $t$; the Semi-conjugate to it $CO$, $c$; and the intermediate Quantities $AP$, $x$; $PM$, $y$; $AD$, $r$; $CD$, $s$.

This being premiſed, becauſe the Triangles $CBE$, $APF$ are ſimilar, therefore $PF$ is $=\frac{nx}{m}$, $AF$ or $DG=\frac{ex}{m}$. And conſequently in the Hyperbola or oppoſite Sections, (*Fig.* 90. and 91.) we have $CG=\frac{ex}{m}\pm s$, $GM$ or $GN=y+\frac{nx}{m}-r$, $PN$ or $GN+GP=y+\frac{2nx}{m}-2r$; $MP\times PN=yy+\frac{2nx}{m}y-2ry$, $\overline{GM}^2=yy+\frac{2nx}{m}y-2ry+\frac{nn}{mm}xx-\frac{2nr}{m}x+rr$. But * $\overline{CD}^2\mp\overline{CK}^2(ss\mp tt):\overline{CG}^2\mp\overline{CK}^2$ $\left(\frac{eexx}{mm}\pm\frac{2esx}{m}+ss\mp tt\right)::\overline{AD}^2(rr):\overline{GM}^2=rr+\frac{eerrxx\pm 2emrrsx}{mmss\mp mmtt}$ $=rr+\frac{eeccxx\pm 2eccmsx}{mmtt}$, by ſubſtituting $\frac{cc}{tt}$ for $\frac{rr}{ss\mp tt}$ the Value * thereof. And comparing the ſaid two Values of $\overline{GM}^2$, we ſhall form the following Equation $yy+\frac{2nx}{m}y-2ry+\frac{nntt-ccee}{mmtt}xx-\frac{2nrtt\mp 2cces}{mtt}x=o$; in which, if you ſubſtitute $cctt$ in the Place of its Value $nutt-ccee$ (where note when $CB$ is half of a ſecond Diameter, you muſt ſuppoſe * a Conjugate Hyperbola to paſs through the Point $B$;) becauſe * $\overline{CE}^2+\overline{CK}^2(ee+tt):\overline{EB}^2(nn)::\overline{CK}^2(tt):\overline{CO}^2(cc.)$ then there will ariſe $yy+\frac{2nx}{m}y-2ry+\frac{cc}{mm}xx-\frac{2nrtt\mp 2cces}{mtt}x=o$, which Equation agrees to all Points of the Section, when the Points $A$, $R$, fall on both Sides the Diameter $CG$, and the Point of Interſection $P$ falls between the Points $A$, $R$.

Now in the ſaid Equation, if you make $y=o$, there will ariſe (by ſtriking out all the Terms affected with $y$) $\frac{cc}{mm}xx-\frac{2nrtt\mp 2cces}{mtt}x=o$, and ſo we ſhall get $x=\frac{2mnrtt\mp 2ccems}{cctt}=AR$; becauſe when $PM(y)$ is equal to nothing, then $AP$ $(x)$ will be $=AR$. Therefore $AP\times PR\left(\frac{2mnrtt\mp 2mcces}{cctt}x-xx\right):MP\times PN\left(yy+\frac{2nx}{m}y-2ry\right)::\overline{CB}^2(mm):\overline{CO}^2(cc)$. For if the Means and Extremes of this Proportion be multiply'd, there will ariſe the preceding Equation. Now

* *Art.* 82, *and* 118.

* *Art.* 82, *and* 118.

* *Art.* 134.

* *Art.* 81, *and* 118.

becauſe the Semidiameters $CO$, $CB$ remain always the ſame, let their Parallels $MN$, $AR$ fall in any Part of the Section: Therefore, *&c.*

I ſhall not here lay down the Calculus for the Ellipſis in particular, becauſe it differs only in ſome Lines from that of the Hyperbola.

### COROLLARY I.

165. IF any two right Lines $MN$, $AR$, terminating in a Conick Section, meet each other in $P$; and if any two other right Lines $FG$, $BD$ be drawn parallel to the two former ones, terminating in the Section likewiſe, and meeting one another in the Point $Q$; then it is manifeſt, that $MP \times PN : AP \times PR :: FQ \times QG : BQ \times QD$. For the two right Lines $AR$, $BD$, being parallel to one another, will be parallel to the ſame right Line $CZ$ given in Poſition; as likewiſe the two right Lines $MN$, $FG$, to the ſame right Line $CT$, alſo given in Poſition. FIG. 92.

### COROLLARY. II.

166. IF there be two Parallels $AR$, $BD$, terminating in a Conick Section, and meeting any right Line $FG$, terminated by the ſame Section in the Points $E$ and $Q$; then I ſay, $FE \times EG : AE \times ER :: FQ \times QG : BQ \times QD$. For if in *Corol.* 1. the Line $MN$ be ſuppos'd to fall in $FG$, it is manifeſt, that the Rectangles $MP \times PN$, $AP \times PR$, will become $FE \times EG$, $AE \times ER$.

### COROLLARY. III.

*For the Circle.*

167. FROM this Theorem may be drawn that moſt noted Property of the Circle, *viz.* that if through any Point $P$, taken within or without a Circle, there be drawn any Number of right Lines $AR$, $MN$, $HL$, &c. terminated by the Circumference, all the Rectangles $AP \times PR$, $MP \times PN$, $HP \times PL$, &c. will be equal to one another: For if the Semidiameters $CB$, $CO$, $CD$, &c. be drawn parallel to theſe Lines, then (by the Theorem) it is manifeſt, that all thoſe Rectangles will be to one another, as the Squares of the ſaid Semidiameters or Radii, which by the eſſential Property of the Circle are all equal to one another. FIG. 93.

COROL-

## COROLLARY IV.

*For the Parabola.*

FIG. 94. 168. IF there be any right Line $MN$ terminating in a Parabola; and if the Diameter $AF$ be drawn through any Point $A$ in the Parabola, meeting the Line $MN$ in the Point $F$: I ſay, the Rectangle $MF \times FN$, is equal to the Rectangle under $AF$ and $CH$, the Parameter of the Diameter $CG$, paſſing through the Middle of $MN$.

For ſuppoſing, in the Theorem, the Line $AP$ to fall in $AF$, then it is manifeſt, that the Line $PF\left(\frac{n}{m}x\right)$ will become equal to nothing, and ſo $\frac{n}{m}=o$. Therefore ſtriking out all the Terms affected with $\frac{n}{m}$ in the Equation for the Parabola, *viz.* $yy+\frac{2nx}{m}y+2ry+\frac{nn}{mm}xx+\frac{2nr}{m}x-\frac{ep}{m}x=o$, we ſhall have this $yy+2ry-\frac{ep}{m}x=o$. But $AF=\frac{ex}{m}$, $CH=p$, and $MF \times FN=yy+2ry$. Whence &c.

I have laid down this Corollary only to ſhew the great Extent of the Theorem it is deduced from; for it may be demonſtrated more eaſy another way, which is thus: $\overline{GM}^2=GC \times CH$, $\overline{AD}^2$ or $\overline{GF}^2=DC \times CH$, and therefore $\overline{GM}^2-\overline{GF}^2$ or $MF \times FN=GC-DC \times CH=AF \times CH$.

## COROLLARY V.

*For the Parabola.*

169. HENCE it is manifeſt,

1. If there be two right Lines $MN$, $EL$, terminating in a Parabola, and parallel to one another; and if through any two Points $A$, $B$, in that Parabola be drawn two Diameters $AF$, $BP$, meeting the Lines $MN$, $EL$, in the Points $F$, $P$: I ſay it is evident, that $MF \times FN : EP \times PL :: AF : BP$. For the Diameter $CG$ which paſſes through the middle of $MN$, paſſes alſo through the middle of $EL$; and conſequently the Rectangle $EP \times PL$ is $= BP \times CH$, and moreover $MF \times FN = AF \times CH$.

2. If any right Line $MN$ terminating in a Parabola, meets two Diameters ($AF$, $BK$,) thereof in the Points $F$, $K$; then we ſhall have this proportion, $MF \times FN : MK \times KN :: AF : BK$.

3. If

3. If any two parallel right Lines $MN$, $EL$, terminating in a Parabola, meet any one of its Diameters ($BP$) in the Points $K$, $P$; then will it be as $MK \times KN : EP \times PL :: BK : BP$.

## COROLLARY VI.

*For the Parabola.*

170. HENCE appears the Way of describing a Parabola passing through three given Points $A$, $M$, $N$, and whose Diameters $AF$, $CG$ shall be parallel to a right Line given in Position; together with the manner of demonstrating, that there can be but one Parabola only that will satisfy the Conditions of the Problem.

For join two of the given Points $M$, $N$, by the right Line $MN$, and through the third $A$ draw a Diameter $AF$ parallel to the Line given in Position, and meeting the Line $MN$ in the Point $F$; also through ($G$) the Middle of $MN$, draw $GC$ parallel to $AF$. This being done, make $MF \times FN : MG \times GN$, or $\overline{GM}^2 :: AF : GC$. And having taken $CH$, a third Proportional to $CG$, $GM$, describe * a Parabola, with the Parameter $CH$, and Diameter $CG$, whose Origin is $C$, and Ordinates are parallel to $MN$; and that will be the Parabola requir'd. * *Art.* 29, *and* 30.

For 1. The Parabola will * pass through the Points $M$, $N$; because by Construction $CH \times CG = \overline{GM}^2$ or $\overline{GN}^2$. 2. It will pass also through the Point $A$, because $MG \times GN : MF \times FN :: CG : FA$. 3. The Diameters $AF$, $CG$, will be parallel to the right Line given in Position. * *Art.* 7, *and* 20.

And because the right Line $CG$, whose Origin is $C$, is a Diameter of the Parabola answering the Conditions of the Problem, and the determinate Line $CH$ is the Parameter of that Diameter; therefore there is but one Parabola only that will satisfy the Problem.

## COROLLARY VII.

*For the Parabola.*

171. IF there be two right Lines $AR$, $MN$, terminating in a Parabola, meeting each other in the Point $P$; and if it be made as $AP \times PR : MP \times PN :: \overline{AP}^2 : \overline{PF}^2$: and the Line $AF$ be drawn; then, I say, the Line $AF$ will be a Diameter. For if the Tangents $CB$, $EB$, be drawn parallel to the right Lines $MN$, $AR$, and the Diameter $CG$ be drawn through ($C$) the Point of Contact meeting $EB$ (produced) in $K$, we shall have $\overline{EB}^2$ or $\overline{KB}^2 : \overline{BC}^2 :: AP \times PR :$ FIG. 88.

$MP \times PN :: \overline{AP}^2 : \overline{PF}^2$, and consequently $KB : CB :: AP : PF$. Therefore the Triangles $KBC$, $APF$, will be similar, and their Sides $AF$, $KC$ parallel: Whence the Line $AF$, being parallel also to the Diameter $CG$, will be a Diameter; because all Diameters * of the Parabola are parallel to one another.

* *Def.* 7, I.

## COROLLARY VIII.

*For the Parabola.*

172. BY means of the last Corollary, a Parabola may be describ'd, which shall pass through four given Points $A$, $M$, $R$, $N$.

For if the said four Points be joined by two right Lines $AR$, $MN$, intersecting each other in the Point $P$; and if you make $AP \times PR : MP \times PN :: \overline{AP}^2 : \overline{PF}^2$, and draw the Line $AF$; then if a Parabola be describ'd * passing through the three Points $A$, $M$, $N$, whose Diameters are parallel to the Line $AF$; this Parabola will be that requir'd. For by the Theorem, the Line $AP$ must meet that Parabola in the Point $R$, being such that $AP \times PR : MP \times PN :: \overline{EB}^2$ or $\overline{KB}^2 : \overline{BC}^2 :: \overline{AP}^2 : \overline{PF}^2$.

* *Art.* 170.

FIG. 95. If the Point $F$ falls beyond the Point $P$, then there may be another Parabola describ'd, which will pass through the four given Points. But it must be here observ'd, that when one of the Points $F$ falls on one of the given Points $M$ or $N$; then there will be but one Parabola answering the Problem; and when both of them fall on the Points $M$, $N$, the Problem will be impossible; because then the Diameter ($AF$) of the Parabola, will meet the Parabola in two Points, which has been * demonstrated to be impossible.

* *Art.* 10.

## COROLLARY IX.

*For the Hyperbola, or opposite Sections.*

FIG. 96, 97. 173. IF there be a right Line $MN$, terminating in an Hyperbola, or the opposite Sections, meeting an Asymptote $CB$ in the Point $Q$, and which is parallel to a right Line given in Position; and if thro' any Point $A$ in the Section, there be drawn a right Line $AP$ parallel to that Asymptote, meeting the Line $MN$ in the Point $P$: I say, the Rectangle $MP \times PN$ to the Rectangle $2\,AP \times PQ$ will be in a given Ratio, let the right Lines $MN$, $AP$ fall in any Part soever of the Section or Sections.

For if in the Theorem (*Fig.* 90, 91.) the Semidiameter $CB$ be supposed to become an Asymptote; then it is * manifest, that each of the three Sides of the Triangle $CBE$ will become infinite. And so if $KS$ be drawn (*Fig.* 96, 97.) through $K$, the End of the Diameter $LK$, (passing through the Middle of $MN$) parallel to $MN$, and meet-

* *Art.* 102.

meeting the Aſymptote $CB$ in $S$; then the Triangle $CKS$ will be form'd, whoſe three Sides are finite, and this will be ſimilar to the Triangle $CBE$; therefore $CK\ (t) : KS$, or * $CO\ (c) ::$ $CE\ (e) : EB\ (n)$; and ſo $ce$ is $= nt$. Now ſubſtituting $nt$ in the Place of its Value $ce$, in the Equation found in the Theorem for the Hyperbola, *viz.* $yy + \frac{2nx}{m}y - 2ry + \frac{nntt-ccee}{mmtt}xx - \frac{2nrtt \mp 2eccs}{mtt}x = o$, there ariſes $yy + \frac{2nx}{m}y - 2ry - \frac{2nrt \mp 2ncs}{mt}x = o$, or $yy + \frac{2nx}{m}y - 2ry = \frac{2nrt \mp 2ncs}{mt}x$. But producing $AD$, (if neceſſary) till it meets the Aſymptote $CB$ in $H$, then the ſimilar Triangles $CKS$, $CDH$, will give this Proportion, *viz.* $CK\ (t) : KS\ (c) :: CD\ (s) : DH = \frac{cs}{t}$. And therefore, $AH$ or $P\mathcal{Q} = \frac{rt \mp cs}{t}$. Whence we have $MP \times PN\ (yy + \frac{2nx}{m}y - 2ry) : 2AP \times P\mathcal{Q}\left(\frac{2rt \mp 2cs}{t}x\right) :: EB\ (n) : CB\ (m) :: KS : CS$. Becauſe in multiplying the Means and Extremes, the aforeſaid Equation will again be produced. Now becauſe the Lines $KS$, $CS$, remain always the ſame, let the Lines $MN$, $AP$ fall in any Part ſoever of the Section; ſince the Diameter $LK$, paſſing through the Middle of $MN$, alſo * paſſes thro' the Middle of all the Parallels to $MN$ terminated by the Section, let them fall any how ſoever. Therefore, *&c.*

*Art.* 113.

*Art.* 145.

This Corollary may be immediately demonſtrated, without having recourſe to the Theorem, after the following manner. Let the given Quantities $CK$ be $= t$, $KS$ or $CO = c$, $CS = m$, and the indeterminate Quantities $CD = s$, $AD$ or $DI = r$, $AP = x$, $PM = y$. Then becauſe the Triangles $CSK$, $APF$, are ſimilar, therefore $PF$ is $= \frac{cx}{m}$, $AF$ or $DG = \frac{tx}{m}$; and therefore $GM$ or $GN = y + \frac{cx}{m} - r$, $CG = \frac{tx}{m} + s$. Now becauſe the Triangles $CKS$, $CDH$, $CG\mathcal{Q}$, are ſimilar; therefore $CK\ (t) : KS\ (c) :: CD\ (s) : DH = \frac{cs}{t} :: CG\left(\frac{tx}{m} + s\right) : G\mathcal{Q} = \frac{cx}{m} + \frac{cs}{t}$. And therefore $M\mathcal{Q} \times \mathcal{Q}N$, or $\overline{G\mathcal{Q}}^2 - \overline{GM}^2 = \frac{2ccsx}{mt} + \frac{ccss}{tt} - yy - \frac{2cxy}{m} + 2ry + \frac{2crx}{m} - rr =$ * $AH \times HI$, or $\overline{DH}^2 - \overline{DI}^2 = \frac{ccss}{tt} - rr$; and by ſtriking out from each Side $\frac{ccss}{tt} - rr$, and bringing over all the Terms wherein $y$ is, to one

FIG. 96.

*Art.* 97.

Side, we ſhall get this Equation, $yy + \frac{2cxy}{m} - 2ry = \frac{2ccsx}{mt} + \frac{2crx}{m}$, which being reduced to a Proportion, gives $MP \times PN\,(yy + \frac{2cxy}{m} - 2ry) : 2AP \times PQ\,(\frac{2csx}{t} + 2rx) :: KS\,(c) : CS\,(m.)$ *W. W. D.*

The Demonſtration is the ſame, for the oppoſite Sections only with the Alteration of ſome Signs.

## COROLLARY X.

*For the Hyperbola, or oppoſite Sections.*

FIG. 98. 174. IT follows from the laſt Corollary,

1. If there be two parallel right Lines $MN$, $HG$, terminating in an Hyperbola, or the oppoſite Sections, and meeting one Aſymptote $CS$ in the Points $Q$, $I$; and if thro' any two Points $(A, B)$ in the Section, there be drawn the two Parallels $AP$, $BD$, to the Aſymptote $CS$, meeting the Parallels $MN$, $HG$, in the Points $P$, $D$; then the Rectangles $MP \times PN$, $2AP \times PQ$, will be to one another, as the Rectangles $HD \times DG$, $2BD \times DI$; and therefore we have $MP \times PN : HD \times DG :: AP \times PQ : BD \times DI$.

2. If there be two parallel right Lines $MN$, $HG$, terminating in an Hyperbola, or the oppoſite Sections, and meeting one Aſymptote $CS$ in the Points $Q$, $I$; and if, through any Point $A$ in the Section, there be drawn $AO$ parallel to $CS$, and meeting the Lines $MN$, $HG$ in the Points $P$, $O$; then we ſhall have the following Proportion, (by conceiving in the laſt Caſe the Line $BD$ to fall in $AP$) $MP \times PN : HO \times OG :: AP \times PQ : AO \times OI :: AP : AO$, becauſe $PQ$ is $= OI$.

3. If there be any right Line $HG$ terminating in an Hyperbola, or the oppoſite Sections, and meeting the Aſymptote $CS$ in the Point $I$; and if thro' any two Points $(A, B)$ in the Section, there be drawn the Lines $AO$, $BD$, parallel to $CS$, and meeting the Line $HG$ in the Points $O$, $D$; then we ſhall have this Proportion, $HO \times OG : HD \times DG :: AO \times OI : BD \times DI$. This is a farther Continuation of the firſt Caſe, in ſuppoſing the Line $MN$ to fall in $HG$.

## COROLLARY XI.

FIG. 92. 175. IF any right Line $BD$, meeting a Conick Section in two Points $B$, $D$, be ſuppos'd to move parallel to itſelf, until it becomes the Tangent $LS$; then it is manifeſt, that the two Points of Interſection $B$, $D$ will coincide in the Point of Contact $L$; and ſo a Point

Point of Contact may be conſider'd as two Points of Interſection that do coincide. Which being premis'd, there ariſes ſeveral Caſes of the 1ſt, 2d, 5th and 10th Corollaries, the Principal of which are as follow.

1. If there be two Tangents, $KS$, $LS$, meeting one another in the Point $S$, and two other right Lines $MN$, $AR$, parallel to thoſe Tangents terminating in the Section, and meeting one another in $P$; I ſay, $MP \times PN : AP \times PR :: \overline{KS}^2 : \overline{LS}^2$. This has been prov'd in the Theorem, with regard to the Parabola: But for the other Sections, if in the firſt Corollary you conceive the Line $FG$ to fall in the Tangent $KS$, and $BD$ in $LS$; then it is plain, that the two Points of Interſection $F, G$, will coincide in the Point of Contact $K$; as likewiſe the two Points $B, D$, in the Point of Contact $L$; and ſo the Rectangles $F\mathcal{Q} \times \mathcal{Q}G$, $B\mathcal{Q} \times \mathcal{Q}D$, will become the Squares $\overline{KS}^2$, $\overline{LS}^2$.

2. In an Ellipſis, or the oppoſite Sections, if there be drawn a Tangent $TX$ parallel to $KS$, and meeting $SL$ in the Point $X$; then we prove, as in *Num.* 1. that $MP \times PN : AP \times PR :: \overline{TX}^2 : \overline{LX}^2$; and therefore $\overline{KS}^2 : \overline{LS}^2 :: \overline{TX}^2 : \overline{LX}^2$; and $KS : SL :: TX : LX$. That is, if two Tangents parallel to $KS$, $TX$, meet a third Tangent $LS$, in the Points $S$, $X$; then we ſhall have this Proportion, $KS : LS :: TX : LX$; or $KS : TX :: LS : LX$.

3. In the Ellipſis, Hyperbola, or oppoſite Sections, if there are two Tangents $KS$, $LS$, meeting one another in the Point $S$; and if the two Semidiameters $CT$, $CZ$, be drawn parallel to thoſe Tangents; I ſay, theſe Tangents will be to one another, as the two Semidiameters. For by the Theorem, $\overline{CT}^2 : \overline{CZ}^2 :: MP \times PN : AP \times PR :: \overline{KS}^2 : \overline{LS}^2$ by *Num.* 1. therefore $CT : CZ :: KS : LS$.

4. If there be two right Lines $AR$, $FG$, terminating in a Conick Section, and meeting the two Tangents $KI$, $LO$, being parallel to them in the Points $I, O$; then I ſay, $FO \times OG : \overline{LO}^2 :: \overline{KI}^2 : AI \times IR$. This is evident by conceiving in *Corol.* 1. that the Line $BD$ becomes the Tangent $LO$; and $MN$, the Tangent $KI$.

5. If there are two Parallels $AR$, $BD$, terminating in a Conick Section, and meeting the Tangent $KH$ in the Points $I, H$; then I ſay, $\overline{KI}^2 : AI \times IR :: \overline{KH}^2 : BH \times HD$, or $\overline{KI}^2 : \overline{KH}^2 :: AI \times IR : BH \times HD$. This is a Continuation of *Corol.* 2. by conceiving the Line $FG$ to fall in the Tangent $KH$.

6. If the Conick Section, in the laſt Number, be ſuppos'd an Hyperbola, and the Tangent $HK$ an Aſymptote; then the Rectangles $BH \times HD$, $AP \times IR$, will become equal to one another. For then the

*Art. 108. the Point of Contact $K$ will be * infinitely distant from the Points $H$, $I$; and consequently the infinite right Lines $HK$, $IK$, whose Difference is only the finite Line $HI$, may be taken as Equals. This has been demonstrated already in the 97th Article; and the Reason why we have repeated it again, is only for proving what we have laid down, and shewing that the same Truths may be discover'd after very different ways.

7. If there be two Tangents $KS$, $LS$, meeting one another in the Point $S$; and if there be a right Line $AR$, terminated by the Section, parallel to one of those Tangents, as $LS$, and meeting the other $KS$, in the Point $I$; then I say, $\overline{KI}^2 : AI \times IR :: \overline{KS}^2 : \overline{LS}^2$. This is manifest by conceiving, in the second Corollary, the Lines $FG$, $BD$, to fall in the Tangents $KS$, $LS$.

8. In an Ellipsis, or the opposite Sections, if there be two parallel Tangents $KI$, $TV$, meeting the Line $AR$, terminating in the Section, in the Points $I$, $V$; then I say, $\overline{KI}^2 : AI \times IR :: \overline{TV}^2 : RV \times VA$. This is a Continuation of the second Corollary, by supposing the Parallels $MN$, $FG$, to fall in the Tangents $TV$, $KI$.

FIG. 94. 9. In a Parabola, if there are two Parallels $MN$, $CH$, one of which touches the Section in $C$, and the other is terminated by the same; and if through any two Points $A$, $B$, of the Section, there be drawn the two Diameters $AF$, $BO$, meeting the Lines $MN$, $CH$, in the Points $F$, $O$; then it is evident, by conceiving (in *Num.* 1, and 2. of *Corol.* 6.) $EL$ to fall in the Tangent $CH$; 1. That $MF \times FN : \overline{CO}^2 :: AF : BO$. 2. That if $FA$ be produced till it meets the Tangent $CH$ in $Q$, then we shall have $MF \times FN : \overline{CQ}^2 :: AF : AQ$.

FIG. 98. 10. If there be two Parallels $MN$, $KT$, whereof one, as $KT$, touches an Hyperbola in $K$, and meets one of the Asymptotes in $S$, and the other $MN$ is terminated by either of the opposite Sections, and meets the same Asymptote in $Q$; and if, thro' any two Points $A$, $B$, in the Section, there be drawn two Parallels $AP$, $BT$, to the Asymptote $CS$, meeting the Lines $MN$, $KT$, in the Points $P$, $T$; then we shall have (by supposing in the three Numbers of *Coroll.* 10. the Secant $GH$ to fall in the Tangent $RT$) 1. The Rectangle $MP \times PN : \overline{KT}^2 :: AP \times PQ : BT \times TS$. 2. By producing $PA$ till it meets $KT$ in $R$, the Rectangle $MP \times PN : \overline{KR}^2 :: AP : AR$. 3. The Square $\overline{KT}^2 : \overline{KR}^2 :: BT \times TS : AR \times RS$.

11. In the opposite Sections, if there be two parallel Tangents $KR$, $LF$, meeting the Asymptote $CS$ in the Points $S$, $V$; and if thro' any two Points $A$, $B$, in the Section, there be drawn two Parallels $AR$, $BF$ to the Asymptote $CS$, meeting the two Tangents in the Points $R$, $F$; then we shall have (by supposing, in *Numb.* 1, and 2.

*Corol.*

*Corol.* 10. the two Secants $MN$, $GH$, to fall in the two Tangents $KR$, $LF$) 1. The Square $\overline{KR}^2 : \overline{LF}^2 :: AR \times RS : BF \times FV$. 2. The Square $\overline{KR}^2 : \overline{LE}^2 :: AR : AE$.

## PROPOSITION XIV.

### Problem.

176. *TO deſcribe an Ellipſis, or two oppoſite Hyperbola's about a given Parallelogram* FGHK; *ſo that one of the Diameters* AB *thereof, being parallel to the two Sides* FK, GH, *be to the Conjugate Diameter* DE, *in the given Ratio of* m *to* n. FIG. 99, 100, *and* 101.

Draw the Lines $AB$, $DE$, biſecting the oppoſite Sides of the given Parallelogram $FGHK$; then it is * manifeſt, that thoſe Lines will be the two Conjugate Diameters of the Section requir'd, and their Point of Interſection will be the Centre thereof; becauſe by one of the Conditions of the Problem, the two Parallels $FG$, $KH$, muſt terminate in the Section, as well as the two other Parallels $FK$, $GH$. This being laid down, if $AB$, $DE$ be taken for thoſe two Conjugate Diameters; and you call the given Quantities $CL$, or $CO$, $a$; $LF$, or $OK$, $b$; and the unknown Quantity $CA$ or $CB$, $t$; then we ſhall have, 1. (when the Section is * an Ellipſis) $BL \times LA\,(tt - aa) : \overline{LF}^2\,(bb) :: AB : DE :: mm : nn$. And therefore $tt = aa + \frac{mmbb}{nn}$. * *Art.* 146. * *Art.* 41, *and* 55.

2. When the Sections are oppoſite ones, $\overline{CL}^2 \mp \overline{CA}^2\,(aa \mp tt) : \overline{LF}^2\,(bb) :: \overline{AB}^2 : \overline{DE}^2 :: mm : nn$: and ſo $tt$ is $= aa - \frac{mmbb}{nn}$, or $tt = \frac{mmbb}{nn} - aa$; *viz.* $= aa - \frac{mmbb}{nn}$, when the Line $AB$ is a firſt Diameter, and $= \frac{mmbb}{nn} - aa$, when the ſame is a ſecond Diameter; and from hence ariſes the following Conſtruction, of which I make three Caſes. * *Art.* 81, *and* 118.

*Caſe* 1. When the Section is an Ellipſis, make the right-angled Triangle $VST$ ſuch; that the Side $ST$ be $= CL$, and the Side $SV = \frac{m}{n} LF$; then if an Ellipſis be deſcrib'd with the Semidiameter $CA = TV$, having the ſame Proportion to its Semi-conjugate $CD$, as $m$ is to $n$. I ſay, that Ellipſis will anſwer the Problem. For 1. The Diameter $AB$ being parallel to the Sides $FK$, $GH$, is to the Conjugate $DE$, in the given Ratio of $m$ to $n$. 2. Becauſe the Triangle $TSV$, is right-angled at $S$, the Square $\overline{TV}^2$ or $\overline{CA}^2\,(tt) = \overline{TS}^2\,(aa) + \overline{SV}^2$ $\frac{mmbb}{}$.

$\left(\frac{mmbb}{nn}\right)$; and therefore $BL \times LA\,(tt-aa) = \frac{mmbb}{nn}$; and consequently we have $BL \times LA\left(\frac{mmbb}{nn}\right) : \overline{LF}^2\,(bb) :: mm : nn :: \overline{AB}^2 : \overline{DE}^2$. Whence it appears, that $LF$ is an Ordinate to the Diameter $AB$; and ſo the Section paſſes thro' the Point $F$. After the ſame manner we demonſtrate that the Section will paſs thro' the three Points $G$, $H$, $K$; becauſe $GL = LF = OK = OH$, and $CO = CL$.

*Caſe* 2. When the Sections are oppoſite ones, and $CL$ is greater than $\frac{m}{n}$ $LF$: form the Right-angl'd Triangle $TSV$, ſuch, that the Side $SV$ be $= \frac{m}{n} LF$, and the Hypothenuſe $VT = CL$; and then deſcribe two oppoſite Hyperbola's, having the Semi-firſt Diameter $CA = TS$, in the ſame Proportion to $CD$, the Semi-conjugate, as $m$ is to $n$.

*Caſe* 3. When the Sections are oppoſite ones, and $CL$ is leſs than $\frac{m}{n} LF$: Then the Right-angl'd Triangle $TSV$ muſt be form'd, having the Side $TS = CL$, and the Hypothenuſe $SV = \frac{m}{n} LF$. And afterwards two oppoſite Hpperbola's muſt be deſcribed with the Semi-ſecond Diameter $CA = TV$, and the Semi-conjugate $CD$, having the ſame proportion to one another as $m$ to $n$.

The Demonſtration of theſe two laſt Caſes is the ſame as that of the firſt; but it muſt be obſerved when $CL$ is $= \frac{m}{n} LF$, the Problem is impoſſible.

## COROLLARY. I.

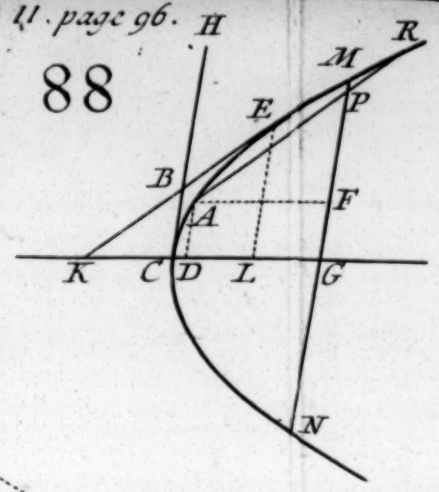
177. BEcauſe the poſition of the two Conjugate Diameters $AB$, $DE$, is determinate, as well as their Magnitude; ſince by the Conditions of the Problem, the ſaid Diameters muſt cut the oppoſite Sides of the Parellelogram, and there can be found but one Value for the Semi-diameters $CA$, $CB$; therefore the Problem can have but one Anſwer.

## COROLLARY II.

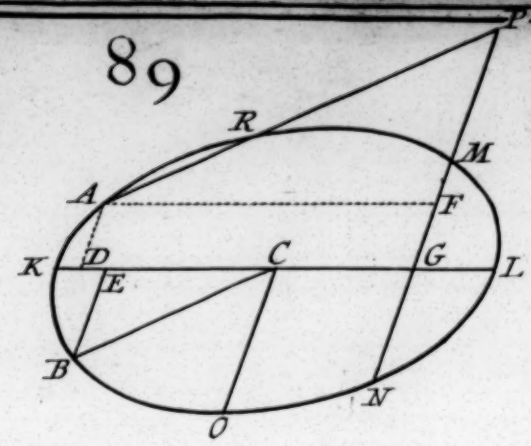
178. HENCE appears the Manner of deſcribing a Conick Section, about a given Parallelogram $FGHK$, and paſſing through a given Point $M$.

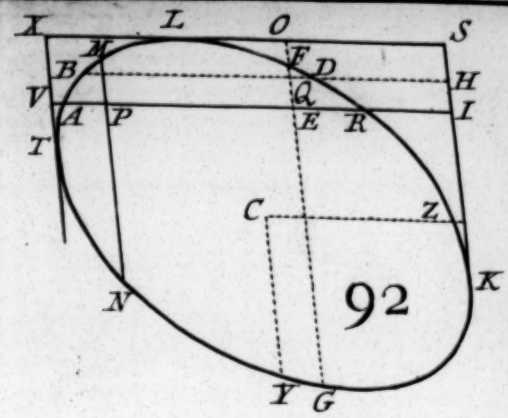
For draw two Conjugate Diameters $AB$, $DE$, biſecting the oppoſite Sides of the Parallelogram; alſo from the given Point $M$ draw the

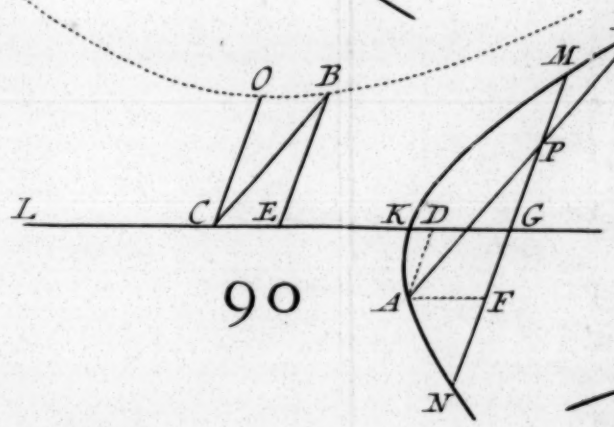
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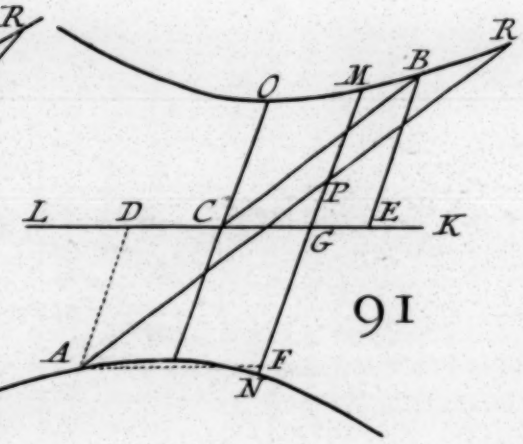
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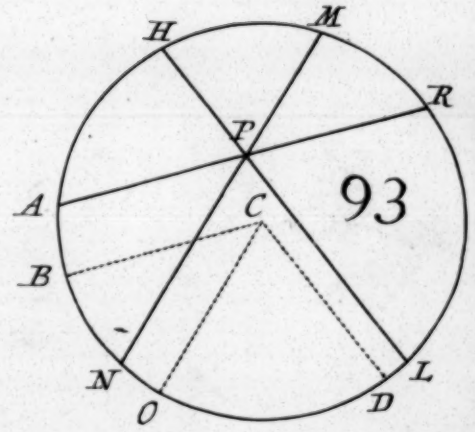

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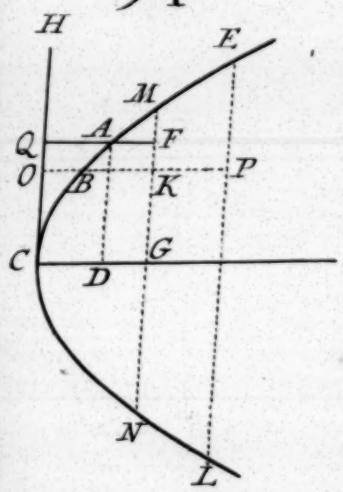
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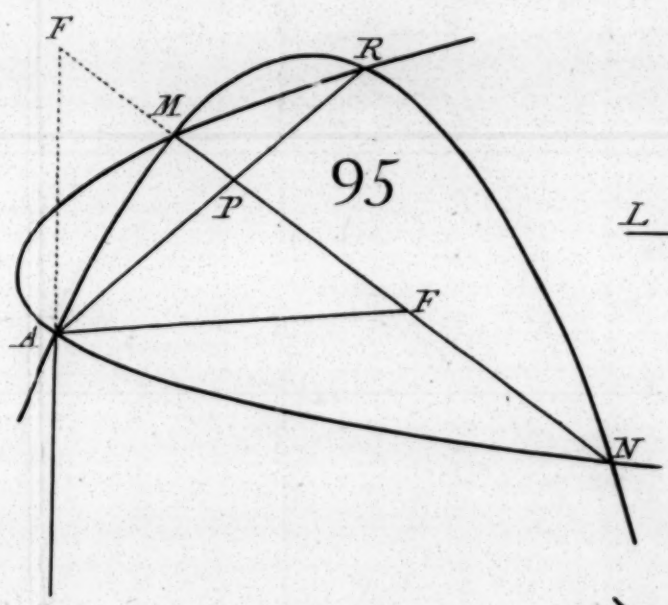

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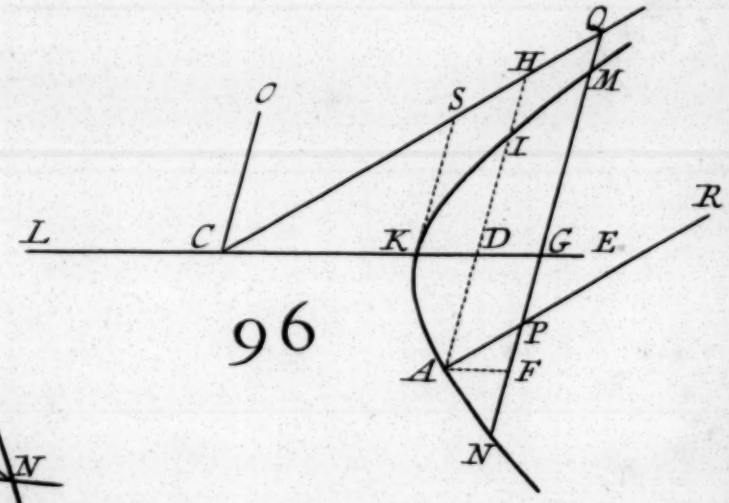
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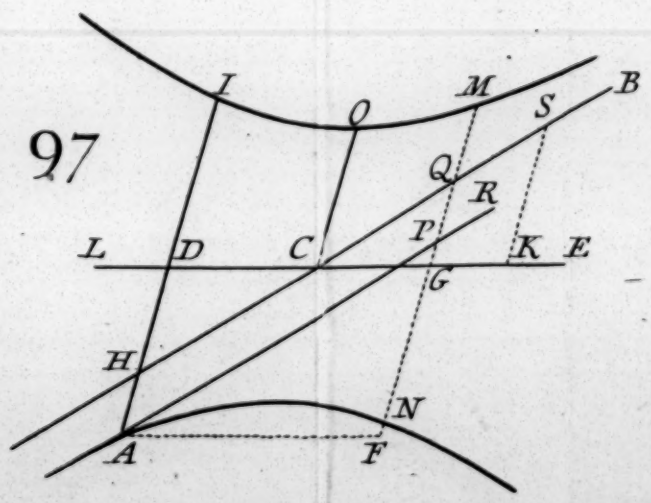

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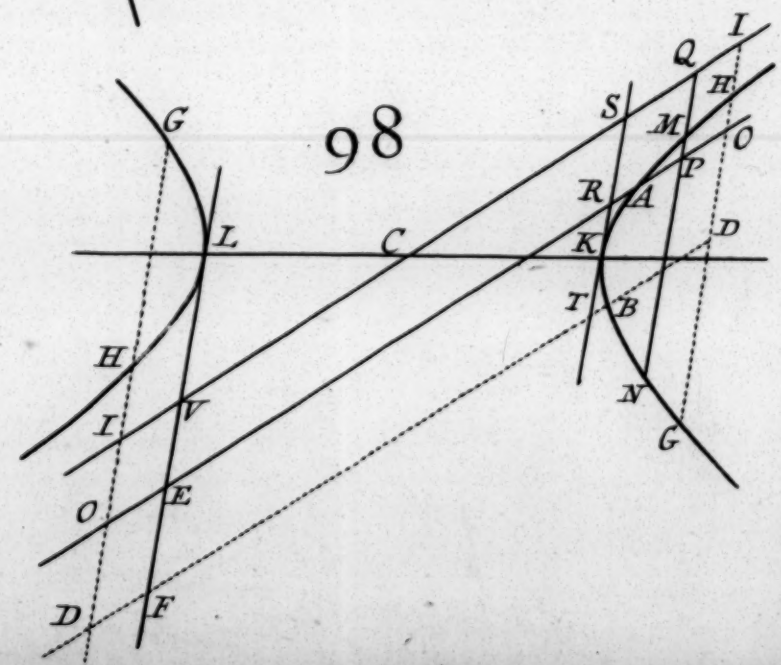
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Ordinate $MP$ to the Diameter $AB$, meeting the oppoſite Sides $FK$, $GH$, in the Points $R$, $Q$, and the Section (which I ſuppoſe to be deſcrib'd) in the Point $N$; then it is evident that $PN = PM$, and ſo $RN = QM$, ſince $PR = PQ$. Therefore the Rectangle $RM \times MQ$ will be equal to the Rectangle $RM \times RN$. But * $FR \times RK : MR \times RN :: RM \times MQ :: \overline{AB}^2 : \overline{DE}^2$. And conſequently the Ratio of the Diameter $AB$, (which is parallel to the Sides $FK$, $GH$,) to the Conjugate $DE$ is given; becauſe the Rectangles $FR \times RK$, $RM \times MQ$, are given. Farther, the Section will be an Ellipſis, when that of the two Ordinates ($MP$, $KO$) to the Diameter ($AB$) falling on the ſame Side the Centre $C$, which is nigheſt the Centre, is greater than that more diſtant; and contrariwiſe, the oppoſite Sections, when that Ordinate is leſſer. Whence it appears, that this Queſtion is brought to that of the laſt Problem. * *Art.* 164.

If the given Point $M$ ſhould fall in one of the Sides of the given Parallelogram, continued out at Pleaſure; then the Problem would be impoſſible; becauſe that Side would meet the Section in three Points, which * cannot be. * *Art.* 149.

### COROLLARY. III.

179. HENCE ariſes a Method of deſcribing a Conick Section ſuch, that the right Line $AB$, given in Poſition, may be one of its Diameters, the given Point $C$ the Centre thereof, and the right Lines $MP$, $KO$, Ordinates to that Diameter.

In the Diameter $AB$, aſſume $CL$, equal to $CO$, and draw $LF$ parallel and equal to $OK$; then it is evident, * that $LF$ will be an Ordinate to the Diameter $AB$, and ſo producing $KO$ to $H$, and $FL$ to $G$, ſo that $OH$ be $= OK$, and $LG = LF$; the equal and parallel right Lines $KH$, $FG$, will be * double Ordinates to the Diameter $AB$. Whence it appears, that the Section muſt be deſcrib'd about the Parallelogram $FGHK$, and muſt paſs through the given Point $M$. And this may be done by the laſt Corollary. * *Art.* 45, 55, 85, *and* 118. * *Art.* 144.

Becauſe this Queſtion is brought to that of the laſt Corollary, which is brought to the Problem in *Coroll.* 1. and ſince this Problem has but one Anſwer, therefore there can be but one Anſwer that will fulfil the Conditions of this Corollary.

## PROPOSITION XV.

### Problem.

180. *TO deſcribe a Conick Section, which ſhall paſs thro' five given Points* F, M, K, G, N; *and to demonſtrate that there can be but one Section that will anſwer the Problem.* FIG. 102, 103.

Join four of the given Points by the two right Lines $FG$, $MN$, meeting one another in the Point $R$; and through the fifth given Point $K$, draw two right Lines $KD$, $KH$, parallel to $FG$, $MN$, meeting them in the Points $E$, $\mathcal{Q}$; then in thoſe Parallels produc'd, (if neceſſary) aſſume the Points $D$, $H$ ſuch, that $MR \times RN : GR \times RF :: ME \times EN : KE \times ED$; and $FR \times RG : MR \times RN :: F\mathcal{Q} \times \mathcal{Q}G : H\mathcal{Q} \times \mathcal{Q}K$. (Obſerving that the Points $K$, $D$, or $K$, $H$, muſt fall on each Side the Point of Concurrence $E$ or $\mathcal{Q}$, when the Points $M$, $N$, or $F$, $G$, likewiſe fall on each Side that ſaid Point, and contrariwiſe.) This being done, draw the right Lines $LI$, $AB$, through the middle Points of the Parallels $DK$, $FG$, and $MN$, $KH$, meeting one another in the Point $C$; then if a Conick Section be deſcrib'd *, with the Line $AB$ given in Poſition as a Diameter, the given Point $C$ as the Centre, and the two right Lines $MP$, $KO$, as Ordinates to that Diameter: I ſay, that Section only will ſatisfy the Conditions of the Problem.

*Art. 179.

*Art. 166. For the two Points $D$, $H$, will be * in the Section paſſing through the five given Points $F$, $M$, $K$, $G$, $N$; and ſo the Lines $LI$, $AB$, will be * Diameters, whoſe Interſection ſhall determine the Centre $C$. Therefore it is manifeſt, that the Line $AB$ given in Poſition, will be a Diameter of the Conick Section paſſing through the five given Points, the Point $C$ the Centre, and the Lines $MP$, $KO$, Ordinates to the Diameter $AB$. And becauſe there is but one Section only that can fulfil theſe Conditions, it is manifeſt, that the ſame will be that requir'd, and there is no other but that.

*Art. 146, and 147.

If it happens that the Diameters $AB$, $LI$, be parallel to one another; then the Section will be * a Parabola, which may be deſcrib'd by *Art.* 170.

*Art. 147.

*The End of the Fourth Book.*

BOOK

# BOOK V.

## *Of the Compariſon of the Conick Sections, and their Segments, with each other.*

### LEMMA I.

181. *IF the Difference of two Quantities does continually diminiſh, ſo that at laſt it becomes leſs than any given Quantity; then will thoſe two Quantities at laſt be equal.*

For if the two Quantities at laſt be not equal, we may aſſign ſome Difference between them; which is contrary to the Hypotheſis.

### LEMMA II.

182. *IF the Ratio of two Quantities be ſuch, that the Antecedent remaining always the ſame, while their Difference and the Conſequent diminiſhing continually, do become leſs than any given Quantity; then will thoſe two Quantities at laſt be equal.*

For the Antecedent will be equal to its Conſequent by the laſt Lemma; and ſo the Quantities, whoſe Ratio they expreſs, will be equal.

### LEMMA III.

183. *IF an Arc* (MN) *of any Curve Line* ABG *be ſuppos'd to be infinitely ſmall, that is, leſs than any given Quantity; and if thro' the Extremities of that Arc there be ſuppos'd to be drawn the Ordinates* MP, NQ, *to the Axis or Diameter* AC, *together with the Parallels* MR, NS *to that Diameter: I ſay, each of the Parallelograms* PQRM, PQNS, *may be taken for the Space* PQNM, *contained between the Ordinates* PM, QN, *the very ſmall right Line* PQ, *and the very ſmall Curve* MN. Fig. 104.

All the Points of any Curve do either continually recede from its Diameter, or continually approach thereto; or elſe that Curve Line is compoſed of ſeveral Parts; ſome of which recede more and more from, and others approach nearer and nearer to the Diameter.

For it is evident, that there is no Part of a Curve Line that can have all its Points equally diſtant from the Diameter; becauſe that Part then would not be curv'd, but a ſtraight Line parallel to that Diameter.

Let us ſuppoſe, 1. That the Arc $MN$ be taken in the Curve $AMB$, all of whoſe Points recede more and more from the Diameter $AC$; then if the Arc $MO$ be taken of a finite Magnitude from $M$ beyond the Point $N$, and the Ordinate $OF$ be drawn parallel to $MP$, as likewiſe the right Lines $OD$, $ME$, parallel to the Diameter $AC$; it is manifeſt, that the Curve-lin'd Space $PFOM$, will be greater than the inſcrib'd Parallelogram $PFEM$, and leſs than the circumſcrib'd Parallelogram $PFOD$. Now if the Point $O$ be ſuppos'd to move along the Curve towards $M$, it is plain, that the Parallelogram $MEOD$, which is the Difference of the circumſcrib'd and inſcrib'd Parallelograms to the Arc $OM$, will diminiſh continually, till at laſt it will become nothing, at the Inſtant the Point $O$ comes to $M$: Therefore, when the Point $O$ is come to $N$, that is, infinitely near to $M$, then the Parallelogram $MEOD$, which becomes $MRNS$, will be leſs than any given Magnitude; and ſo, by *Lemma* 1. the Parallelograms $PQRM$, $PQNS$, will then become equal to one another; and conſequently each of them equal likewiſe to the curvilineal Space $PQNM$. Therefore, *&c.*

2. Suppoſe the very ſmall Arc $MN$, to be taken in the Curve $BMG$, all the Points of which approach nearer and nearer to the Points of the Diameter $CG$; then it is manifeſt, that the Demonſtration in this Caſe will be the ſame as in the laſt, only here the circumſcrib'd Parallelogram $PQNS$, becomes an inſcrib'd one.

3. Let the Curve $ABG$ be compos'd of ſeveral Portions, ſome of which, as $AB$, recede more and more from the Diameter $AG$; and contrariwiſe others, as $BG$, accede nearer and nearer to it; then, I ſay, the Points, as $B$, that ſeparate thoſe Parts, cannot fall in the Arcs $MN$; for if this were poſſible, the Point $B$ would be nearer to $M$ than the Point $N$ is; which is contrary to the Suppoſition; and ſo it is manifeſt, that this laſt Caſe is contain'd in either the firſt or ſecond Caſe.

## COROLLARY I.

184. HENCE, if any Ordinate $CB$ be drawn at pleaſure parallel to $PM$; and if the Part of the Curve $AB$, be ſuppos'd to be divided into an infinite Number of infinitely ſmall Arcs, as $MN$; then the Space $ACB$, contain'd under the right Lines $AC$, $CB$, and the Part of the Curve $AB$, will be equal to the Sum of all the Parallelograms, as $PQRM$, or $PQNS$. It is evident farther, that the Space

Space *MPCB*, contained under the Right Lines *MP*, *PC*, *CB*, and the Part of the Curve *MB* will be equal to the Sum of all the Parallelograms (*PQRN*) that can be form'd in that Space; and the ſame is to be underſtood of the whole Curve *ABG*.

### Corollary II.

185. IF there be any Figure *CMDOC*, included between two Parallels *CE*, *DF*; and if any where between thoſe Parallels there be ſuppos'd two right Lines *MO*, *NL*, infinitely near to one another, and likewiſe parallel to *CE*, *DF*; then, I ſay, the Space or Part (*OMNL*) of the Figure *CMDOC* included between thoſe Parallels, will be equal to the Rectangle under one of them, as *MO*, and their Diſtance *MR*, or *OS*. For if the Perpendicular *AB* be drawn to the Parallels *CE*, *DF*, meeting the Parallels *MO*, *NL*, in the Points *P*, *Q*; then it is manifeſt, by the Lemma *, that the Space *PMNQ* is equal to the Rectangle *PMRQ*, and the Space *POLQ* to the Rectangle *POSQ*; and conſequently the Space *OMNL* is equal to the Rectangle *OMRS*, or *OM*×*PQ*. Fig. 105. *Art. 183.

### Corollary. III.

186. IT is manifeſt by the laſt Corollary, that if there be any two Figures *CMDOC*, *EGFHE*, included between two Parallels *CE*, *DF*, being ſuch, that if a right Line *MH* be any where drawn between *CE*, *DF*, parallel to them; the Parts *MO*, *GH*, of that Line included by in Figures *CMDOC*, *EGFHE*, be in a given Ratio; then, I ſay, the two Figures, that is, the Spaces contain'd between them are alſo to one another in a given Ratio. For ſuppoſing another Parallel *NK* infinitely near to *MH*, and drawing a Perpendicular *AB* to the Parallels *CE*, *DF*, meeting the Parallels *MH*, *NK*, in the Points *P*, *Q*; then it is evident, (by the laſt Corollary) that the Space *OMNL* is equal to the Rectangle *OM*×*PQ*. And moreover, the Space *GHKI* equal to the Rectangle *GH*×*PQ*. Therefore thoſe two Spaces will be to one another, as *MO* to *GH*; and becauſe this happens always, let the Line *MH* be drawn in any Place: It follows, that the Sum of all the ſmall Spaces *MNLO*, that is, the Space *CMDOC* will be to the Sum of all the ſmall Spaces *GHIK*, that is, to the Space *EGFHE*, in a given Ratio.

After the ſame manner we prove, that the Part (*MDO*) of the Figure *CMDOC*, is to the correſpondent Part (*GFH*) of the other Figure *EGFHE*, in a given Ratio; as alſo the remaining Parts *CMO*, *EGH*.

Hence, if the given Ratio be a Ratio of Equality, that is, if the Parts (*MO*, *GH*) of the right Line *MH*, are always equal to one another; then the Spaces *CMDOC*, *EGFHE*, and their correſpondent Parts *MDO*, *GFH*, aad *CMO*, *EGH*, will be equal to one another.

## LEMMA IV.

FIG. 106. 187. *IF* MN *be an infinitely ſmall Arc of any Curve Line; and if* MT, NT, *be two Tangents meeting one another in the Point* T, *and if the Line* MN *be a Chord, and the right Line* NS *be drawn perpendicular to the Tangent* MT *produced; then, I ſay, the Arc* MN *may be taken for its Chord* MN, *or for the Sum of the two Tangents* MT, NT; *or elſe for the right Line* MS.

Every Cure is concave neceſſarily one way, or made up of many Parts, whereof ſome are concave one way, and the others concave the contrary way. Now the Points ſeparating thoſe Parts cannot be *Art. 183. * had in infinitely ſmall Arcs, as *MN*; becauſe then they would be nearer to the Point *M* than *N* is; which is contrary to the Suppoſition. Therefore we can ſuppoſe always, that the Arc *MN* is a Part of a Curve, that is concave one way.

Now if the Arc *MO* be taken in the Curve having a finite Magnitude; and if there be drawn the Chord *OM*, the Tangent *OG*, and the right Line *OD* parallel to *NS*; then it is manifeſt, 1. That the Tangent *MD* is leſs than the Chord *MO*, becauſe the Triangle *MDO* is right-angled at *D*, and conſequently the ſame is leſſer ſtill than the Arc *MNO*; wherefore the Arc *MNO*, and its Chord *MO*, are each greater than *MD*, and each of them leſs than the Sum of the two Tangents *MG*, *OG*. 2. Becauſe the Arc *MNO* is concave but one way, therefore if a Tangent *TR* be drawn through any Point *N* in the Arc *MO*; the Points T, *R*, wherein that Tangent meets the Tangents *MG*, *OG*, will be ſituate between the Points *M*, *G*; and *O*, *G*; and ſo the Angle *OGD*, is greater than the Angle *RTG*, or *NTS*.

This being ſuppos'd, if the right Lines *ME*, *MF*, be drawn parallel to *OG*, *NT*, meeting the right Line *DO* in the Points *E*, *F*; and if the Point *O* be ſuppos'd to move along the Curve towards the Point *M*; then it is viſible, that the Angle *OGD*, or the Angle *EMD*, will diminiſh continually, until it vaniſhes at the ſame time as the Point *O* comes to *M*; becauſe then the Tangent *OG* will coincide with the Tangent *MD*; and therefore the Line *ME* diminiſhes continually, until it becomes at laſt equal to *MD*. Therefore, when the Point *O* is come to *N*, that is, when the ſame is infinitely near to the Point *M*; the Line *ME*, being then in *MF*, will differ from the Tangent *MD*, only by a

Magnitude leſs than any given one. And ſo the Lines *TN*, *TS*, will be equal to one another. Therefore the two Tangents *MT*, *TN*, taken together, will be then equal to the Right Line *MS*, or to the Arc *MN*, or elſe to the Chord *MN*. *W. W. D.* * *Art.* 182.

## Corollary I.

188. BEcauſe the Angle *FMD*, or *NTS*, equal to it, is infinitely ſmall, in the Suppoſition of the Point *N* being infinitely near to the Point *M*; therefore the Internal Angle *NMT*, of the Triangle *MTN*, being leſs than the External Angle *NTS*, will be alſo infinitely ſmall, that is, leſs than any given Angle: And ſo there can no Right Line be drawn through the Point *M*, in the Angle *TMN*. Whence it appears that the two Right Lines *MT*, *MN*, do coincide; and ſo any Tangent may be look'd upon as a Right Line paſſing thro' two Points of a Curve Line infinitely near to each other.

## Corollary II.

189. IF any Curve Line be ſuppoſed to be divided into an infinite Number of infinitely ſmall Arcs as *MN*: Then it is evident, that if the Chords of thoſe Arcs be taken for the Arcs themſelves, there will ariſe a Polygon of an infinite Number of Sides, each Side being infinitely ſmall, which may be taken for that Curve Line, as no wiſe differing there-from. * And farther, if the ſmall Sides of that Polygon be produced both ways, they will be Tangents to that Curve; ſince every of them do paſs through two Points infinitely near to each other. * *Art.* 187.

## Scholium.

190. IT muſt here be obſerv'd, that what I have ſaid with regard to the Tangents to the Conick Sections, does not extend to any other Curves but thoſe that are Concave the ſame way, as are the Conick Sections: * Whereas that Definition of Tangents laid down above, takes in the Tangents to all Curves in general, and is the Foundation of the Method for drawing Tangents, explain'd in my Treatiſe, *Des Infiniment petits*, which I dare affirm is as ſimple and general as can be deſired. A ſhort Specimen of which may be ſeen at the End of this Book. * *Art.* 26, 61, 124.

## DEFINITIONS.

I.

FIG. 107, 108, 109. Two Segments of any two Curve Lines *B A d*, *b a d*, are called *similar*, if when any Right-lin'd Figure *B M N O D*, being inſcribed within one of them, we can inſcribe always a ſimilar Right-lin'd Figure *b m n o d*, in the other.

2.

Two Conick Sections are ſaid to be *ſimilar*, when any Segment *B A D*, being taken in the one, we can aſſign always a ſimilar Segment *b a d* in the other.

3.

The Diameters *A P*, *a p*, in two Conick Sections, are ſaid to be *ſimilar*, when they make the ſame Angles *A P M*, *a p m*, with their Ordinates *P M*, *p m*.

### COROLLARY.

191. HENCE it appears that the leſſer the Sides *B M*, *M N*, &c. *b m*, *m n*, &c. are, the greater is their Number, and the nearer do the ſimilar Right-lin'd Figures *B M N O D*, *b m n o d*, approach to the Segments *B A D*, *b a d*, in which they are inſcrib'd; ſo that at laſt the ſaid ſimilar Right-lin'd Figures will become equal to thoſe Segments, *viz.* when each * of the Sides is infinitely ſmall, and their Number conſequently infinite. Therefore the ſimilar Segments *B A D*, *b a d*, are to one another as the Squares of their Chords, *B D*, *b d*, being Homologous Sides; and the Parts of the Curves *B A D*, *b a d*, as theſe Chords.

*Art. 189.

## PROPOSITION I.

### Theorem.

FIG. 107. 192. *IF there be two Parabolas* A M, a m, *having two ſimilar Diameters* A L, a L, *ſituate in the ſame ſtraight Line: So that the Ordinates* P M, p m, *be parallel to one another,; and if the fix'd Point* L *be aſſum'd in that Line within the Parabola, ſo that* L A *be to* L a, *as the Parameter* (A G) *of the Diameter* (A L) *of the Parabola* A M, *to the Parameter* (a g) *of the Diameter* (a L) *of the Parabola* a m: *Then I ſay, if a Right Line* L M, *be drawn from the fix'd Point* L *to any Point* M *of the Parabola* A M; *that Line will meet the other Parabola* a m, *in one Point* m, *being ſuch that* L M : L m : : L A : L a.

Draw the Ordinate $MP$, and call the given Quantities $LA, a$; $La, b$; $AG, p$; and the indeterminate Quantities $AP, x$; $PM, y$: then we ſhall have this Proportion, *viz.* $LA\ (a) : La\ (b) :: AG\ (p) : ag = \frac{bp}{a}$. But if $ap$ be taken in the Diameter $(aL)$ of the Parabola $am$ equal to $\frac{bx}{a}$, and the Ordinate $pm$ be drawn; then it is plain, * that $\overline{pm}^2 = pa \times ag\ \left(\frac{bbpx}{aa}\right) = \frac{bbyy}{aa}$ by putting $yy$ for $px$ *; and ſo $pm$ is $= \frac{by}{a}$. Therefore $PM\ (y) : pm\ \left(\frac{by}{a}\right) :: LP\ (a - x) : Lp\ \left(b - \frac{bx}{a}\right)$. And conſequently the Line $LM$ will paſs through the Point $m$, the Extremity of the Ordinate $pm$, that is, the ſaid Line will cut the Parabola $am$ in that Point: And ſo becauſe the Triangles $LPM$, $Lpm$, are ſimilar, therefore $LM : Lm :: PM\ (y) : pm\ \left(\frac{by}{a}\right) :: LA\ (a) : La\ (b)$. *W. W. D.*

* *Art.* 6, *and* 20.

* Ibid.

## Corollary. I.

193. If any Segment $BAD$, be taken in the Parabola $AM$; and if the right Lines $LB$, $LD$, be drawn, meeting the other Parabola $am$ in the Points $b$, $d$, and the Chord $bd$ be drawn; then, I ſay, the Segment $(bad)$ of the Parabola $am$, is ſimilar to the Segment $BAD$ of the Parabola $AM$. For if any right-lin'd Figure $BMNOD$ be inſcrib'd in the Segment $BAD$; then, by drawing the right Lines $LM, LN$, $LO$, meeting the other Parabola in the Points $m, n, o$; it is manifeſt, that the Triangles $LBM$, $Lbm$; $LMN$, $Lmn$; $LNO$, $Lno$; $LOD$, $Lod$; $LBD$, $Lbd$, will be ſimilar: and ſo the Sides $BM$, $bm$; $MN$, $mn$; $NO$, $no$; $OD$, $od$; $BD$, $bd$, will be parallel, and always in the ſame Ratio, each to its Correſpondent; becauſe all the right Lines $LB$, $LM$, $LN$, $LO$, $LD$, are divided in the ſame Ratio in the Points $b$, $m$, $n$, $o$, $d$: therefore the right-lin'd Figures $BMNOD$, $bmnod$, are ſimilar. And ſince it is manifeſt, that this Demonſtration is the ſame, let the right-lin'd Figure inſcrib'd in the Segment $BAD$ be what it will; therefore the Segments $BAD$, $bad$, are * ſimilar; and ſo conſequently are * the Parabola's $Am$, $am$, alſo.

* *Def.* 1.

* *Def.* 2.

### COROLLARY II.

194. HENCE, if a double Ordinate $EF$ be drawn in the Parabola $AM$, meeting the other Parabola $am$ in the Points $e, f$; then the Segments ($EAF$, $eaf$,) of the Parabola's $AM$, $am$, will be ſimilar.

### COROLLARY III.

195. ALL Parabola's are ſimilar; for if $AL$, $aL$ be taken in the Diameters of two different Parabola's, being to one another as the Parameters $AG$, $ag$; and if the Diameter $La$ be ſuppos'd to be plac'd in the Diameter $LA$, ſo that the Points $L$, $L$, coincide with one another, and the Ordinates $PM$, $pm$, be parallel; then, by drawing a right Line $LM$, from the fixed Point $L$ to any Point $M$ of the Parabola $AM$, it is manifeſt, that this Line will meet the other Parabola $am$ in one Point $m$, being ſuch, that $LM : Lm :: LA : La$. Therefore, * *&c.*

* *Art.* 193.

### COROLLARY IV.

196. HENCE, if $AL$, $aL$, be taken in the Diameters of two different Parabola's, being to one another as the Parameters of thoſe Diameters; and if the double Ordinates $EF$, $ef$, be drawn through the Points $L, L$; then the Segments ($EAF$, $eaf$,) of the two Parabola's $AM$, $am$, will be ſimilar.

### COROLLARY V.

197. IF two Segments $BAD$, $bad$, are ſimilar, and one of them, as $BAD$, be the Segment of ſome one Parabola; then, I ſay, the other Segment $bad$ will be a Segment of ſome other Parabola; and ſo among all the Curves poſſible, there can be none but Parabola's that can be ſimilar to a given Parabola. For if the ſmall Segment $bad$ be ſo placed within the great one $BAD$, that the Chords $bd$, $BD$, be parallel; and if any two ſimilar right-lin'd Figures $BMNOD$, $bmnod$, be inſcrib'd in thoſe Segments; then it is plain, that the homologous Sides $BM$, $bm$; $MN$, $mn$, &c. of the ſaid two Figures will be parallel, becauſe the Angles $DBM$, $dbm$; $BMN$, $bmn$, &c. are equal. And drawing $LM$, $LN$, $LO$, thro' $L$ the Point of Concurrence of the right Lines $Bb$, $Dd$, which join the Ends of the parallel Chords $BD$, $bd$, (being the two homologous Sides given); then the ſaid Lines $LM$, $LN$, $LO$, will paſs through the

the correſpondent Points *m*, *n*, *o*, wherein they will be divided in the ſame Ratio as *L B* is in *b*, or *L D* in *d*; becauſe *B D* : *b d* :: *L B* : *L b* :: *B M* : *b m* :: *L M* : *L m* :: *M N* : *M n* :: *L N* : *L n* :: *N O* : *n o* :: *L O* : *L o* :: *O D* : *o d*.

Now if the Diameter (*L A*) of the Parabola *A M* be drawn thro' the Point *L*; and if the ſame Parabola *a m* be divided in *a*, in the ſame Ratio as *L B* is in *m b*, or *L D* in *d*; and if you deſcribe * the Parabola *a m* with the Diameter *a L*, and the Parameter *a g*, (being to the Parameter *A G* of the Diameter (*A L*) of the Parabola *A M*, as *L a* is to *L A*,) whoſe Ordinates *p m* are parallel to the Ordinates *P M* of the other Parabola; then it is * manifeſt, that the ſaid Parabola will paſs through all the Points *b*, *m*, *n*, *o*, *d*, dividing all the right Lines *L B*, *L M*, *L N*, *L O*, *L D*, in the given Ratio of *B D* to *b d*. And becauſe the ſame Reaſon holds, let the Number of Sides of the ſimilar right-lin'd Figures *B M N O D*, *b m n o d*, as likewiſe their Magnitude, be what they will; therefore the Parabola *a m* paſſes thro' every Point, as the Segment *b a d* does; and ſo that Segment will be a Part thereof. *W. W. D.*

* *Art.* 161.

* *Art.* 192.

## PROPOSITION II.

### Theorem.

198. *IF there be an Ellipſis, or Hyperbola* A M, *one of whoſe firſt Diameters is the Line* A H, *and the Line* A G *its Parameter*; *and if the fix'd Point* L *be taken in that Diameter (produced in the Hyperbola) as alſo the Points* a, h, *ſuch that* L A : L H :: L a : L h. *And again, if there be another Ellipſis or Hyperbola, whoſe firſt Diameter is the Line* a h, *and Parameter to that Diameter the Line* a g, *which is to* A G, *as* a h *to* A H; *and if the Ordinates* p m *to the ſame be parallel to the Ordinates* P M *of the other Section* A M: *Then if any right Line* L M, *be drawn from the fixed Point* L *to any Point* M *in the Section* A M, *I ſay, that Line will meet the other Section* a m *in the Point* m, *ſo that* L M : L m :: L A : L a: *that is, all right Lines drawn from the fixed Point* L *to the Section* A M, *will be divided in the ſame Ratio by the Section* a m. Fig. 108, 109.

*We are to prove, that* L M : L m :: L A : L a.

Draw the Ordinate *M P*, and call the given Quantities *L A*, *a*; *L a*, *b*; *A H*, 2 *t*; and the indeterminate Quantities *A P*, *x*; *P M*, *y*: then we ſhall have this Proportion, *L A* (*a*) : *L a* (*b*) :: *L H* : *L h* :: *L H* $\pm$ *L A*, or *A H* (2 *t*) : *L h* $\pm$ *L a*, or $ab = \frac{2bt}{a}$. Now if *a p* be taken in the Diameter (*a h*) of the Section *a m* equal to $\frac{bx}{a}$, and the Ordinate

dinate $p\,m$ be drawn; then it is * manifeſt, that $AP \times PH\ (2tx \mp xx) : \overline{PM}^2\ (yy) :: AH : AG :: ab : ag :: ap \times ph \left(\frac{2bbtx \mp bbxx}{aa}\right) : \overline{pm}^2 = \frac{bbyy}{aa}$, and ſo $pm = \frac{by}{a}$. Therefore $PM\ (y) : pm \left(\frac{by}{a}\right) :: LP\ (a - x) : Lp\ \left(b - \frac{bx}{a}\right)$. And conſequently the Line $LM$ will paſs through $m$ the Extremity of the Ordinate $p\,m$, that is, the ſaid Line will cut the Section $am$ in that Point. Therefore, becauſe the Triangles $LPM$ $Lpm$, are ſimilar, $LM : Lm :: PM\ (y) : pm \left(\frac{by}{a}\right) :: LA\ (a) : La\ (b)$. *W. W. D.*

* *Art.* 42, 55, 81, *and* 118.

## COROLLARY I.

199. IF $BAD$ be any Segment of the Section $AM$, and the right Lines $LB$, $LD$, be drawn meeting the other Section $am$ in the Points $b, d$; and if the Chord $bd$ be drawn likewiſe; then, I ſay, the Segment $(bad)$ of the Section $am$, is ſimilar to the Segment $(BAD)$ of the Section $AM$; and therefore, if a double Ordinate $EF$ be drawn in the Section $AM$, thro' the Point $L$, meeting the other Section in the Points $e, f$; the Segments $EAF$, $eaf$, of two Ellipſes or Hyperbola's $AM$, $am$, will be ſimilar. This may be demonſtrated after the ſame manner, as is done for the Parabola in *Art.* 193, and 194.

## COROLLARY II.

200. ALL Ellipſes or Hyperbola's $AM$, $am$, having two ſimilar Diameters $AH$, $ah$, in the ſame Ratio with their Parameters $AG$, $ag$, are ſimilar to one another. For if $AL$, $aL$, be taken in the ſame Ratio as the Diameters $AH$, $ah$; and if the Diameter $ah$ be ſuppos'd to be laid in $AH$, ſo that the Points $L, L$ coincide; and the Ordinates $pm$, $PM$ be parallel to one another; then the right Line $LM$ being drawn from the fix'd Point $L$ to any Point in one Section $AM$, it is evident, that the ſaid Line will always meet the other Section $am$ in the Point $m$, ſo that $LM : Lm :: LA : La$. Therefore, * *&c.*

* *Art.* 199.

## COROLLARY III.

201. HENCE, if there be two Ellipſes, or Hyperbola's $AM$, $am$, whoſe two ſimilar Diameters $AH$, $ah$, have the ſame Ratio as their Parameters $AG$, $ag$; and if $AL$, $aL$, be taken in the ſame Ratio

Ratio as the Diameters $A\,h$, $a\,h$, and the double Ordinates $E\,F$, $e\,f$, be drawn thro' the Points $L$, $L$; then it is manifeſt, that the Segments $E\,A\,F$, $e\,a\,f$, of the two Sections $A\,M$, $a\,m$, are ſimilar.

## COROLLARY IV.

202. IF two Segments $B\,A\,D$, $b\,a\,d$, be ſimilar; and one of them be the Segment of an Ellipſis or Hyperbola $A\,M$, any one of whoſe Diameters is the Line $A\,H$, and $A\,G$ the Parameter thereof; then, I ſay, the other Segment $b\,a\,d$ will be that of ſome other Ellipſis or Hyperbola $a\,m$, having the Line $a\,h$ ſimilar to $A\,H$, for one of the Diameters thereof, and the Ratio of this Diameter to its Parameter $a\,g$, the ſame as that of the Diameter $A\,H$ to its Parameter $A\,G$. For if the Segment $b\,a\,d$ be ſo plac'd within the Segment $B\,A\,D$, that the Chord $b\,d$ be parallel to the Chord $B\,D$, and the Lines $B\,b$, $D\,d$, do concur in the Point $L$ of the Diameter $A\,H$, (which is poſſible always) and if there be inſcrib'd in both of the ſaid Segments any ſimilar right-lin'd Figures; then we can prove, as in the Parabola (*Art.* 197.) that the right Lines $L\,M$, $L\,N$, $L\,O$, will paſs through the correſpondent Points $m$, $n$, $o$, and will be divided by theſe Points in the ſame Ratio as $L\,B$ is in $b$, or $L\,D$, in $d$.

Now if $L\,A$, $L\,H$, be divided in the ſame Ratio in the Points $a$, $h$, as $L\,B$ is in $b$; and if an Ellipſis, or Hyperbola $a\,m$, be deſcrib'd * with the Diameter $a\,h$, and Parameter $a\,g$, (being to the Parameter $A\,G$ of the Diameter $A\,H$, as $L\,a$ to $L\,A$, or $a\,b$ to $A\,H$) whoſe Ordinates $p\,m$, be parallel to the Ordinates ($P\,M$) of the other Ellipſis or Hyperbola $A\,M$; then it is * evident, that that Section will paſs through all the Points $b$, $m$, $n$, $o$, $d$, which divide all the right Lines $L\,B$, $L\,M$, $L\,N$, $L\,O$, $L\,D$, in the given Ratio of $b\,d$ to $B\,D$. And becauſe this Reaſoning is the ſame always, be the Number of Sides of the right-lin'd ſimilar Figures $B\,M\,N\,O\,D$, $b\,m\,n\,o\,d$, as likewiſe their Magnitude, what they will: Therefore the Ellipſis, or Hyperbola $a\,m$, paſſes through all the Points that the Segment $b\,d$ does; and ſo that Segment is a Part or Portion of the ſame.

* *Art.* 161.

* *Art.* 198.

## COROLLARY V.

203. HENCE, if there be two ſimilar Ellipſes, or Hyperbola's $A\,M$, $a\,m$; and if $A\,H$ be any Diameter of the Section $A\,M$; then we can have always a Diameter ($a\,b$) of the Section $a\,m$, which is in the ſame Ratio to its Parameter $a\,g$, as $A\,H$ to its Parameter $A\,G$; and ſo the ſimilar Diameters $A\,H$, $a\,b$, will be in the ſame Ratio as the Diameters, which are Conjugates to them: And becauſe there can be but two Pair of Conjugate Diameters * in an Ellipſis or Hyper-

* *Art.* 66, *and* 128.

Hyperbola making the ſame Angles with each other; and ſince the ſaid Diameters do only differ in Poſition, their Magnitude being the ſame; therefore in ſimilar Ellipſes or Hyperbola's, all Conjugate Diameters making the ſame Angles, will be to one another in the ſame Ratio; obſerving to take the two greater of the Conjugate Diameters for the Antecedents of the Ratio's, and the leſſer for the Conſequents.

## PROPOSITION III.

### Theorem.

FIG. 110, 111. 204. *IF any two Parallels* B D, E F, *be drawn in a Conick Section terminated by the ſame; and if their Extremities be joined by two right Lines* B E, D F; *I ſay, the Segments* B M E B, D M F D, *contain'd under Portions of the Section, and the right Lines joining the Extremities of the Parallels, will be equal to one another.*

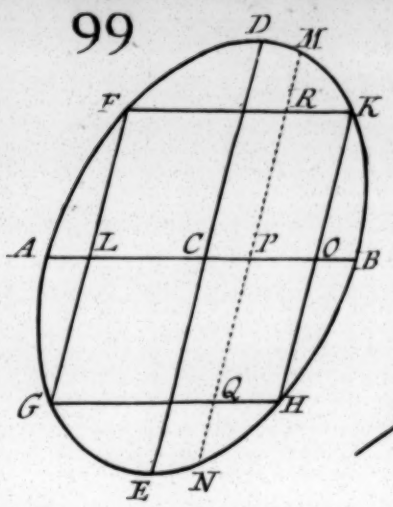
Produce the Chords *B E*, *D F*, meeting each other in the Point *G*, and draw the right Line *G H* through the ſaid Point *G*, and (*H*) the Middle of the Line *B D*; then the Line *G H* will biſect *E F* (parallel to *B D*) in the Point *K*; as alſo any other parallel (to the ſame Line *B D*) as *O O* in the Point *P*. Therefore the Line *H K*, will be
*Art. 146. a Diameter; * and the Parallels *B D*, *E F*, double Ordinates to the ſame; and ſo if a Line be drawn through any Point *P*, in that Diame-
*Art. 144. ter parallel to the Ordinates; that Line will meet * the Section in two Points *M*, *M*, equally diſtant from *P*. Whence it appears, that the Parts *M O*, *O M*, of any Parallel (*M M*) to *B D*, included within the Segments *B M E B*, *D M F D*, are always equal to one another, let that Parallel be any where drawn between the Lines *B D*, *E F*.
*Art. 186. Therefore it is manifeſt, * that thoſe two Segments ſhall be equal.

If the Chords *B E*, *D F*, be parallel; then the right Line *H K* muſt be drawn thro' *H* the Middle of *B D*, parallel to thoſe Chords, and the Demonſtration will be the ſame as before.

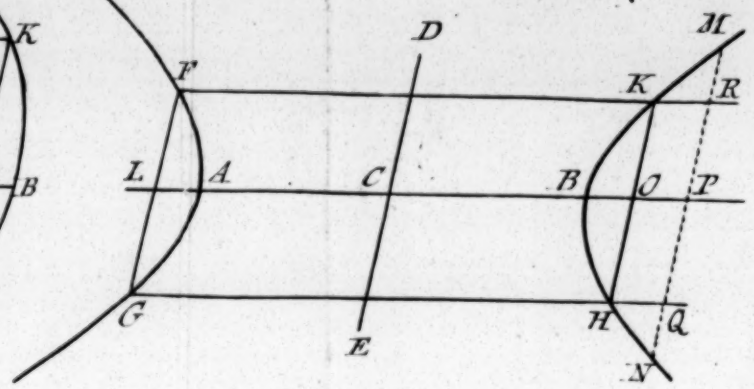
### COROLLARY I.

FIG. 110. 205. BEcauſe *P M* is equal always to *P M*, it follows, 1. That the Conick Trapezia *K H B E*, *K H D F*, are equal to one another. 2. (When the Line *B D* does not cut the Section, but touches it in the Point *A*) the Conick Trilineal Figures *A K E*, *A K F*, are equal; and ſo are likewiſe the Segments *A E M A*, *A F M A*: Becauſe the Triangle *A E F*, is divided into two equal Parts by the Diameter *A K*, paſſing through the Middle of *E F*.

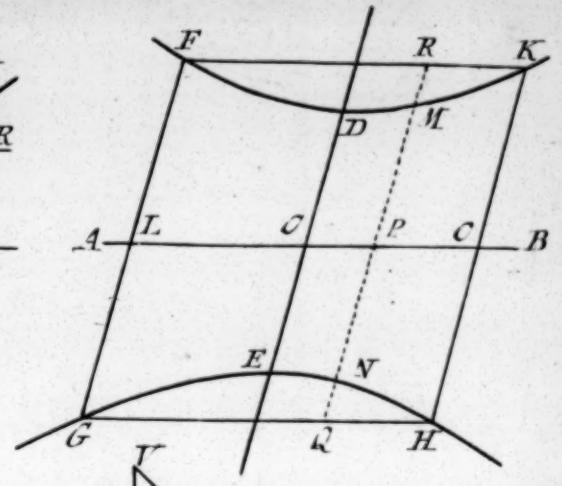
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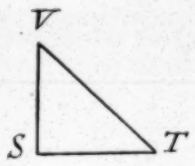


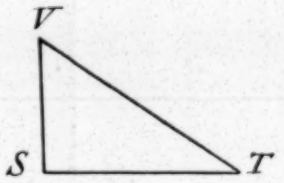

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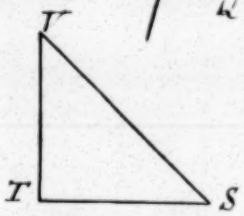


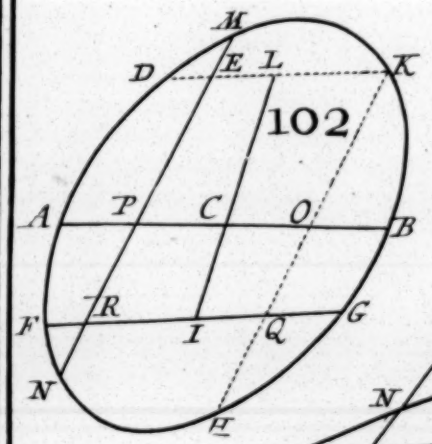

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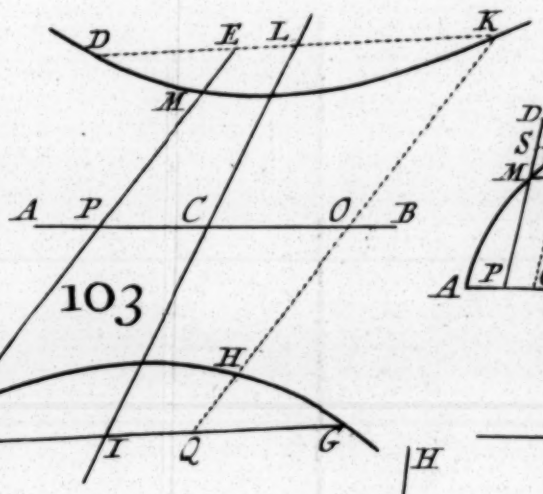



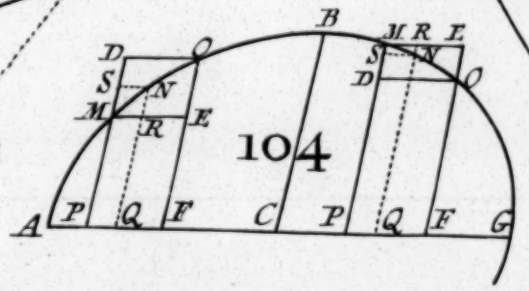



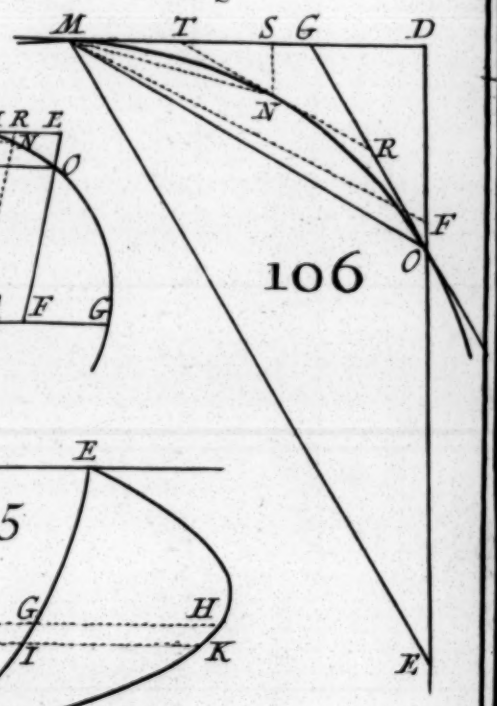

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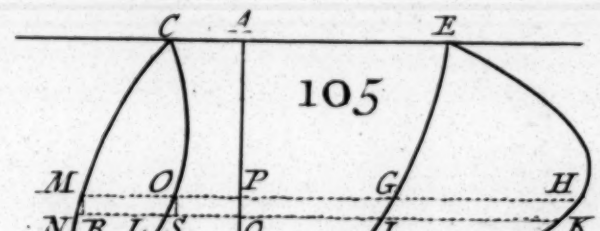

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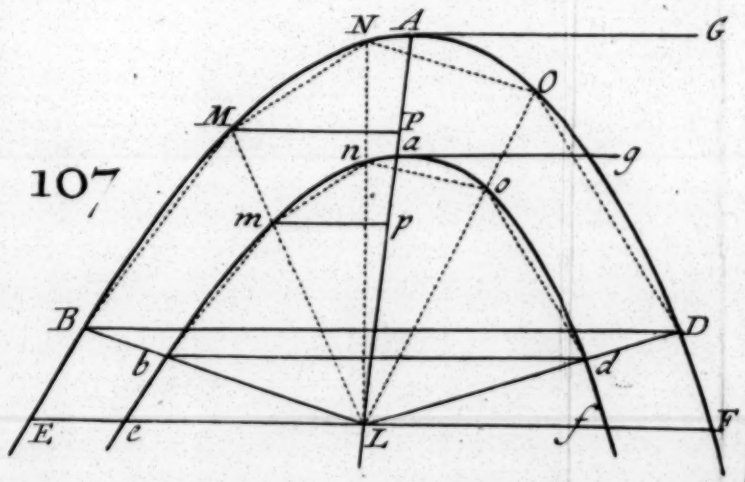

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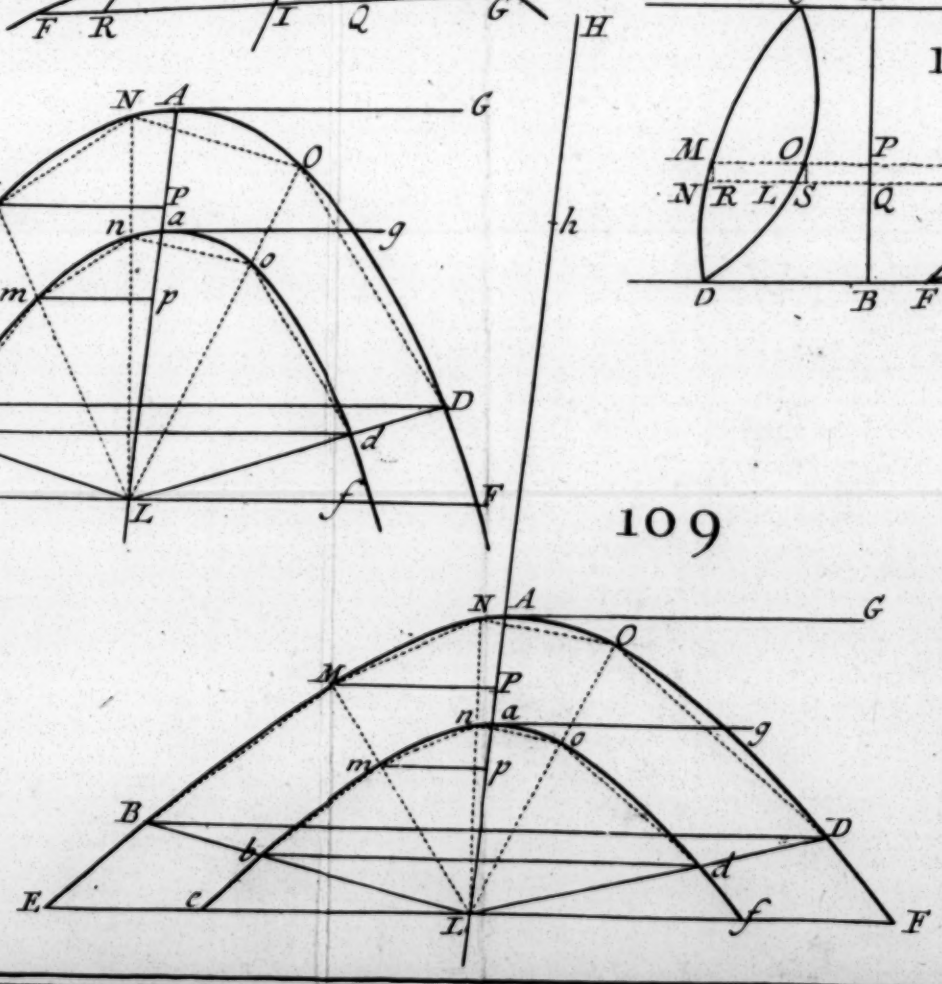

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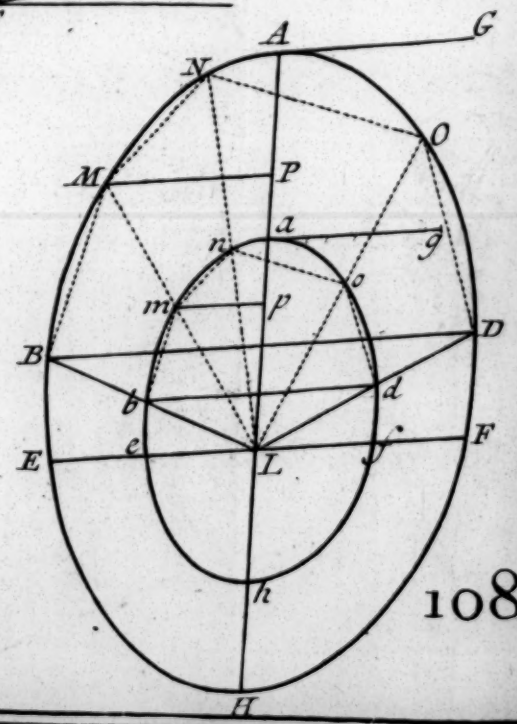

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### COROLLARY II.

206. IF the Section be a Parabola, Ellipſis, or Hyperbola; and the right Lines $BF$, $DE$, be drawn through the Extremities of the Parallels $BD$, $EF$, cutting one another between thoſe Parallels; then the Segments $BFDAB$, $DEBAD$, will be equal to one another. For the Triangles $BFD$, $BED$, being between the ſame Parallels $BD$, $EF$, and having the ſame Baſe $BD$, are equal to one another; and ſo if the Segment $DMFD$ *plus* the Segment $BADB$ be added to one Side, and the Segment $BMEB$ (which is equal to $DMFD$) *plus* the ſame Segment $BADB$, to the other; then the wholes $BFDAB$, $DEBAD$, will be equal to one another. Fig. 110.

### COROLLARY III.

207. HENCE appears the manner of drawing two right Lines $DG$, $DF$, from a Point $D$ in a Conick Section, ſuch, that they ſhall cut off from that Section two Segments $DGED$, $DFBD$, each equal to a given Segment $BEDB$. For draw the right Lines $BD$, $DE$, and draw $BG$ parallel to $DE$, and $EF$ parallel to $BD$, meeting the Section in the Points $G$, $F$; then it is manifeſt, * by joining the right Line $DF$, that the Segment $DFBD$, is equal to the Segment $BEDB$; becauſe $DB$, $EF$, are parallel. And in like manner by joining $DG$, the Segment $DGED$ is equal to the Segment $BEDB$, becauſe $BG$, $DE$, are parallel. Fig. 112. * *Art.* 206.

If the Point given happens to fall upon one End of the given Segment $DGED$; then a right Line $GF$ muſt be drawn through the other End, parallel to the Tangent paſſing through the Point $D$; which being done, if the Chord $DF$ be drawn through the Point $F$, (wherein $GF$ meets the Section,) and the given Point $D$; it is plain, that the Segment $DFBD$ will be equal to the given Segment $DGED$.

Hence, in this laſt Caſe, there can be but one Segment $DFBD$, equal to the given Segment $DGED$; becauſe any other Segment having the given Point $D$, as one of the Extremes thereof, will be greater or leſs than the Segment $DFBD$, according as the other Extreme is nearer to, or farther from $D$ than $F$ is. Therefore, if two Segments $DGED$, $DFBD$, having one common Extreme $D$, are equal between themſelves; and if a right Line be drawn through the Point $D$, parallel to the right Line $GF$, joining their other Ends; then will that Line touch the Section in the Point $D$.

COROL-

### COROLLARY IV.

208. FROM the laſt Corollary ariſes a new and very eaſy way of drawing a Tangent through a Point *D* given in a Conick Section, which is as follows.

Draw any two right Lines *D B*, *B E*, through the Point *D*, meeting the Section in the Points *B*, *E*; moreover, through the Points *B*, *E*, draw the right Lines *BG*, *EF*, parallel to *D E*, *B D*, meeting the Section in the Points *G*, *F*, which join by the Line *G F*; then if a right Line be drawn thro' the Point *D*, parallel to *G F*, that Line will touch the Section in the Point *D*; becauſe the Segments; *D G E D*, *D F B D*, being each equal to the Segment *B E D B*, will be equal to one another.

## PROPOSITION IV.

### Theorem.

Fig. 113, 114, 115. 209. *IN an Ellipſis, Hyperbola, or oppoſite Sections, if there be two right Lines* BD, EF *parallel between themſelves, and terminating in the Section; and if the Semidiameters* CB, CE, CD, CF, *be drawn from the Centre* C: *I ſay, the Hyperbolick or Elliptick Sectors* CBE, CDF, *will be equal to one another.*

For if the Diameter *CK* be drawn through *H*, *K*, the Middles of *B D*, *E F*; then the Triangles *CHB*, *CHD*, and *CKE*, *CKF*, will be equal; becauſe they have the ſame Vertex *C*, and their Baſes *HB*, *HD*, and *KE*, *KF*, are equal. Therefore *(Fig.* 114.*)* $KHBE + CBE = CKE - CHB = CKF - CHD = KHDF + CDF$; and *(Fig.* 113, 115.*)* $KHBE - CBE = \pm CHB \mp CKE = \pm CHD \mp CKF = KHDF - CDF$. And becauſe the Conick Trapezia *KHBE*, *KHDF*, are * equal; therefore the Elliptick or Hyperbolick Sectors *CBE*, *CDF*, are equal alſo.

*Art. 205.

### COROLLARY I.

Fig. 113, 114. 210. IF the Section be an Ellipſis or Hyperbola; and if the Line *BD*, being parallel to *EF*, becomes a Tangent in *A*; then it is plain, that the Sectors *CAE*, *CAF*, will be equal to one another. For producing the Semidiameter *CA*, until it meets the Line *EF* in the Point *K*, the ſaid Line *EF* will be biſected by the Point *K*, and conſequently the Triangles *CKE*, *CKF*, will be equal. And the Trilineal Conick Figures *AKE*, *AKF*, are * ſo alſo. Whence, *&c.*

*Art. 205.

### COROLLARY II.

211. HENCE, if any Hyperbolick or Elliptick Sector *CEF*, be requir'd to be divided into two equal Parts; there is no more to do, but draw the Semidiameter *CA*, biſecting the Chord (*EF*) of that Sector in the Point *K*. From whence we can prove again, that the Sectors *CBE*, *CDF*, are equal to one another, if *BD* be parallel to *EF*. For ſince by this Means, the Sectors *CAE*, *CAF*, and *CAB*, *CAD*, are equal to one another: Therefore the Sectors *CBE*, *CDF*, being the Differences between them muſt needs be equal.

## PROPOSITION V.

### Theorem.

212. *IF there be a Semicircle* ADH, *whoſe Diameter* AH *is the firſt Axis of an Ellipſis* ABH; *and if a Perpendicular be let fall from any Point* N *in the Periphery of the Circle to the Axis meeting the ſame in* P, *and the Ellipſis in the Point* M; *and laſtly, if the right Lines* CM, CN, *be drawn from the Centre* C. *I ſay, the Elliptick Sector* CAM, *is to the circular Sector* CAN, *as* CB, *the half of the ſecond Axis of the Ellipſis is to* CA *or* CD, *the half of the firſt Axis.* FIG. 116.

For $\overline{PM}^2 : \overline{CB}^2 :: AP \times PH : AC \times CH$, or $\overline{CA}^2$, by * the Property of the Ellipſis. And $\overline{PN}^2 : \overline{CD}^2 :: AP \times PH : AC \times CH$, or $\overline{CA}^2$, by the Property of the Circle; therefore $\overline{PM}^2 : \overline{CB}^2 :: \overline{PN}^2 : \overline{CD}^2$; or $\overline{PM}^2 : \overline{PN}^2 :: \overline{CB}^2 : \overline{CD}^2$. And extracting the ſquare Roots, $PM : PN :: CB : CD$ or $CA$. And ſince this is ſo always, let the Perpendicular $PMN$ fall any how; therefore * the whole Elliptick Space $ABHA$, is to the Semicircle $ADHA$, and the Part or Portion $APM$ of that Space to the Part or Portion $APN$ of the Semicircle, as $CB$ to $CD$, or $CA$. But the right-angled Triangle $CPM$, is to the right-angled Triangle $CPN$, having the ſame Altitude, as the Baſe $PM$ is to the Baſe $PN$, that is, as $CB$ is to $CD$, or $CA$; and conſequently the Elliptick Space $APM$ *plus*, or *minus* the Triangle $CPM$ (*viz. plus* when $AP$ is leſs than $AC$, and *minus* when it is greater) that is, the Elliptick Sector $CAM$ will be to the circular Space $APN$ *plus* or *minus* the Triangle $CPN$, that is, to the circular Sector $CAN$, as $CB$ is to $CD$, or $CA$. *W. W. D.*

* *Art.* 42, *and* 55.

* *Art.* 186.

### COROLLARY I.

213. BEcauſe the Sector ($CAN$) of the Circle is equal to the Rectangle under the Arc $AN$, and one half the Radius $CA$, or $CD$: Therefore the Elliptick Sector $CAM$ is equal to the Rectangle under the ſame Arc $AN$, and the one half of $CB$.

### COROLLARY II.

214. IF thro' any Point $G$, (beſides $P$) in the firſt Axis ($AH$) be drawn a Perpendicular to that Axis, meeting the Ellipſis in the Point $E$, and the Circle in the Point $F$; I ſay, the Elliptick Sectors $ACE$, $ACM$, are to one another, as the circular Sectors $ACF$, $ACN$. For $ACM : ACN :: CB : CD$. And moreover $ACE : ACF :: CB : CD$. And therefore $ACM : ACN :: ACE : ACF$; and $ACM : ACE :: ACN : ACF$. Whence, if it be requir'd to find the Elliptick Sector $ACM$, which may be to the Elliptick Sector $ACE$, in a given Ratio, you need only find the circular Sector $ACN$, that may have that given Ratio to the Sector $ACF$, or elſe (which is the ſame thing) divide the Arc $ANF$, or Angle $ACF$, into that given Ratio.

## PROPOSITION VI.

### Theorem.

FIG. 117, 118. 215. *IF there be two Hyperbola's* AM, AN, *or* BM, DN, *having the Point* C, *as a Centre common to them both, the right Line* CA, *a Semidiameter to them both, and any two right Lines* CB, CD, *ſituate in the ſame Line, ſemi-conjugate Diameters to* CA, CA; *and if through any Point* P *in the Semidiameter* CA *(produced, if neceſſary) there be drawn a right Line parallel to* CD, *meeting the Hyperbola's in the Points* M, N; *as alſo the right Lines* CM, CN, *from the Centre* C *to the ſaid Points* M, N; *I ſay, the Hyperbolick Sectors* CAM, CAN, *or* CBM, CDN, *will be to one another, as the Semiconjugate Diameters* CB, CD.

* *Art.* 81, *and* 118. By the Property * of the two Hyperbola's $AM$, $AN$, or $BM$, $DN$, we have the two following Proportions, *viz.* $\overline{PM}^2 : \overline{CB}^2 :: \overline{CP}^2 \mp \overline{CA}^2 : \overline{CA}^2 :: \overline{PN}^2 : \overline{CD}^2$. And conſequently $\overline{PM}^2 : \overline{PN}^2 :: \overline{CB}^2 : \overline{CD}^2$. And extracting the ſquare Roots, $PM : PN :: CB : CD$. And ſince we have always this Proportion, let the Parallel $PMN$ be any where drawn; therefore * the Hyperbolick Spaces $APM$, $APN$, or $CPMB$, $CPND$, are to one another, as $CB$ to $CD$. But the Triangles $CPM$, $CPN$,

* *Art.* 186.

*CPN*, are to one another, as their Baſes *PM*, *PN*, (becauſe they are ſituate between the ſame Parallels *CD*, *PN*), or as the Semiconjugate Diameters *CB*, *CD*. And conſequently (*Fig.*117.) *CB* : *CD* : : *CPM* — *APM* : *CPN* — *APN* : : *CAM* : *CAN*. Or elſe (*Fig.* 118.) *CB* : *CD* : : *CPMB* — *CPM* : *CPND* — *CPN* : : *CBM* : *CDN*. *W. W. D.*

### COROLLARY.

216. IF the two Semi-conjugate Diameters *CA*, *CD*, be equal to one another; then *AN*, or *DN*, will be an equilateral Hyperbola. And if we did know how to ſquare the Hyperbolick Sectors *CAN*, or *CDN*, then ſhould we have likewiſe the Quadrature of the Sectors *CAM*, or *CBM*, whoſe Baſes are Parts (*AM*, or *BM*) of ſome other Hyperbola, and *CD* the Conjugate Diameter thereof equal to any given Magnitude: Becauſe the Relation of the Hyperbolick Sectors *CAM*, *CAN*, or *CDN*, *CBM*, being expreſs'd by the right Lines *CD*, *CB*, is given. Therefore, if we could get the Quadrature of the equilateral Hyperbola, we ſhould have likewiſe the Quadrature of any other Hyperbola: Juſt as the Quadrature of all Ellipſes might be had by having * the Quadrature of the Circle. **Art.* 212.

## PROPOSITION VII.

### Theorem.

217. *IF the Parts* CK, CL *be aſſum'd in one Aſymptote* (CN) *of any Hyperbola* EBDF, *having the ſame Ratio, as any two other Parts* CG, CH, *of the ſame Aſymptote; and if the Lines* GF, HD, KB, LE, *be drawn parallel to the other Aſymptote* CP, *meeting the Hyperbola in the Points* F, D, B, E; *and laſtly, if the Semidiameters* CF, CD, CB, CE, *be drawn; I ſay, the two Hyperbolick Sectors* CBE, CDF, *will be equal to one another.* FIG. 119.

Draw the two right Lines *BD*, *EF*, meeting the Aſymptotes in the Points *M*, *O*, *N*, *P*; then becauſe *KB*, *HD*, and *LE*, *GF* are parallel, we have the two following Proportions, *MB* : *MK* : : *DO* : *CH*, and *NE* : *NL* : : *FP* : *CG*. And therefore, *MK* = *CH*, and *NL* = *CG*, becauſe * *MB* = *DO*, and *NE* = *FP*. But (by the Hypotheſis) *CG*, or *LN* : *CH* or *KM* : : *CK* : *CL* : : *LE* : * *KB*. and therefore *LN* : *LE* : : *KM* : *KB*. And conſequently the Lines *NE*, *MB*, that is, *EF*, *BD*, are parallel. Therefore the Hyperbolick Sectors *CBE*, *CDF*, are * equal to one another. *W. W. D.* **Art.* 95. **Art.* 100. **Art.*209.

### COROLLARY I.

218. IF the Parts $CK$, $CL$, of one Aſymptote $CN$, be in the ſame Proportion, as any two Parts $CS$, $CT$, of the other Aſymptote $CP$; and if the Lines $KB$, $LE$, and $SD$, $TF$, be drawn parallel to the ſaid Aſymptotes; then it is plain, that the Hyperbolick Sectors $CDF$, $CBE$, ſhall be equal alſo to one another. For drawing $FG$, $DH$, parallel to the Aſymptote $CP$, we have * this Proportion, *viz.* $CG : CH :: HD$, or $CS : GF$, or $CT$* $:: CK : CL$. Therefore, *&c.*

*Art. 100.
* *Hyp.*

### COROLLARY II.

219. IF $CK$ be taken in the aforeſaid Aſymptote $CN$, equal to a third Proportional to any two Parts $CG$, $CH$, of the ſame; then we can prove, after the ſame manner as in the Theorem, that the Line $BF$ is parallel to the Tangent paſſing through the Point $D$; and ſo * the Hyperbolick Sectors $CFD$, $CDB$, are equal to one another. Therefore, if any Number of Parts, $CG$, $CH$, $CK$, $CL$, &c. be taken in a continued Geometrical Progreſſion in one Aſymptote $CN$, and the right Lines $GF$, $HD$, $KB$, $LE$, &c. be drawn from them parallel to the other Aſymptote; then the Hyperbolick Sectors $CFD$, $CDB$, $CBE$, &c. are every of them equal to one another.

*Art. 210.

### COROLLARY. III.

220. HENCE, if $CH$ be the firſt of two mean Proportionals between $CG$, $CL$. And if the right Lines $GF$, $HD$, $LE$, are parallel to the Aſymptote $CP$; then the Sector $CDF$, to the Sector $CFE$, will be as 1 to 3. And if $CH$ be the firſt of three mean Proportionals between $CG$, $CL$; the Sector $CDF$, will be to the Sector $CFE$, as 1 to 4. And univerſally, if $m$ denotes any whole Number, and $CH$ be the firſt of as many mean Proportionals between $CG$, $CL$, as the Number $m-i$ contains Units; then the Sector $CDF$ will be to the Sector $CFE$, as $i$ is to the Number $m$.

### SCHOLIUM.

221. HENCE we may give a very exact Idea of thoſe Numbers in Arithmetick, call'd Logarithms; and likewiſe ſhew their great Uſe in facilitating extremely Arithmetical Operations, wherein large Numbers are concern'd.

Let $CG$ expreſs Unity, $CL$ the Number 10, and ſuppoſe the Hyperbolick Sector $CFE$ to be divided into 10000000000 equal Parts; then

then if there be a Table divided into two Columns, in the firſt of which are orderly contain'd all the natural Numbers 1, 2, 3, 4, 5, 6, *&c.* and in the other, artificial Numbers ſtanding againſt them, expreſſing the Number of Parts that the Hyperbolick Sector *CDF* does contain, with reſpect to the Number of Parts contain'd in the Sector *CEF*; then the artificial Numbers are call'd the *Logarithms* of the natural Numbers anſwering to them. This being premis'd,

1. If any two natural Numbers *CH*, *CK*, be propos'd to be multiply'd by one another, you need but take their Logarithms in the Table expreſs'd by the Sectors *CFD*, *CFB*; and then adding the two Logarithms together, you will have the Logarithm expreſſing the Sector *CFE*, againſt which in the firſt Column ſtands the natural Number *CL*, which is the Product of the Multiplication of the two Numbers *CH*, *CK*.

2. If it be propos'd to divide the Number *CL* by the Number *CK*, you need only ſubſtract the Logarithm (*CFB*) of the Diviſor *CK*, from the Logarithm (*CFE*) of the Dividend, and the Remainder *CBE*, or *CFD*, will be the Logarithm of the Quotient *CH*.

3. If it be propos'd to extract any Root of the Number *CL*, for Example, the Cubick; then you need only divide the Logarithm (*CFE*) thereof into three equal Parts, and you will have the Logarithm *CFD*, againſt which ſtands the Number *CH*, which is the Cube Root ſought.

All this follows from the Equality of the Hyperbolick Sectors *CFD*, *CBE*, and *CFD*, *CDB*, *CBE*, &c. when $CG : CH :: CK :: CL$, and $CG : CH :: CH : CK :: CK : CL ::$ &c. Therefore by ſuch a Table it is manifeſt, that Arithmetical Operations in great Numbers may be vaſtly abbreviated, and ſo the Logarithms are of great Uſe in Trigonometry and Aſtronomy, *&c.*

Becauſe the Relation of the Hyperbolick Sectors *CFD*, *CFB*, *&c.* to the Sector *CFE*, cannot be expreſs'd exactly in Numbers; therefore that Relation is expreſs'd in Numbers nearly; and by means of theſe Numbers, (call'd *artificial* ones) and the natural Numbers ſet againſt them is a Table of Logarithms compos'd, which has all the Properties we have here explain'd. Now according to the Suppoſition, that the Sector *CFE* being the Logarithm of *CL* (10) contains 10000000000 equal Parts, we ſhall find that the Parallelogram *CGFT*, contains more than 4342944818 of thoſe Parts, and leſs than 4342944819. Therefore any Hyperbolick Sector *CBF*, is to the Parallelogram *CGFT*, nearly as the Logarithm of the Number *CK* (taken to ten Places of Figures beſides the Characteriſtick) is to the Number 4342944819.

PRO-

## PROPOSITION VIII.

### Theorem.

FIG. 120. 222. *IF in each of the Aſymptotes there be taken the Parts* CG, CL, *and* CR, CS, *being ſuch that* $\sqrt[m]{CG} : \sqrt[m]{CL} :: \sqrt[n]{CR} : \sqrt[n]{CS}$; *and if the right Lines* GF, LE, RT, SV, *be drawn parallel to the Aſymptotes; I ſay, the Sector* CFE *will be to the Sector* CTV, *as* m *is to* n: *The Letters* m, n, *denoting any whole Numbers at pleaſure.*

For if you make $\sqrt[m]{CG} : \sqrt[m]{CL} :: CG : CH$, and $\sqrt[n]{CR} : \sqrt[n]{CS} :: CR : C\mathcal{Q}$. And if the right Lines $HD$, $\mathcal{Q}N$, be drawn parallel to the Aſymptotes; then it is manifeſt, * that the Hyperbolick Sectors $CFD$, $CTN$, ſhall be equal to one another; becauſe * $CG : CH :: CR : C\mathcal{Q}$. But by the Nature of Geometrical Progreſſions, the Line $CH$ will be the firſt of as many mean Proportionals between $CG$ and $CL$, as the Number $m-i$ contains Units, and the Line $C\mathcal{Q}$, the firſt of as many mean Proportionals between $CR$ and $CS$, as the Number $n-i$ contains Units. Therefore * $CFE : CFD :: m : i$, and $CTN$, or $CFD :: CTV : i : n$. And conſequently the Sector $CFE$ is to the Sector $CTV$, in the Ratio compounded of $m$ to $i$, and of $i$ to $n$, that is, as the Number $m$ is to the Number $n$. *W. W. D.*

*Art.* 218. *Hyp.* *Art.* 220.

### COROLLARY.

223. HENCE, if any Hyperbolick Sector $CFE$ be given, as alſo any Point T in the Hyperbola; and if it be requir'd to find ſome other Point $V$ in the ſaid Section, ſo that the Sector $CFE$ be to the Sector $CTV$, as $m$ is to $n$; then you muſt aſſume $CS$, ſuch that $\sqrt[m]{CG} : \sqrt[m]{CL} :: \sqrt[n]{CR} : \sqrt[n]{CS}$, or (which comes to the ſame) $\sqrt[m]{\overline{CG}^{n}} : \sqrt[m]{\overline{CL}^{n}} :: CR : CS$; that is, you muſt take $CS = CR \times \sqrt[m]{\frac{\overline{CL}^{n}}{\overline{CG}^{n}}}$.

## PROPOSITION IX.

### Theorem.

FIG. 121. 224. *IF the right Lines* BK, FG, *be drawn through the Extremities* B, F, *of any Hyperbolick Sector* CBF, *parallel to one Aſymptote* CS, *and terminating in the other* CL; *I ſay, the Hyperbolick Sector* CBF *is equal*

*equal to the Hyperbolick Space* BKGF, *contain'd between the Parallels* BK, FG, *the Part* GK *of the Aſymptote* CL, *and the Portion* BF *of the Hyperbola.*

For if the Triangle *CGA* be taken from the equal * Triangles *CKB*, *CGF*, (the Point *A* being the Interſection of the two right Lines *FG*, *CB*,) and if to the two Remainders *BKGA*, *CAF*, there be added the Hyperbolick Space *BAF*; then on one Side we ſhall have the Space *BKGF*, equal to the Sector *CBF* on the other. *W. W. D.* * *Art.* 99.

### Corollary I.

225. IF the Lines *BQ*, *FO*, be drawn parallel to the Aſymptote *CL*, and terminating in the Aſymptote *CS*; then we can prove by the like Reaſon, that *CBF* is equal to the Hyperbolick Space *BQOF*; from whence it appears, that the Spaces or Hyperbolick Trapezia *BKGF*, *BQOF*, are equal to one another.

### Corollary II.

226. HENCE, whatever has been demonſtrated in the 217th, 218th, 219th, 220th, 221ſt, 222d, and 233d Articles of Hyperbolick Sectors, extends to the aforeſaid Trapazia, becauſe theſe are equal to the Sectors.

## PROPOSITION X.

### Theorem.

227. *IF there be two Hyperbola's* BMF, HND, *having the ſame Aſymptotes* CL, CS; *and if through any two Points* G, K, *in one Aſymptote, there be drawn the right Lines* GDF, KHB, *parallel to the other*; *I ſay, the Hyperbolick Space* HKGD *to the Hyperbolick Space* BKGF, *is as the Power of the Hyperbola* HND, *to the Power of the Hyperbola* BMF. FIG. 122.

For through any Point *P* in the Part *GK*, draw a Parallel to *GD* or *KH*, meeting the Hyperbola *BMF*, in the Point *M*, and the Hyperbola *HND*, in the Point *N*; and call the Powers of the Hyperbola's *HND*, *BMF*, *aa*, *bb*; and the indeterminate Quantity *CP*, $x$; then will * *PN* be $= \frac{aa}{x}$, and $PM = \frac{bb}{x}$; and ſo $PN : PM :$ *aa* : *bb*. And becauſe this is ſo always, let the Point *P* be any where taken between *G* and *K*, therefore * the Hyperbolick Space *HKGD*, to the Hyperbolick Space *BKGF* is, as *aa* to *bb*. *W. W. D.* * *Art.* 101. * *Art.* 186.

COROL-

### COROLLARY.

228. WHEN the Powers of the Hyperbola's $HND$, $BMF$, are to one another, as the Number $m$ to the Number $n$; then in the Hyperbola $HND$ we can find always an Hyperbolick Trapezium $RSVT$, equal to the Hyperbolick Trapezium $GKBF$, in the other Hyperbola $BMF$; the right Lines $CG$, $CK$, $CR$, being given. For it is manifeſt, * that the Trapezium $GKHD$, to the Trapezium $GKBF$, is as $m$ to $n$; and ſo the whole Difficulty conſiſts in finding the Trapezium $RSVT$, in the Hyperbola $HND$, which ſhall be to the Trapezium $GKHD$, as the Number $n$ to $m$. And this may be done * in taking $CS$ ſuch, that $\sqrt[m]{CG^n} : \sqrt[m]{CK^n} :: CR : CS$.

*Art. 222, and 225.

*Art. 223, and 226.

### DEFINITIONS.

4.

Fig. 123. If there be an indefinite right Line $AC$, whoſe Origin is the fix'd Point $A$; and if there be a Curve $AMB$ ſuch, that a right Line $MP$ being drawn from any Point $P$ in the ſame, making a given Angle $APM$ with $AC$, and if, the indeterminate Quantities $AP$, $PM$, being call'd $x$, $y$, we have always $ax = yy$ (where the Letter $a$ denotes a given Line); then it is plain, * that the Curve $AMB$ is a Parabola, the right Line $AC$ a Diameter, the right Line $PM$ an Ordinate to that Diameter, and the given Line $a$ the Parameter. But now, if the Nature of the Curve $AMB$ be ſuppos'd to be expreſs'd by this Equation $y^3 = aax$, or this $y^3 = axx$; then that Curve is call'd a *Cubick Parabola*, or a *Parabola of the third Degree*. Becauſe the Power of one of the indeterminate Quantities $x$, $y$, ariſes to the third Degree. In like manner, if the Equation be $y^4 = a^3 x$, or $y^4 = ax^3$; then the Curve $AMB$, is called a *Parabola of the fourth Degree*. Becauſe one of the indeterminate Quantities, as $y$, ariſes to the fourth Degree, and ſo of others to Infinity.

* *Art.* 19.

5.

Fig. 124. If there be a right Line $AC$ (as in the laſt Definition) and the fixed Point $A$ be the Origin; and if there be a Curve Line $BM$ ſuch, that drawing the right Line $MP$ from any Point $M$ thereof, making a given Angle $APM$ with $AC$, and calling $AP$, $x$, $PM$, $y$; we have always $xy = aa$ (where the Letter $a$ denotes a Line given); then it is plain,

* that that Curve will be an Hyperbola, and the right Line $AC$ one of its Aſymptotes, the Line $AD$ (parallel to $PM$) the other, and the Square $aa$ the Power thereof. But if the Equation expreſſing the Nature of the Curve $BM$ be $xxy = a^3$; then that Curve is called *a Cubick Hyperbola*, or an *Hyperbola of the third Degree*; becauſe the Product $xxy$ of the two indeterminate Quantities $x$ and $y$, is of three Dimenſions. Farther, if the Equation be $x^3y = a^4$: then the Curve $BM$ is an *Hyperbola of the fourth Degree*. Becauſe the Product $x^3y$ is of four Dimenſions: And ſo of others to Infinity. * *Art.* 101.

## Corollary I.

229. IF the Letter $m$ denoting any whole Number, be the Exponent of the Power of the indeterminate Quantity $AP$ $(x)$; and if likewiſe the Letter $n$ be the Exponent of the other indeterminate Quantity $PM$ $(y)$; then it is evident, that the Equation $y^n = x^m \times a^{n-m}$; or (for Brevity's Sake, making the given Quantity $a = 1$) $y^n = x^m$, expreſſes the Nature of Parabola's of all the Degrees to Infinity. In like manner the Equation $x^m y^n = a^{m+n}$, or (making $a = 1$) $x^m y^n = 1$, expreſſes generally the Nature of all Kinds of Hyperbola's. Fig. 123, 124.

## Corollary II.

230. IF any indefinite right Line $AD$ be drawn thro' $A$ the fix'd Origin of the Line $AC$, parallel to $PM$; and if $MK$ be drawn parallel to $AC$, meeting $AD$ in the Point $K$, and the indeterminate Quantities $AR$, $KM$, be call'd $x$, $y$; then it is evident, that the indeterminate Quantity $x$, which expreſs'd before the Line $AP$, or $MK$, will now be $y$; and contrariwiſe $y$, which expreſs'd $PM$ or $AK$, will now be $x$. From whence it follows,

1. If the Curve $AMB$ be a common Parabola, then the Equation thereof will be $yy = ax$, or $xx = ay$, according as the Points of the Parabola regard the Points of the Line $AC$ or $AD$; if that Curve be a Cubick Parabola, then the Equation expreſſing the Nature thereof will be $y^3 = aax$, when the Points thereof regard the Line $AC$, or $x^3 = aay$, when the Points thereof regard thoſe of the Line $AD$; and $y^n = x^m a^{n-m}$, or $x^n = y^m a^{n-m}$, will expreſs in general the Nature of the Parabola $AMB$, according as the ſame regards the right Line $AC$ or $AD$, where $n$ is ſuppoſed to exceed $m$. Fig. 123.

2. The common Hyperbola is always expreſs'd by this Equation $xy = aa$, whether it regards the Line $AC$ or $AD$, the Cubick Hyperbola by this $xxy = a^3$, when it reſpects $AC$, and by this $xyy = a^3$, when it regards the other Line $AD$. And laſtly, $x^m y^n = a^{m+n}$ Fig. 124.

or $x^n y^m = a^{m+n}$, will expreſs in general the Nature of the Hyperbola, according as the Points thereof have regard to thoſe of the Line $AC$, or of $AD$.

### COROLLARY. III.

231. HENCE it is manifeſt, that there are two Cubick Parabola's, one of which is expreſs'd by the Equation $y^3 = a\,a\,x$, or $x^3 = a\,a\,y$, and the other by $y^3 = a\,x\,x$, or $x^3 = a\,y\,y$; whereas there is but one Cubick Hyperbola $x\,x\,y = a^3$, or $x\,y\,y = a^3$. For the indeterminate Quantities $x$ and $y$, can be combined but four ways for expreſſing the Nature of the Cubick Parabola, and but two for expreſſing the Nature of the Cubick Hyperbola. And becauſe the four firſt Equations appertain to two different Curves, and the two laſt to the ſame; therefore, *&c.* And after this manner may be found the Number of Parabola's or Hyperbola's of the fourth Degree, fifth Degree, *&c.*

### COROLLARY IV.

Fig. 124. 232. THE indefinite right Lines $AC$, $AD$, may not only be Aſymptotes of the common Hyperbola, but moreover, Aſymptotes of any other Hyperbola of whatſoever Degree. For let the general Equation, expreſſing the Nature of any Hyperbola, be $x^m y^n = a^{m+n}$, or $y^n = \frac{a^{m+n}}{x^m}$ $(AP = x,\ PM = y)$ when the Points of it are referr'd to thoſe of the Line $AC$; then it is manifeſt, that the more $AP\,(x)$ increaſes, the more, on the contrary, will $y^n$, and conſequently $PM\,(y)$ diminiſh; ſo that when $x$ is infinitely great, $PM\,(y)$ will become nothing; that is, the Hyperbola $BM$, and the Line $AC$, being both infinitely produced, do continually approach nearer and nearer to one another, until they meet one another at an infinite Diſtance; and ſo that Line will be an Aſymptote. Now if the Points of the ſame Hyperbola be referr'd to thoſe of the Line $AD$, then we ſhall have $x^n y^m = a^{m+n}$, or $y^m = \frac{a^{m+n}}{x^n}$. $(AK = x,\ KM = y)$; from whence it follows, that the more $AK\,(x)$ increaſes, the more doth $KM\,(y)$ diminiſh, and ſo the Line $AD$ is an Aſymptote alſo of the ſame Hyperbola.

## PROPOSITION XI.

### Problem.

Fig. 123. 233. *IT is requir'd to draw the Tangent* MT *to the Point* M *given in the Cubick Parabola* AMB, *whoſe Nature is expreſſed by the Equation* $y^3 = a\,x\,x$.

Suppoſe

Suppoſe the Arc $MN$ to be infinitely ſmall, and draw $NQ$, parallel to $PM$, and $MR$ parallel to $AC$; then the ſmall Triangle $MRN$, will be ſimilar to the great one $TPM$; becauſe the ſmall Arc $MN$ may be taken * for a Part of the Tangent $TM$ produced. This being laid down, call the Sub-tangent $(TP)$ ſought, $s$; the ſmall right Line $PQ$, or $MR$, $e$; and then we ſhall have $RN = \frac{ey}{s}$; becauſe the Triangles $TPM$, $MRN$, are ſimilar. Now, if the Cube of $QN\,(y + \frac{ey}{s})$ be put for $y^3$ in the Equation $y^3 = axx$, expreſſing the Nature of the Curve $AMB$, and the Square of $AQ\,(x + e)$ for $xx$; then we ſhall form this Equation $y^3 + \frac{3ey^3}{s} + \frac{3eey^3}{ss} + \frac{e^3y^3}{s^3} = axx + 2eax + eea$, which expreſſes the Relation of $AQ$ to $QN$. And if the Equation $y^3 = axx$ be taken from this Equation, and the remaining Equation be divided by $e$; then there will ariſe $\frac{3y^3}{s} + \frac{3ey^3}{ss} + \frac{eey^3}{s^3} = 2ax + ea$; and ſtriking out all the Terms wherein $e$ happens, becauſe $PQ\,(e)$ being infinitely ſmall, thoſe Terms are infinitely little in reſpect of the others; and then we ſhall get $\frac{3y^3}{s} = 2ax$; and therefore $PT\,(s) = \frac{3y^3}{2ax} = \frac{3}{2}x$, in ſubſtituting $y^3$ for $axx$ the Value thereof. *W. W. D.*

* *Art.* 189.

SCHOLIUM.

234. IF Regard be had to the aforegoing Proceſs, it will appear by ſubſtituting, inſtead of the Power of $y$, a like Power of $y + \frac{ey}{s}$, that there is only Occaſion for the two firſt Terms of that Power; for all the other Terms being multiply'd by the Powers of $e$, have either $e$, or Powers of $e$, in the laſt Equation found at the End of the Operation; and ſo theſe Terms muſt conſequently vaniſh. Underſtand the ſame in putting for the Power of $x$, a like Power of $x + e$. But if all the Powers of the Binomial $x + e$, be ſucceſſively formed; then $x^2 + 2ex$ will be the two firſt Terms of the ſecond Power; $x^3 + 3exx$, the two firſt Terms of the third; $x^4 + 4ex^3$, the two firſt Terms of the fourth; $x^5 + 5ex^4$, the firſt Terms of the fifth; and ſo on. So that the two firſt Terms of any Power $(m)$ of $x + e$, will be $x^m + mex^{m-1}$. After the ſame manner it will be found, that the two firſt Terms of any Power $(n)$ of the Binomial $y + \frac{ey}{s}$, will be $y^n + \frac{ney^n}{s^n}$.

### COROLLARY.

235. HENCE we have a general Expreſſion for the Sub-tangent $PT(s)$ of all Kinds of Parabola's, by means of the general Equation $y^n = x^m a^{n-m}$, or (making $a = 1$) $y^n = x^m$ expreſſing the Nature of all Sorts of Parabola's.

For ſubſtitute the two firſt Terms of the Power $(n)$ of $y + \frac{ey}{s}$, that is, $y^n + \frac{ney^n}{s}$, for $y^n$ in the general Equation $y^n = x^m$; and the two firſt Terms of the Power $(m)$ of $x + e$, that is, $x^m + m e x^{m-1}$ for $x^m$; and then we ſhall get $y^n + \frac{ney^n}{s} = x^m + m e x^{m-1}$. And ſubſtracting the firſt Equation from this, and then dividing by $e$, we ſhall have $\frac{ny^n}{s} = m x^{m-1}$; and therefore $s = \frac{ny^n}{mx^{m-1}} = \frac{n}{m} x$, by putting $x^m$ for $y^n$.

## PROPOSITION XII.

### Problem.

Fig. 124. 236. *TO draw Tangents to all Kinds of Hyperbola's.*

The ſame Preparation being made, as in the aforegoing Propoſition, ſubſtitute the two firſt Terms of the Power $(m)$ of $A\mathcal{Q}$ $(x + e)$ that is, $x^m + m e x^{m-1}$ for $x^m$, in the general Equation $x^m y^n = a^{m+n}$ expreſſing the Relation of $AP(x)$ to $PM(y)$: and the two firſt Terms of the Power $(n)$ of $\mathcal{Q}N\left(y - \frac{ey}{s}\right)$ that is, $y^n - \frac{ney^n}{s}$ for $y^n$: and ſo by Multiplication we ſhall get this Equation $x^m y^n + m e y^n x^{m-1} - \frac{ney^n x^m}{s} - \frac{mney^n x^{m-1}}{s} = a^{m+n}$ expreſſing the Relation of $A\mathcal{Q}$ to $\mathcal{Q}N$. And ſubtracting the firſt Equation from this, and then dividing by $e y^n$, and there will come out $m x^{m-1} - \frac{nx^m}{s} - \frac{mnex^{m-1}}{s} = 0$. And ſtriking out the Term $\frac{mnex^{m-1}}{s}$, being incomparably ſmall with reſpect to the two others, becauſe the infinitely ſmall Line $P\mathcal{Q}$ $(e)$ multiplies the ſame, and we ſhall get $PT(s) = \frac{nx^n}{mx^{m-1}} = \frac{n}{m} x$.

COROL-

### COROLLARY.

237. HENCE, if it be requir'd to draw a Tangent $MT$ to the given Point $M$ in any Parabola or Hyperbola of whatſoever Degree, the Nature of the Parabola being expreſs'd by $y^n = x^m a^{n-m}$, and of the Hyperbola by $x^m y^n = a^{m \times n}$ : you need only aſſume the Sub-tangent $PT = \frac{n}{m} AP$, on the ſame Side the Point $A$ as $P$ is, when the Curve is a Parabola, and on the other Side when the ſame is an Hyperbola. FIG. 123, 124.

## PROPOSITION XIII.

### Theorem.

238. *LET there be a Parabola* AMB *of any Kind, whoſe Nature is expreſs'd by the Equation* $y^n = x^m a^{n-m}$ : *and ſuppoſe the right Line* BC *to be drawn from any Point* B *in the ſame, making a given Angle* (ACB) *with* AC : *and compleat the Parallelogram* ACBD. *I ſay, the circumſcrib'd Parallelogram* ACBD *is to the Parabolick Space* ACBMA, *contained under the ſtraight Lines* AC, CB, *and the Part* AMB *of the Parabola, as* m + n *is to* n. FIG. 125.

*We are to prove, that* ACBD : ACBMA :: m + n : n.

Suppoſe an Arc ($MN$) of the Parabola $AMB$ to be infinitely ſmall, or (as ſome pleaſe to ſpeak) indefinitely ſmall, that is, ſo very ſmall as to be leſs than any given Part of that Parabola; and draw the right Lines $MP$, $N\mathcal{Q}$, parallel to $BC$; and $MK$, $NL$, parallel to $AC$, forming the ſmall Parallelogram $MRNS$: Alſo draw the Tangent $MT$, meeting the Diameter $AC$ in the Point $T$, from which draw a right Line parallel to $CB$, meeting the Lines $MK$, $NL$ in the Points $F$, $G$. This being done, the ſmall Arc $MN$ may be taken * for one of the infinitely ſmall Sides of the Polygon, making up the Part or Portion ($AMB$) of the Parabola, and the Tangent $MT$, for that ſmall Side continu'd out; ſo that we have two right-lin'd Triangles $NRM$, $MPT$, which are ſimilar: Whence $NR$ or $MS : RM :: MP : PT$ or $MF$. And therefore the Parallelogram $PMR\mathcal{Q}$ is equal to the Parallelogram $FMSG$; becauſe the Angles $PMR$, $FMS$, are equal, and the Sides about theſe Angles are reciprocally proportional. *Art. 189.

But * $MF$ or $PT = \frac{n}{m} AP$ or $\frac{n}{m} MK$. Therefore alſo the Parallelogram $FMSG$, or its equal $PMR\mathcal{Q} = \frac{n}{m} KMSL$. And ſince this is always ſo, let the ſmall Arc $MN$ be taken any where on the Portion *Art. 137.

of the Parabola; therefore the Sum of all the ſmall Parallelograms $PMRQ$, that is, the * Trilineal Parabolick Figure $ACBMA$ is $= \frac{n}{m} ADBMA$ the Sum of all the ſmall Parallelograms $\frac{n}{m} KMSL$. Whence $ADBMA : ACBMA :: m : n$. And conſequently $ADBMA + ACBMA$, or $ACBD : ACBMA :: m + n : n$. *W. W. D.*

* *Art.* 184.

### COROLLARY.

239. HENCE it is evident, that the Trilineal Parabolick Figure $APM$ to the circumſcrib'd Parallelogram $APMK$, is as $n$ to $m + n$; and ſo the Parabolick Trapezium $MPCB$ is $= \frac{n}{m+n} ABCD - \frac{n}{m+n} APMK$; becauſe $ACBMA = \frac{n}{m+n} ACBD$, and $APM = \frac{n}{m+n} APMK$.

## PROPOSITION XIV.

### Theorem.

Fig. 126. 240. *LET there be an Hyperbola* BMO, *of any Kind, whoſe Nature is expreſs'd by the Equation* $x^m y^n = a^{m+n}$; *and ſuppoſe the right Line* BC *to be drawn from any Point* B *therein parallel to one of the Aſymptotes* AD, *and terminating in the other* C, *and compleat the Parallelogram* ACBD; *I ſay, the ſaid inſcrib'd Parallelogram* ACBD *is to the Hyperbolick Space* ECBMO, *contain'd under the determinate right Line* BC, *the right Line* CE *indefinitely produced towards* E, *and the Part* BOM *of the Hyperbola infinitely produced towards* O, *as* m—n *is to* n.

*We are to prove, that* ACBD : ECBMO :: m—n : n.

The ſame Preparation being made as in the laſt Propoſition, we prove, as we have done there, that the ſmall Parallelogram $PMRQ$ is $= \frac{n}{m} KMSL$. And becauſe this is always ſo, let the infinitely ſmall Arc $MN$ be ſuppos'd to be taken in any Part ($BMO$) of the Hyperbola; therefore the Sum of all the little Parallelograms $PMRQ$, that is, the Space * $ECBMO$ is $= \frac{n}{m} EADBMO$, the Sum of all the ſmall Parallelograms $\frac{n}{m} KMSL$. Whence we have $EADBMO : ECBMO :: m : n$; and therefore $EADBMO - ECBMO$, or $ACBD : ECBMO :: m—n : n$. *W. W. D.*

* *Art.* 184.

COROL-

### COROLLARY I.

241. HENCE it is manifeſt, that the Hyperbolick Trapezium $CPMB$ is $= \frac{n}{m-n} ACBD - \frac{n}{m-n} APMK$; becauſe $ECBMO$ is $= \frac{n}{m-n} ACBD$, and for the ſame Reaſon the Space $EPMO$ is $= \frac{n}{m-n} APMK$.

### COROLLARY II.

242. HENCE, 1. If $m$ be greater than $n$; then the Relation of the inſcrib'd Parallelogram $ACBD$ to the Space $ECBMO$ indefinitely extended towards $E$, will be expreſs'd always by poſitive Numbers; and ſo in this Caſe we have always the abſolute Quadrature of that Space. 2. When $m=n$, which happens in the common Hyperbola; then the Relation of the Parallelogram $ACBD$ to the Hyperbolick Space $ECBMO$ is as $o$ to 1; that is, the ſaid Space is infinite in reſpect of the inſcrib'd Parallelogram $ACBD$. 3. When $m$ is leſs than $n$, then the inſcrib'd Parallelogram $ACBD$ will be to the Hyperbolick Space $ECBMO$, as a negative Number is to a poſitive one; and therefore the Ratio of that Space to the Parallelogram $ACBD$, is (allowing the Expreſſion) more than infinite. But it muſt be obſerv'd, in this Caſe, that the Hyperbolick Space included by the right Line $DB$, the Aſymptote $AD$ infinitely produced towards $D$, and the Hyperbola $OMB$ infinitely produced towards $B$, will be to the inſcribed Parallelogram $ACBD$, as $m$ is to $n-m$, that is, the ſaid Space is ſquarable: For if the indeterminate Quantities $(x)$ be aſſumed on the Aſymptote $AD$, inſtead of $AC$, then the Equation of the Hyperbola will become * $x^n y^m = a^m \times n$. *Art. 230.

## PROPOSITION XV.

### Theorem.

243. *IF there be any Curve* AMB *within the right Angle* CAD, *and the right Line* MT, *touches the ſame in any Point* M *taken at pleaſure; and if there be ſome other Curve* HFE, *within the Angle* DAH, *(adjacent to the Angle* CAD*) ſuch, that, the Line* FM *being drawn from any Point* F *therein parallel to* AC, *meeting the Line* AD *in* K, *and the Curve* AMB *in* M, *the Line* AK *may be to* MT, *always as ſome conſtant Line* a *to* KF: *I ſay, if the Line* EB *be drawn through any Point* D *of the* FIG. 127.

*the Line* AD, *parallel to* AC, *and terminated by the two Curves, then the Space* ADEFH *will be equal to the Rectangle under the Curve* AMB, *and the conſtant Quantity* a.

*We are to prove, that* ADEFH = AMB × a.

Suppoſe $MN$ to be an infinitely ſmall Arc taken any where in the Curve $AMB$, and draw the right Lines $MF$, $NG$, parallel to $AC$, meeting the right Line $AD$ in the Points $K$, $L$, and the Curve $HFE$, in the Points $F$, $G$; alſo draw the right Lines $FS$, $MR$, parallel to $AD$, and produce $RM$ till it meets $AC$ in $P$. This being done, the two ſimilar right-angled Triangles $MPT$, $MRN$, give this Proportion, *viz.* $MR : MN :: MP$ or $AK : MT :: a : KF$. And therefore $KF \times MR$, that is, the ſmall Rectangle $FKLS$ is $= MN \times a$. And becauſe this is ſo always, let $MN$ be taken any where at pleaſure in the Curve $AMB$; therefore the Sum of all the ſmall Rectangles $KLSF$, that is, * the Space $ADEFH$, will be equal to the Sum of all the ſmall Rectangles $MN \times a$, that is, equal to the Rectangle under the Curve $AMB$, and the conſtant Quantity $a$. *W. W. D.*

*Art. 184.

## COROLLARY I.

244. HENCE the Rectangle under the Portion $AM$, and the conſtant Quantity $a$, is equal to the Space $AKFH$; and the Rectangle under the Portion $MB$, and the ſame Quantity $a$, is equal to the Space $KDEF$.

## COROLLARY II.

245. IF the Curve $AMB$ be ſuppoſed to be a Cubick Parabola, expreſs'd by $y^3 = axx$, ($AP$ being $= x$, and $PM = y$) then will * $PT$ be $= \frac{3}{2}x$; and becauſe the Triangle $MPT$ is right-angled, the Hypothenuſe $MT$ will be $= \sqrt{yy + \frac{9}{4}xx}$. But by the Property of the Curve $HFE$, it muſt be as $MP\,(y) : MT\,(\sqrt{yy + \frac{9}{4}xx}) :: a$ $KF$. And ſo there ariſes $\overline{KF}^2 = aa + \frac{9aaxx}{4yy} = aa + \frac{9}{4}ay$, by ſubſtituting $y^3$ for $axx$. Whence the Curve $HFE$ is a Parabola in this Caſe, the right Line $AD$ being the Axis; the Point $O$ falling on the other Side $D$ with reſpect to $A$, (ſo that $AO$ be $= \frac{4}{9}a$) being the Origin thereof, and the Parameter $= \frac{9}{4}aa$; for by the Property of that Parabola, the Square of the Ordinate $KF$ will be * equal to the Rectangle under $KO$, and the Parameter $\frac{9}{4}a$, that is, in analytick Terms, $\overline{KF}^2 = aa + \frac{9}{4}ay$. And becauſe the parabolick Trapezia $ADEH$, $AKFH$, are * ſquarable; therefore we have the Rectification of the Curve $AMB$, or of any one of the Parts ($AM$) thereof.

*Art. 233.

*Art. 19.

*Art. 239.

If

If you have a mind to expreſs the Value of the Part $AM$, you muſt obſerve, firſt, that $AH$ is $= a$; becauſe $\overline{AH}^2 = AO \times \frac{9}{4}a = aa$. Then calling the Tangent $MT$, $t$; and the Line $AK$ or $MP$, $y$; and there ariſes $KF = \frac{at}{y}$, and the Parabolick Trapezium $FKAH$, or $* \frac{2}{3} FK \times KO - \frac{2}{3} HA \times AO$ will be $= \frac{2}{3} at + \frac{8aat}{27y} - \frac{8}{27}aa =$ $AM \times a$. That is, (dividing by $a$) the Portion $AM$ fought will be $= \frac{2}{3} t + \frac{8at}{27y} - \frac{8}{27} a$. From whence ariſes the following Conſtruction. *Art. 239.

Draw the Tangent $MT$ from any given Point $M$ in the Cubick Parabola $AMB$, meeting the Line $AK$ drawn through the Origin $A$ of the Axis $AC$ perpendicular to the ſame, in the Point $Q$, and one $AK$ aſſume $AV = \frac{8}{27} a$; then if $VC$ be drawn parallel to $MT$, meeting the Axis in $C$, and a Circle be deſcrib'd about the Centre $V$, with the Radius $VA$, cutting $VC$ in $X$; I ſay, the Part $AM$ of the Cubick Parabola $AMB$, will be equal to the Sum of the two right Lines $MQ$, $CX$.

For becauſe the Triangles $TPM$, $TAQ$, are ſimilar, it is plain, that $MQ$ is $= \frac{2}{3} MT$ $(t)$, ſince $AP = \frac{2}{3} PT$; and becauſe the Triangles $MPT$, $VAC$, are ſimilar, we have this Proportion $MP(y) : MT(t) :: AV\left(\frac{8}{27} a\right) : VC = \frac{8at}{27y}$, and therefore $CX = \frac{8at}{27y} - \frac{8}{27} a$. Whence, *&c.*

## PROPOSITION XVI.

### Theorem.

246. *LET there be an Equilateral Hyperbola* EAF, *together with a Parabola* NCS; *and let* C *be the Centre of the Hyperbola, the Line* CA *half of its firſt Axis, and the Line* CA *produced beyond* C, *the Axis of the Parabola, having a Line the double of* CA *for the Parameter thereof, and the Point* C *for its Origin. Then if a right Line* NE *be drawn through any Point* N *in the Parabola* NCS, *parallel to* CA, *meeting the Hyperbola* EAF *in the Point* E, *and its ſecond Axis* CL *in the Point* L. *I ſay, the Hyperbolick Space* CLEA, *included between the right Lines* AC, CL, LE, *and the Portion of the Hyperbola* EA *is equal to the Rectangle under the Portion of the Parabola* CN, *and the right Line* AC. FIG. 128.

Through any Point $M$ in the Portion of the Parabola $CN$, draw $MG$ perpendicular to the Tangent $MT$, as alſo $MB$ parallel to $CA$, meeting the Hyperbola in $B$, and the ſecond Axis $CL$ in $H$; then the Lines $MG$, $HB$, will be equal to one another. For drawing the Ordinate $MP$, we have * $PG = CA$; and ſince the Triangle $MPG$ is right-angled, the Square $\overline{MG}^2 = \overline{PM}^2 + \overline{PG}^2 = \overline{CH}^2 + \overline{CA}^2 = {}^*\overline{HB}^2$; becauſe $EAF$ is an equilateral Hyperbola; and ſo $MG = HB$. But the ſimilar right-angled Triangles $TPM$, $MPG$, give this Proportion, $MP$ or $CH : MT :: PG$ or $CA : MG$ or $HB$. Whence, * *&c.*

* *Art.* 24.

* *Art.* 127.

* *Art.* 143.

## COROLLARY I.

247. HENCE it is manifeſt, that the Hyperbolick Trapezium $HLEB$ is equal to the Rectangle under the Portion of the Parabola $MN$, and $CA$ the half of the Parameter of the Axis.

## COROLLARY II.

248. IF any two Parallels $BD$, $EF$, be drawn in the equilateral Hyperboly $EAF$; and if the right Lines $BM$, $EN$, $DR$, $FS$, be drawn through their Extremities parallel to $AC$, meeting the ſecond Axis of the Hyperbola in the Points $H$, $L$, $K$, $O$; then if the right Lines $BE$, $DF$ be drawn, the Difference of the Rectangles $AC \times MN$, $AC \times RS$, will be equal to the Difference of the right-lin'd Trapezia $HLEB$, $KOFD$.

For the Rectangle $AC \times MN$ is * equal to the Hyperbolick Trapezium $HLEB$; and conſequently the Rectangle $AC \times MN$ *plus* the Hyperbolick Segment $DF$ will be equal to the right-lin'd Trapezium $KOFD$. And ſo becauſe the two Hyperbolick Segments $EB$, $DF$, are * equal to one another, the Difference of the Rectangles $AC \times MN$, $AC \times RS$, will be equal to the Difference of the right-lin'd Trapezia $HLEB$, $KOFD$. *W. W. D.*

* *Art.* 247.

* *Art.* 204.

## COROLLARY. III.

249. THE ſame Things being premis'd as in the laſt Corollary; if it be made as $2AC : LH :: BH + LE : m$; then it is plain, that the Rectangle $AC \times m = \frac{1}{2} LH \times \overline{BH + LE}$, that is, equal to the right-lin'd Trapezium $HLEB$. In like manner, if it be made as

as $2AC : KO :: KD + FO : n$; then $AC \times n$ will be equal to the right-lin'd Trapezium $KOFD$. And consequently the Difference * of the Rectangles $AC \times MN$, $AC \times RS$, will be equal to the Difference of the Rectangles $AC \times m$, $AC \times n$; that is, dividing by $AC$, the Difference of the Parabolick Arcs $MN$, $RS$, will be equal to the Difference between the right Lines $m$, $n$. Whence it appears, that ſtraight Lines may be found equal to the Difference of an infinite Number of Parabolick Arcs, ſuch, as $MN$, $RS$.

* *Art.* 248.

## *The End of the Fifth Book.*

# BOOK VI.

## *Of the Conick Sections conſider'd in the Solid.*

## CHAP. I.

### *Of the Three Conick Sections in general.*

#### DEFINITIONS.

I.

FIG. 129. IF there be any immoveable Point *S* aſſum'd without a Plane in which the Circle *VXY* is deſcrib'd, and if a right Line *SZ* drawn through that Point, and infinitely produced both ways, moves quite round the Circumference of the Circle; then each of the two Superficies produced by the Motion of the indefinite right Line *SZ*, is call'd ſeparately a *Conick Superficies*, and both of them conjunctly *oppoſite Conick Superficies*, or *only oppoſite Superficies*.

2.

The immoveable Point *S*, common to both the oppoſite Superficies, is call'd the *Vertex*.

3.

The Circle *VXY*, *the Baſe*.

4.

The Solid comprehended under the Baſe *VXY*, and that Part of the Conick Superficies between the Baſe and the Vertex *S*, is called a *Cone*.

5.

The Line *SX* drawn from the Vertex *S* to any Point *X* in the Baſe, is a *Side* of the Cone.

6. The

6.

The Line *SO* drawn from the Vertex *S*, through *O* the Centre of the Baſe, is call'd the *Axis* of the Cone.

7.

If the Axis be perpendicular to the Plane of the Baſe, then the Cone is call'd a *right Cone*; and if the Axis be not, then the Cone is call'd a *Scalene Cone*.

8.

If a Conick Superficies be cut by a Plane (*F A G*) not paſſing thro' the Vertex *S*, or parallel to the Plane of the Baſe *V X Y*; the Curve *FAG*, form'd by the Concurrence of that Plane and the Conick Superficies, is a *Conick Section*. Fig. 130, 131, 132.

9.

If a Plane (*SDE*) be drawn through the Vertex (*S*) of a Cone, parallel to the Plane of a Conick Section; the indefinite right Line *D E*, formed by the Concurrence of that Plane, and the Baſe of the Cone, is call'd a *Directrix*.

10.

A Conick Section (*FAG*) is call'd a *Parabola*, when the Directrix *D E* touches the Circular Baſe of the Cone; an *Ellipſis*, when the ſame falls quite without that Baſe; and an *Hyperbola*, when it falls within [or cuts] that Baſe.

But in this laſt Caſe, if the Plane of the Section be continued, the ſame will meet and cut the oppoſite Conick Superficies; and the Curve *K M H* form'd thereby, is call'd an oppoſite Hyperbola with regard to the former Hyperbola *F A G*; and both of them together are call'd *oppoſite Hyperbola's*, or *oppoſite Sections*. Fig. 132.

11.

If any ſtraight Line, in the Plane of a Conick Section, meets that Section in one Point only, and being both ways infinitely continued, does not cut or fall within the Section; then that Line is call'd a *Tangent*; and the Point wherein it meets the Section, is named the Point of *Contact*. Fig. 130, 131, 132.

COROL.

### COROLLARY I.

FIG. 130. 250. IN the Parabola, all the Sides of the Cone being produced indefinitely will neceſſarily meet the Plane thereof, except the Side *S D* only, which is drawn from the Vertex *S* through the Point *D*, wherein the Directrix touches the Baſe; becauſe there can be only that Side in the Plane *S D E*, parallel to the Plane of the Section, and all the other Sides cut the ſame in the Point *S*. From whence it appears, that the Parabola is a Curve of an infinite Extenſion, which doth not return into itſelf.

### COROLLARY II.

FIG. 131. 251. IN an Ellipſis, all the Sides of the Cone being produced (if neceſſary) will meet the Plane thereof; becauſe all the Sides of the Cone meet the Plane *S D E* parallel thereto in the Point *S*. From whence it appears, that the Ellipſis includes a Space, and returns into itſelf.

### COROLLARY III.

FIG. 132. 252. IN the oppoſite Hyperbola's, all the Sides of the Cone, except only *S D*, *S E*, which are drawn from the Vertex *S* to the Points *D*, *E*, wherein the Directrix cuts the Baſe, being both ways infinitely produced, muſt neceſſarily meet their Plane; becauſe there can be but theſe two Sides that fall in the Plane *S D E*, which is parallel to the Plane of the Hyperbola's, and all the other Sides cut that Plane in the Point *S*. Farther, the Sides of the Part of the Cone *S D V E* do form the Points of the Hyperbola *F A G*, and the Sides of the Parts *S D Y E* being produced beyond the Vertex *S*, do form the Points of the oppoſite Hyperbola *K M H*. From whence it appears, that each of the oppoſite Sections are infinite, and do not return into themſelves, no more than the Parabola.

## PROPOSITION I.

### Theorem.

FIG. 132. 253. *IF two oppoſite Superficies be cut by a Plane* S a m, *paſſing through the Vertex* S; *I ſay, the common Sections of that Plane, and the oppoſite Superficies, will be two ſtraight Lines* S a, S m, *both ways indefinitely extended from* S.

For

For let *a m* be the common Section of the cutting Plane, and the Plane of the Baſe; then it is plain, that *a m* will cut the Baſe in two Points *a*, *m*; becauſe the Plane *S a m* falls within the Conick Surface. And ſo if the Sides *S a*, *S m*, be drawn, and produced indefinitely both ways from the Vertex *S*; then it is manifeſt by the Generation of oppoſite Superficies, that thoſe Sides will be the two common Sections of the ſaid Superficies, and the cutting Plane *S a m*. *W. W. D.*

### Corollary I.

254. BEcauſe that Part of the Line *a m*, joining the two Points *a*, *m*, falls within the Baſe, and all the reſt of that Line without the ſame; therefore, if the Plane *S a m* be ſuppos'd indefinitely extended every way, that Part thereof included in the Angle *a S m*, as alſo in the oppoſite vertical Angle, will fall within the two oppoſite Superficies; and all the reſt of that plane will fall without the ſaid two Superficies.

### Corollary II.

255. HENCE if any two Points (*A*, *M*) of a Conick Section be join'd by a ſtraight Line, that Line will fall within the Section, and being both ways produced, will fall without the ſame. For drawing the Sides *S a*, *S m*, from the Vertex *S* through the Points *A*, *M*, and drawing a Plane through theſe Sides; it is manifeſt, that the Line *A M* falls in that Part of this Plane included in the Angle *a S m*, and all the reſt of that Line will be in the Part of the Plane falling in the adjoining Angles. Fig. 130.

### Corollary III.

256. IF through the Vertex of the Cone *S*, there be drawn a right Line parallel to the Line *A M*, terminating in any Conick Section; then it is evident by the laſt Corollary, that that Line *S H* will fall in one of the Angles adjoining to the Angle *a S m*, that is, it will fall without the Conick Superficies; and ſo the ſame will meet the Plane of the Baſe in ſome Point without the Circumference, or elſe will be parallel to the Baſe.

### Corollary IV.

257. IT is manifeſt (by *Corol.* 1.) that if any two Points (*A*, *M*) of two oppoſite Hyperbola's be join'd by a right Line, then that Line will fall within the ſaid Hyperbola's; and if the ſame be both ways Fig. 132.

ways continu'd, it will fall without them: for drawing the Sides *Sa*, *Sm*, thro' the Vertex (*S*) and the Points *A*, *M*, and drawing a Plane through thoſe Points and the Vertex both ways indefinitely extended from *S*; then it is evident, that that Part of this Plane included in the Angle *ASM*, wherein the Line *AM* falls, is contain'd between thoſe two Superficies; and likewiſe that the Part of the ſaid Plane included between the two adjoining Angles, (wherein are the Continuations of the Line *AM*) falls within thoſe two Superficies. And becauſe the Line *AM* is the common Section of the Plane *Sam*, and that of the two oppoſite Hyperbola's; therefore, *&c.*

### COROLLARY V.

258. IT is manifeſt by the 2d and 4th Corollaries, that a right Line can meet a Conick Section, or the oppoſite Sections, in two Points only, and not more.

## PROPOSITION II.

### Theorem.

FIG. 129. 259. *IF either of the two oppoſite Superficies be cut by a Plane* ouxy *parallel to the Baſe* OVXY; *I ſay, the Section made by that Plane, and the Conick Superficies, is a Circle; and the Point* o, *wherein that Plane meets the Axis* SO, *(produced on the other Side the Vertex* S, *if neceſſary) is the Centre thereof.*

For if through any Point *X* in the Baſe, there be drawn the Radius *XO* to the Centre *O*, as alſo the Side *XS* from the Vertex *S*, meeting the Plane *ouxy* in the Point *x*; then the Lines *OX*, *ox*, will be parallel to one another: Becauſe they are the common Sections of the two parallel Planes *OVXY*, *ovxy*, and the Plane *SOX* (produced on the other Side the Vertex, if neceſſary) therefore the Triangles *OSX*, *oSx*, will be ſimilar, and conſequently this Proportion will be had always, *viz.* *SO* : *OX* :: *So* : *ox*. Now becauſe the two firſt Terms of this Proportion are ſtanding Quantities, therefore the fourth Term *ox* will be a ſtanding Quantity, let the Point *x* be taken any where at pleaſure; and conſequently the Curve *vxy* is the Circumference of a Circle, and the Point *o* is the Centre.

### COROLLARY.

260. HENCE the Baſe of a Cone may be put in any Place deſir'd, according as it is found moſt proper ſo to do. And ſo when the Section is a Parabola or Hyperbola, it is placed commonly ſo as

to

to cut the Section; but when the Section is an Ellipſis, it is ſometimes placed ſo as to cut the ſame, and ſometimes ſo that the Ellipſis be above it.

## PROPOSITION III.

### Theorem.

261. *IF through any Point* A, *taken in the Parabola* FAG, *there be drawn the indefinite right Line* AB *in the Plane thereof, within the Cone, parallel to the Side* SD, *and paſſing through the Point* D, *wherein the Directrix* DE *touches the Baſe; I ſay, that Line* AB *falls entirely within the Section, and being infinitely produced towards* B, *will never after meet the ſame.* FIG. 130.

For if the Plane *SAB* be drawn thro' the Vertex *(S)* of the Cone, and the Line *AB*; this Plane will form two Sides of the Cone by its Concurrence with the Superficies, one of which will be always the Line *SD*, becauſe *AB* is parallel to it; and the other, the Line *Sa* which paſſes thro' the Point *A*. But the Plane *DSa*, contain'd between the Sides *SD*, *Sa*, infinitely produced towards *D*, *a*, does fall within the Conick Superficies: And conſequently * the Line *AB*, which is always in that Plane, being parallel to the Side *SD*, ſhall fall wholly within the Parabola, and will never after meet it, tho' infinitely produced towards *B*. * *Art.* 254.

## PROPOSITION IV.

### Theorem.

262. *IF through any Point* A *taken in the Parabola* FAG, *there be drawn the right Line* AM *in the Plane thereof within the Cone, not being parallel to the Side* SD, *paſſing through the Point* D, *wherein the Directrix* DE *touches the Baſe; I ſay, that Line* AM *will meet the Parabola in ſome other Point* M. FIG. 130.

For if the Plane *SAM* be drawn through the Vertex *S*, and that Line; then this Plane will fall within the Conick Superficies, and will not paſs through the Side *SD*; and therefore the ſame will form * the two Sides *Sa*, *Sm* of the Cone, one of which, as *Sa*, paſſes thro' the Point *A*; and the other *Sm*, is not parallel to the Plane of the Section, becauſe *(Hyp.)* there is no Side but *SD*, which is parallel thereto. Therefore the Side *Sm* being produced (if neceſſary) will meet the Plane of the Parabola in ſome Point *M*, which the Line *AM*, formed by the Concurrence of the Plane *aSm*, and that of the Parabola, paſſes through. And it is manifeſt, that the Point *M* is * *Art.* 253.

in the Parabola *FAG*, becauſe it is both in the Plane of the Section, and in the Superficies of the Cone. Whence, *&c.*

## PROPOSITION V.

### Problem.

FIG. 133, 134, 135. 263. *TO draw a Tangent, as* AF, *from a Point* A *given in a Conick Section.*

Through the Point *A*, and the Vertex (*S*) of the Cone, draw the right Line *SA*, meeting the Plane of the Baſe in the Point *a*, and draw the Tangent *Eaf* to the Point *a* in the Baſe; then the Line *AF*, made by the Concurrence of the Plane *SEaf* (produced beyond the Vertex, if neceſſary) and the Plane of the Section will be the Tangent ſought.

For becauſe the Tangent *Eaf* falls entirely without the Baſe, the Point *a* therein being excepted; therefore the Plane *SEaf*, indefinitely produced both ways from the Vertex *S*, will meet the oppoſite Superficies only in the Line *Sa* both ways indefinitely produced, and all the reſt of that Plane falls quite without the Superficies. And conſequently the Line *AF*, form'd by the Concurrence of the Plane *SEaf*, and the Plane of the Section, hath only the Point *A*, wherein the Line *Sa* meets the Plane of the Section, common to either of the oppoſite Superficies; and does fall wholly without the Section, that Point being only excepted. Therefore, *&c.*

### COROLLARY I.

264. BEcauſe there can be but one Tangent *Eaf* drawn to the Point *a* in the Baſe of the Cone, therefore alſo there can be but one Tangent *AF* drawn to a Point (*A*) given in a Conick Section.

### COROLLARY II.

265. FROM hence ariſes the Manner of drawing a Tangent *AF*, parallel to a right Line (*MN*) given in Poſition in the Plane of a Conick Section, or the oppoſite Sections. For if *SE* be drawn through the Vertex (*S*) of the Cone, parallel to *MN*, this Line will either meet the Directrix *DE* in ſome Point *E*, or elſe be parallel thereto; becauſe this Line *SE* will be parallel to the Plane of the Section, and will fall conſequently in the Plane *SDE*. Now if *SE* meets *DE* in the Point *E*, falling without the Circular Baſe of the Cone; draw the Tangent *Eaf* from the Point *E* to the Circle, and then it is manifeſt, that the common Section of the Plane *SEaf*, and the

the Plane of the Section, *viz.* $AF$ is a Tangent, and will be parallel to the Line $MN$; becauſe the two Sections ($AF$, $SE$,) of the parallel * Planes $MAN$, $SED$, made by the touching Plane $SEaf$, are paral- * *Hyp.* lel to one another, as well as * $SE$, $MN$.

## COROLLARY III.

266. THE ſame Things being premis'd as in the laſt Corollary, it follows, FIG. 133.

1. That in the Parabola, the Problem is impoſſible, when the Line $MN$ given in Poſition, is parallel to the Side $SD$ paſſing through the Point $D$, wherein the Directrix $DE$ touches the Baſe: For then, ſince the Point $E$ falls in $D$, there can be no Tangent, but the Directrix $DE$ drawn through that Point: And ſince the Plane paſſing thro' the Vertex, and the Directrix $DE$, is * parallel to the Plane of the * *Def.* 9. Parabola, therefore there can be no Tangent form'd, becauſe theſe two Planes cannot cut one another. But when the Line given in Poſition is not parallel to the Side $SD$, there may be drawn always one Tangent $AF$ parallel to that Line, and no more: for the Point $E$ falling then without the Baſe of the Cone, we can draw always $Eaf$, $EDL$, to that Baſe; the latter of which coinciding with the Directrix, is of no uſe for forming a Tangent in the Plane of the Section; but by means of the former $Eaf$, we may find always ſome Tangent $AF$ by the Concurrence of the Plane $SEaf$, and the Plane of the Section, and that will be the Tangent ſought. The ſame muſt be underſtood when the Line $SE$ is parallel to the Directrix, for the Tangent $Eaf$ will then become parallel to the Directrix; and ſo becauſe there can be drawn but one Tangent parallel to the Directrix, ſince the Directrix itſelf touches the Baſe in the Point $D$, therefore, *&c.*

2. In the Ellipſis, there can be drawn always two Tangents $AF$, FIG. 134. $BG$, parallel to the Line $MN$ given in Poſition, and conſequently parallel to one another. For becauſe all the Points of the Directrix $DE$ fall without the Baſe; therefore there can be drawn always two Tangents $Eaf$, $Ebg$, from the Point $E$ to that Baſe, not coinciding with the Directrix; and by means of theſe and the common Sections of the Planes $SEaf$, $SEbg$, and the Plane of the Section, the two Tangents $AF$, $BG$, will be form'd parallel to $MN$. The ſame muſt be underſtood, when the Line $SE$ is parallel to the Directrix; for then, inſtead of the Tangents $Eaf$, $Ebg$, drawn from the Point $E$ in the Directrix, you need only draw two Tangents parallel to the Directrix, which is poſſible always.

3. In the oppoſite Sections, the Problem is impoſſible when the FIG. 135. Point $E$ falls within the Baſe of the Cone, becauſe then there can no Tangent be drawn from that Point to the Baſe. But when the Point

*E* falls without the Baſe, then there may be found always two Tangents *AF*, *BG*, parallel to the Line *MN* given in Poſition; for becauſe the Directrix cuts the Baſe, there can be drawn always two Tangents *Eaf*, *Ebg*, from the Point *E* to the Baſe, falling on both Sides the Directrix, by means of which the Interſection of the Planes *SEaf*, *SEbg*, and the Plane of the Section, will form the two Tangents *AF*, *BG* ſought. The ſame muſt be underſtood, when the Line *SE* is parallel to the Directrix *DE*; for inſtead of the two Tangents *Eaf*, *Ebg*, you need only draw two Tangents parallel to the Directrix, which is always poſſible.

In this laſt Caſe it muſt be obſerv'd, that the parallel Tangents *AF*, *BG*, appertain always to the oppoſite Hyperbola's, and never to one and the ſame; for this is evident, becauſe the Tangents (*Eaf*, *Ebg*) to the Baſe do fall neceſſarily on both Sides the Directrix *DE*.

## COROLLARY IV.

267. IT follows from the laſt Corollary, (1.) That in the Parabola and Hyperbola, there cannot be two Tangents parallel to one another; and contrariwiſe, in the Ellipſis and oppoſite Sections, if any Tangent *AF* be given in Poſition, there can always be drawn another (*BG*) parallel to it.

(2.) If the Line *MN* given in Poſition, be bounded by a Conick Section; then, in the Parabola, there can be drawn always ſome Tangent *AF* parallel thereto; and in the Ellipſis and oppoſite Sections, two Tangents *AF*, *BG*; becauſe the Line *SE*, drawn through the Vertex (*S*) parallel to *MN*, will * meet the Plane of the Baſe, either in ſome Point *E* without the Circumference, or be parallel thereto.

* *Art.* 256.

## DEFINITIONS.

12.

FIG. 133. In a Parabola, if through any Point *A* you draw the right Line *AB* within the Cone parallel to the Side *SD*, paſſing through the Point *D*, wherein the Directrix *DE* touches the Baſe; the ſaid Line *AB* is call'd a *Diameter*, and the Point *A* the *Origin* [or Vertex] thereof.

13.

FIG. 134, 135. In the Ellipſis or oppoſite Sections, any right Line *AB*, joining the Points of Contact of two parallel Tangents *AF*, *BG*, is call'd a *Diameter*; and the Points *A*, *B* are the Extremities thereof.

14. If

14.

If through any Point *P*, in any Diameter (*A B*) of a Conick Secti Fig. 133, on, there be drawn the right Line *M N*, (meeting the Section in the 134, 135. Points *M*, *N*,) parallel to the Tangent, paſſing through the Origin (*A*) of that Diameter in the Parabola, and through either of the Extremities thereof in the other Sections; this Line *M N* is an *Ordinate* on both Sides to the Diameter *A B*, and either of its Parts *P M*, or *P N*, is an *Ordinate* to that Diameter.

15.

That Diameter, which is at right Angles to its Ordinates, is called an *Axis*.

COROLLARY.

268. IT is manifeſt by *Def.* 12. (1.) That all Diameters in a Parabola are parallel to one another, becauſe they are all parallel to the Side of the Cone *S D*, drawn through the Point *D*, wherein the Directrix *D E* touches the Baſe. (2.) That there can be drawn but one Diameter through a Point given in the Plane of a Parabola, becauſe there can only one Line be drawn through that Point parallel to the Side *SD*.

## PROPOSITION VI.

### Problem.

269. *A NY Diameter* A B *of a Conick Section, together with its Ordinate* P M, *being given, to deſcribe the Section.* Fig. 136, 137, 138.

Draw any Plane (the Plane of the Scheme *A P M* excepted) thro' the Ordinate *P M*, and draw the indefinite right Line *P a* through the Point *P* in that Plane perpendicular to *PM*, and deſcribe a Circle about any Point *C* in that Line with the Radius *CM*. This being done,

1. When the Section is to be a Parabola, From one of the Points *a*, *D*, wherein the Circle cuts the Perpendicular *P A*, *viz.* *a*, draw the right Line *a A* through the Origin (*A*) of the Diameter *A B*, meeting the right Line *D S* drawn from the other Point *D*, in the Point *S*; then if a Conick Superficies be deſcrib'd with the Point *S*, for the Vertex, and the Circle *D M a N* for a Baſe: I ſay, the Concurrence of that Superficies, and the Plane *APM*, will form the Parabola *MAN* ſought. For if the Lines *D E*, *a f*, be drawn through the Extremities Fig. 136.

of

of the Diameter $Da$ parallel to $PM$; then it is manifeſt, that theſe Lines will be Tangents, becauſe $PM$ is * perpendicular to $Da$. But the Plane $SDE$ paſſing through the Vertex of the Cone $S$, and the Tangent $DE$, is parallel to the Plane $APM$; ſince $SD$, $DE$, are * parallel to $AP$, $PM$: therefore * the Section $MAN$, form'd by the Plane $APM$ in the Conick Superficies, will be a Parabola, and the Line $AB$, a Diameter thereof. Farther, the touching Plane $Saf$ forms * the Tangent $AF$, in the Plane $APM$, which Tangent will be parallel to $PM$; becauſe $AF$ is the common Section of the two Planes $Saf$, $APM$, which paſs through the Parallels $af$, $PM$; and conſequently * the Line $PM$ will be an Ordinate to the Diameter $AB$.

* *Hyp.*
* *Hyp.*
* *Def.* 10, *and* 12.
* *Art.* 263.
* *Def.* 14.

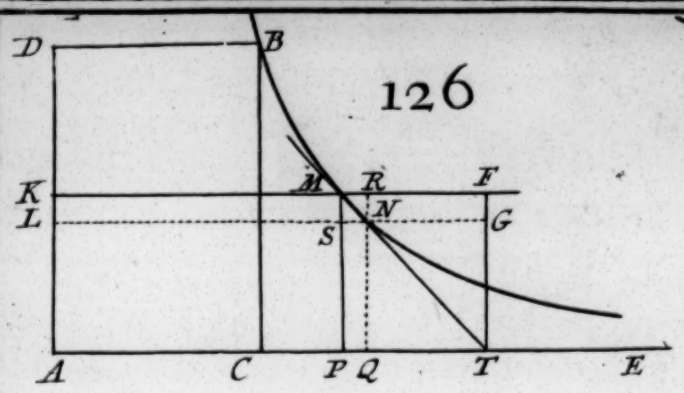
FIG. 137, 138. 2. When the Conick Section is to be an Ellipſis or Hyperbola. Draw the right Lines $aA$, $bB$, from the Points $a$, $b$, (wherein the indefinite Perpendicular $Pa$ cuts the Circle) through the Extremities $A$, $B$, of the Diameter $AB$, meeting one another in the Point $S$. And then if a Cone be deſcrib'd, with the Vertex $S$, and the Baſe $aMbN$; I ſay, the Plane $APM$ will form the requir'd Section $MAN$, in the Superficies of the Cone. For if $SD$ be drawn parallel to $AB$ the Diameter of the Section, meeting $ab$ the Diameter of the Baſe in $D$; and if $DE$, $af$, $bg$, be drawn through $D$, and the Extremities $a$, $b$, parallel to $PM$; then it is manifeſt, that the Plane $SDE$ is parallel to the Plane $APM$, and ſo $DE$ will be * the Directrix. But in the Ellipſis, the Point $D$ falls on the Diameter $ab$ produced without the Circle; ſince the Diameter ($AB$) of the Section, falls in the Angle $aSb$, form'd by the Sides of the Cone $Sa$, $Sb$; and contrariwiſe, in the Hyperbola the Point $D$ falls within the Circle, becauſe then the Diameter ($AB$) of the Section falls in the Angle $aSB$, adjoining to the Angle $aSb$. Therefore, by the 10th Definition, the Section $MAN$ is an Ellipſis in the firſt Caſe, and an Hyperbola in the ſecond. Farther, the Tangent $AF$, paſſing through the Extremity ($A$) of the Diameter $AB$, being the common Section of the touching Plane $Saf$, and the cutting Plane $APM$ (paſſing thro' the Parallels $af$, $PM$,) will be parallel to $PM$. In like manner, the Tangent $BG$, being the common Section of the touching Plane $Sbg$, and the cutting Plane $APM$ (paſſing thro' the Parallels $bg$, $PM$) will be parallel alſo to $PM$; therefore the Line $AB$ is * a Diameter, and $PM$ is an Ordinate to the ſame.

* *Def.* 9.
* *Def.* 13, *and* 14.

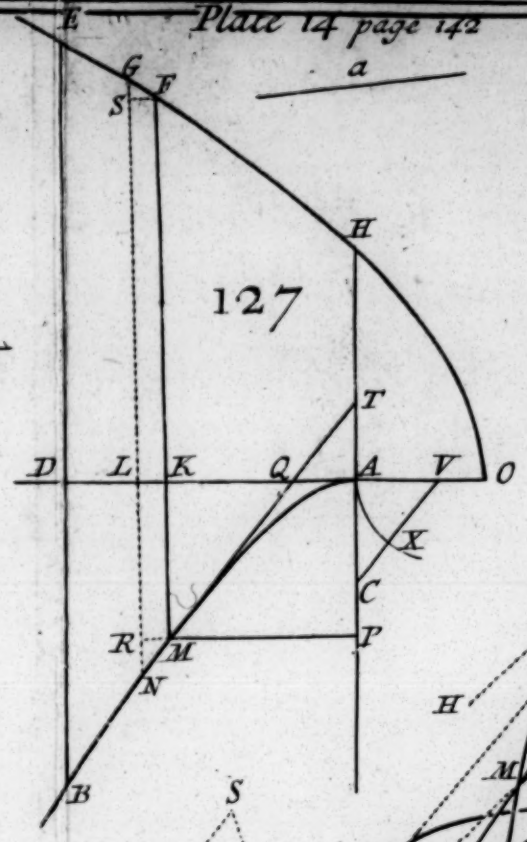
In the Ellipſis, it may happen that the Lines $Aa$, $Bb$, be parallel between themſelves; in which Caſe, you may take any other Point in the Line $ab$, for the Centre of the Circle $aMbN$.

DEFI-

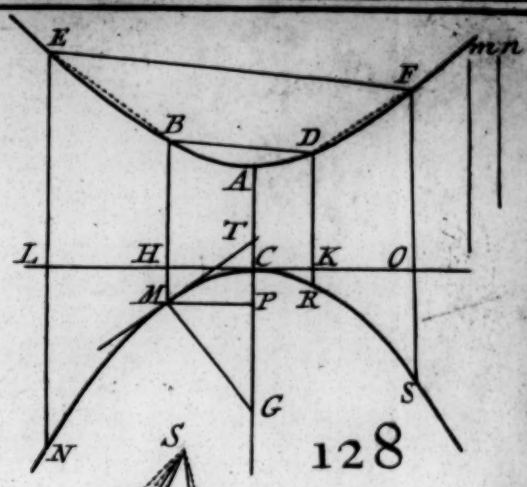
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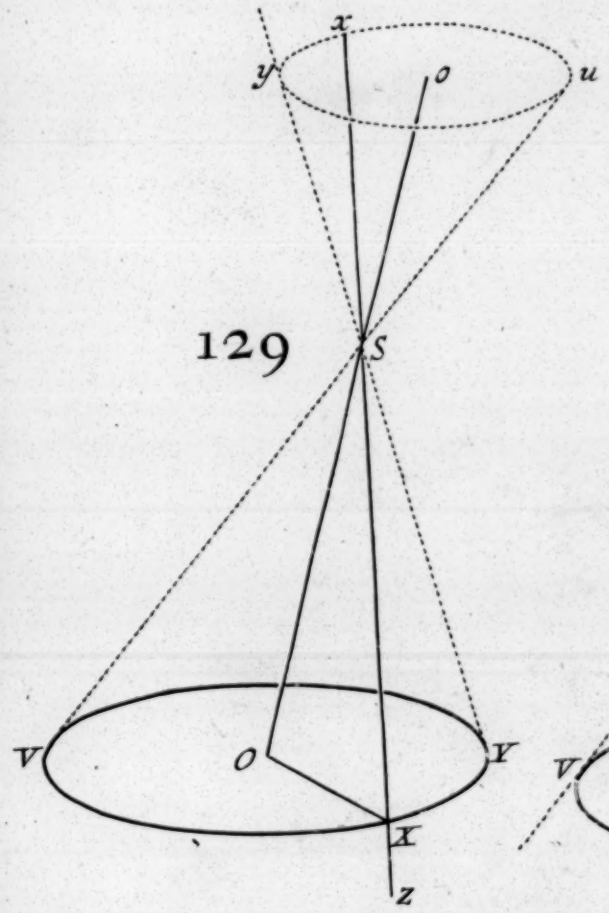

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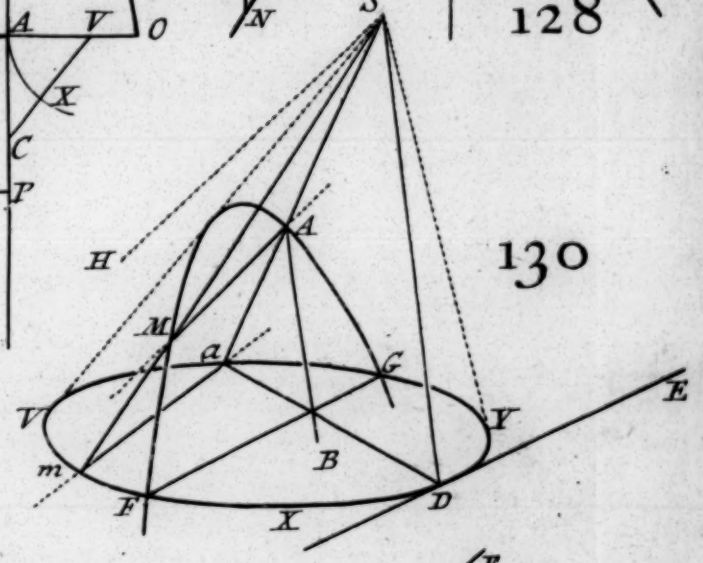

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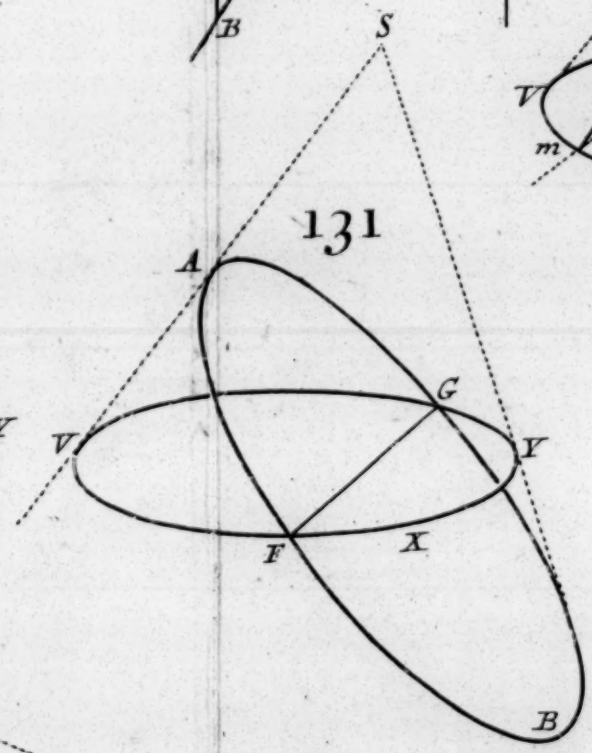

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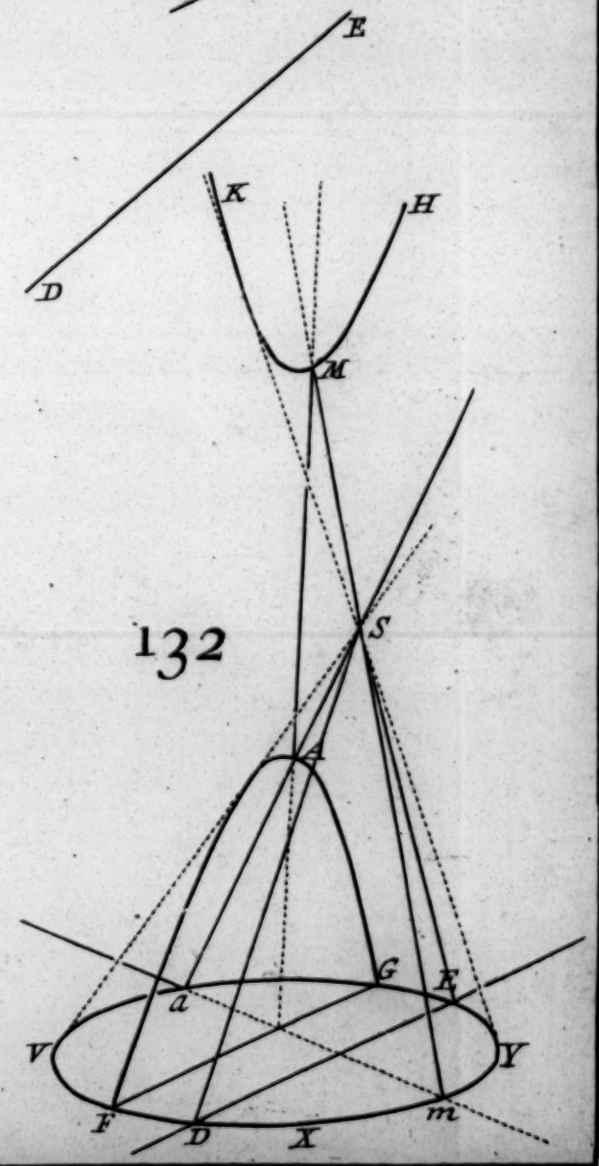

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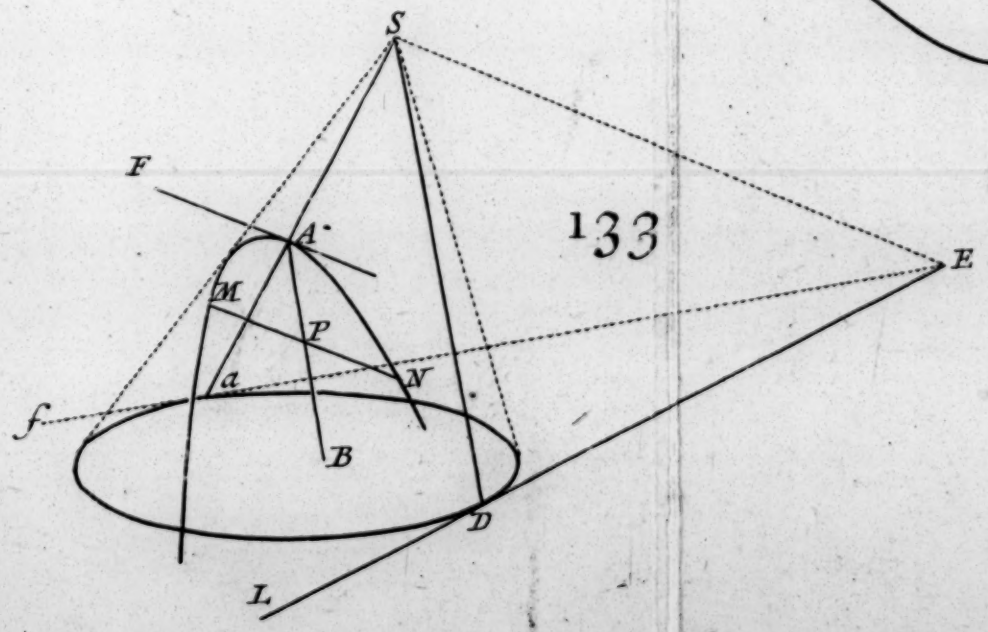

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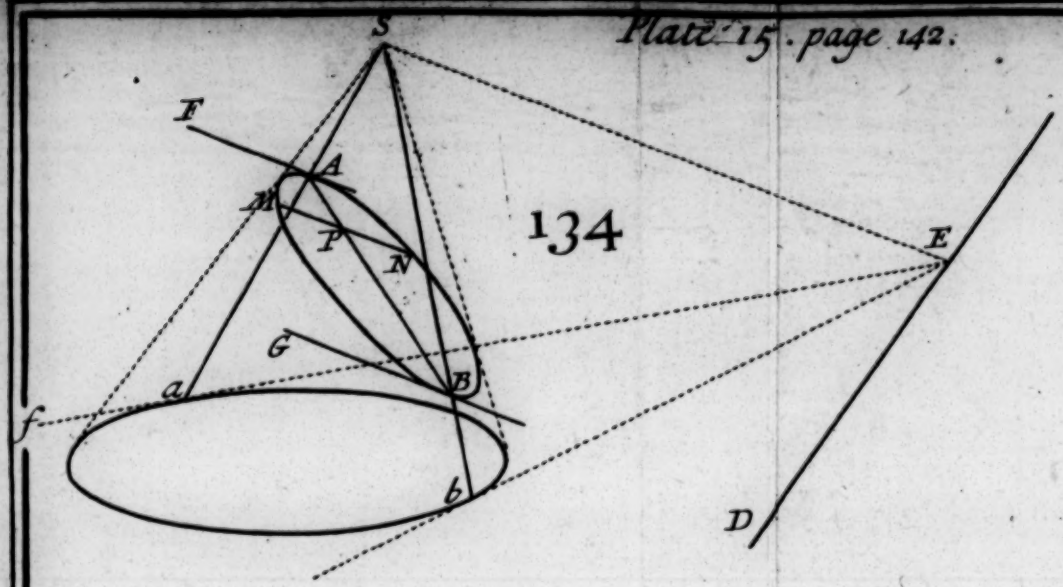

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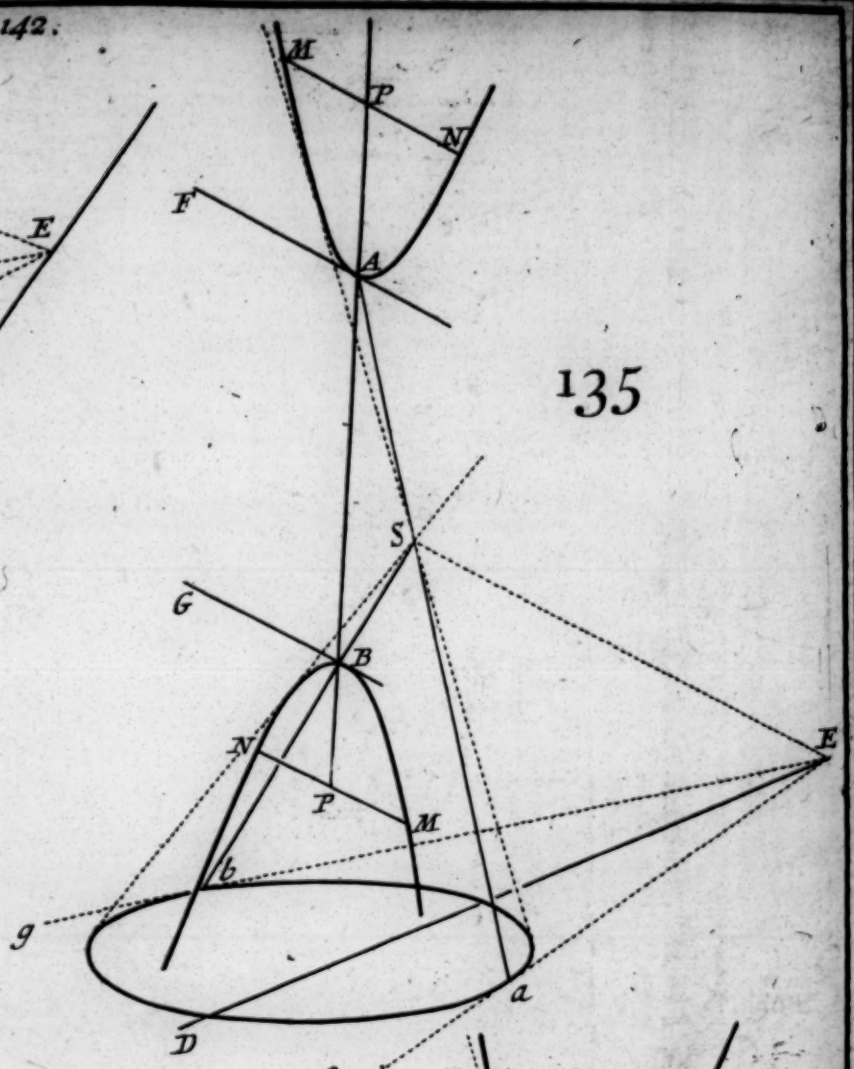

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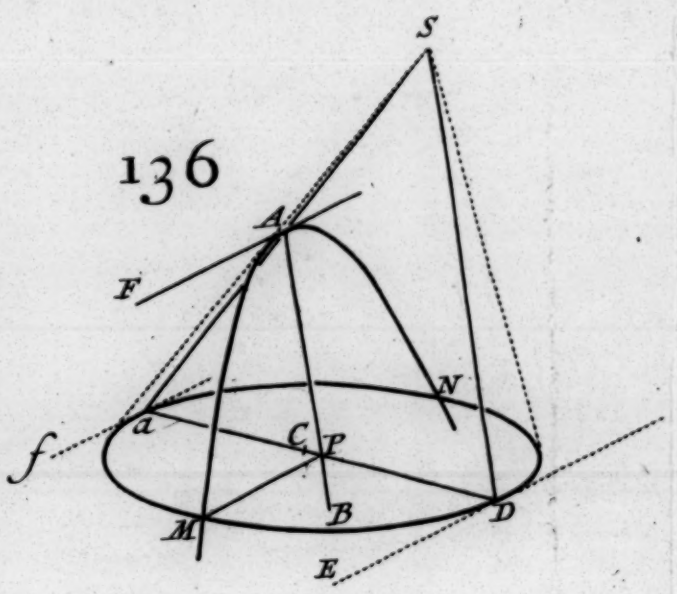
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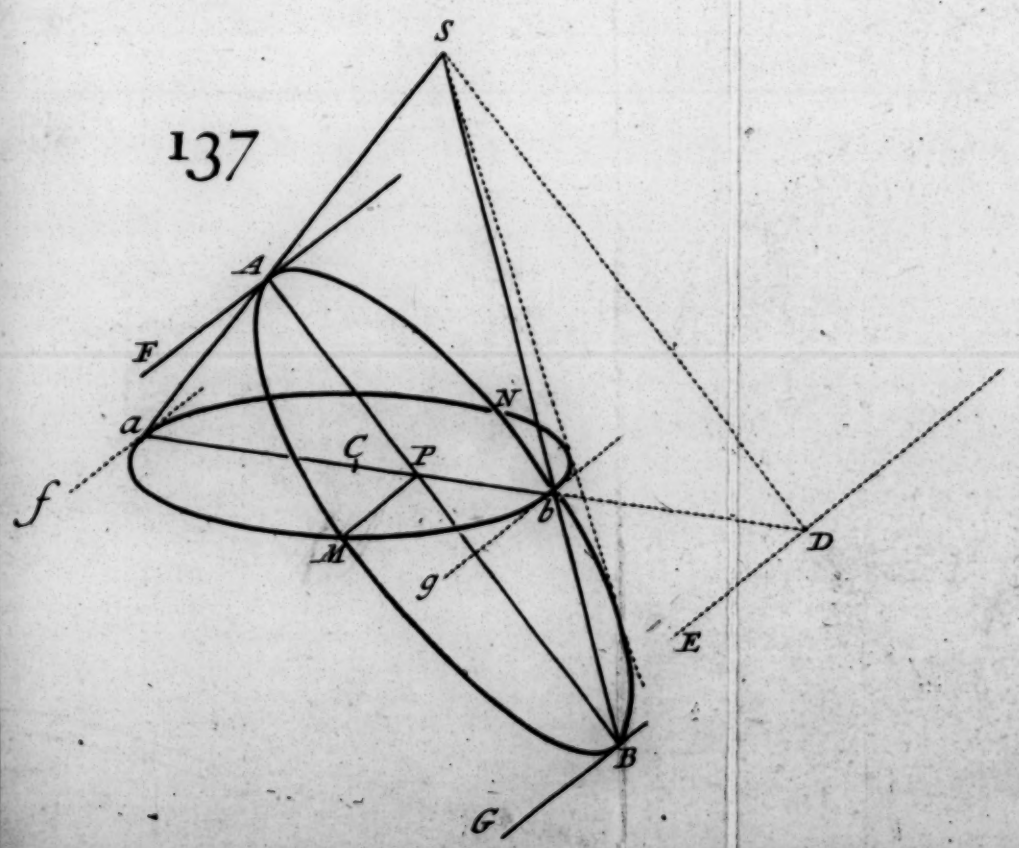

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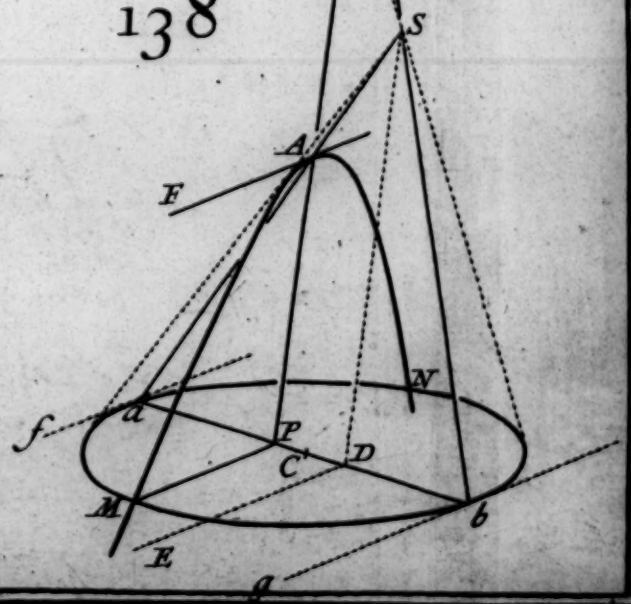

136




137




138



## DEFINITION.

16.

If two Tangents $DH$, $EK$, be drawn thro' the two Points ($D$, $E$) wherein the Directrix cuts the Baſe, when the Section is an Hyperbola; and if two Planes $SDH$, $SEK$, be drawn through the Vertex $S$, and thoſe two Tangents; the two indefinite right Lines $CH$, $CK$, form'd by the Concurrence of thoſe Planes, and the Plane of the Hyperbola's, are call'd *Aſymptotes*. Fig. 139.

## COROLLARY.

270. IF thro' the Point of Contact $D$, there be drawn the Side $DS$, indefinitely produced both ways from $S$; then it is evident, that the Plane $SDH$ will touch the oppoſite Superficies in that Line; becauſe all the Points of the Tangent $DH$, except $D$, do fall without the Circumference of the Baſe. But the Plane $SDE$ paſſing through the Vertex $S$, and the Directrix $DE$, being * parallel to the Plane of the oppoſite Hyperbola's, the common Sections $SD$, $CH$, of thoſe two Planes $SD$, $CH$, and the Plane $SDH$, will be parallel between themſelves; and therefore the Aſymptote $CH$ will fall quite without (and between) the oppoſite Superficies, and conſequently will both ways leave the oppoſite Sections entirely without meeting them. The ſame may be prov'd of the other Aſymptote $CK$, becauſe the two Aſymptotes $CH$, $CK$, are form'd by the Planes $SDH$, $SEK$, falling on both Sides the ſame Conick Superficies, and the Superficies oppoſite to it; therefore all the Points of the Hyperbola $FAG$, are contain'd in the Angle $HCK$, and all the Points of the oppoſite Hyperbola in the Angle vertically oppoſite thereto. * Def. 9.

## PROPOSITION VII.

### Theorem.

271. *IF the right Line* B A *be drawn through any Point* B, *taken in one Aſymptote* CK, *parallel to the other Aſymptote* CH; *I ſay, this Line will meet one of the oppoſite Hyperbola's in ſome Point* A *only, and being indefinitely produced, will be ever after within the ſame.* Fig. 139.

Becauſe the Lines $BA$, $SD$, are parallel to the ſame Line $CH$, they will be parallel between themſelves; and ſo both of them will be in the ſame Plane cutting the oppoſite Superficies, ſince that Plane muſt paſs through one Side $SD$ of the Cone, and make an Angle with the Plane $SDH$, which touches the Cone in that Side $SD$. Therefore

the Plane of the Parallels $BA$, $SD$, will form two Sides of the oppoſite Cones; one of which is the Side $SD$, and the other the Side $Sa$, which neceſſarily ſhall cut the Line $BA$ in ſome Point $A$, becauſe the ſame is ſituate in the Plane paſſing through the Parallels $SD$, $AB$, and does cut $SD$ in $S$. Whence, becauſe the Point $A$ is in one of the Conick Superficies, and alſo in the Plane of the oppoſite Sections, this Point will be one Point of the Hyperbola. Moreover, becauſe the Line $BA$ being indefinitely produced towards $A$, falls wholly in the Plane $DSa$, contain'd between the Sides $DS$, $Sa$, when the Point $A$ appertains to the Hyperbola $FAG$, and in the vertical Angle $ASd$, when the ſame appertains to the oppoſite Hyperbola; therefore the Line $AB$ falls wholly within one of the Conick Superficies, and conſequently alſo within the Hyperbola, which is the Section thereof.

### COROLLARY.

272. HENCE it appears, that no Line can be drawn between the Hyperbola $FAG$, and its Aſymptote $CH$ parallel to that Aſymptote. And becauſe the Line $BA$ divides the Hyperbola into two indefinite Parts or Portions, one of which falls neceſſarily wholly within the Space contain'd between the Parallels $BA$, $CH$; therefore the more $CB$ diminiſhes, the more does the Point $A$ advance in that Portion, even until $CB$ becomes leſs than any given Magnitude; that is, if an Hyperbola and its Aſymptote be indefinitely continu'd, they will approach always nearer and nearer to each other, till at laſt their Diſtance will become leſs than any given Magnitude, and yet they will * never meet.

**Art.* 270.

## PROPOSITION VIII.

### Problem.

FIG. 140. 273. *THE Aſymptotes* CH, CK, *of any Hyperbola* FAG, *together with any one Point* F *therein being given, to deſcribe the Hyperbola.*

Draw any right Line $HK$ through the given Point $F$, terminating in the Aſymptotes, and draw any Plane (except the Plane $HCK$ of the Scheme) thro' that Line, in which draw $MN$ from the Point $P$, the Middle of $HK$ perpendicular to $HK$; and about any Point $O$ therein, as a Centre, and with the Radius $OF$, deſcribe a Circle $FMN$. From the Points $H$, $K$, draw the two Tangents $HD$, $KE$ to that Circle, and through the Points of Contact $D$, $E$, draw the Lines $DS$, $SE$, parallel to the Aſymptotes $CH$, $CK$, meeting each other in the Point $S$. Then if a Conick Superficies be deſcrib'd with the Vertex $S$, and

and the Baſe *FMN*; I ſay, the Concurrence of that Conick Surface and the Plane *HCK*, will form the Hyperbola *FAG* requir'd.

For by the Property of the Circle *FMN*, it is manifeſt, 1. That the Chord *FG* is biſected in the Point *P* by the Diameter *MN* being * at right Angles to it; and therefore, (becauſe $FH = PK$, by Conſtruction) $FH$ is $= GK$, and $GH = FK$; and conſequently $GH \times HF = FK \times KG$. 2. $GH \times HF = \overline{HD}^2$, and $FK \times KG = \overline{KE}^2$, and ſo $HD = KE$. 3. If the Tangents *HD*, *KE*, be produced meeting one another in the Point *Q*, the Parts *DQ*, *EQ*, will be equal to one another; and ſo $DQ : EQ :: DH : EK$. Whence it appears, that the Line *DE*, joining the Points of Contact of the Tangents *HD*, *KE*, will be parallel to the Line *HK*, and the Plane *SDE* to the Plane *CHK*: Therefore the Line *DE* will be * the Directrix; and becauſe the ſame cuts the Baſe in two Points, the Conick Section *FAG* will be * an Hyperbola. It is farther manifeſt, that this Hyperbola will paſs through the given Point *F*, becauſe this Point is both in the Conick Superficies and in the Plane *HCK*, being the Plane of the Hyperbola; and the Lines *CH*, *CK*, will be the Aſymptotes of that Hyperbola, as being * the Common Sections of the touching Planes *SDH*, *SEK*, and the Plane of the Hyperbola.

* *Hyp.*

* *Def.* 9.

* *Def.* 10.

* *Def.* 15.

It may ſo happen that the Tangents *DH*, *HK*, be parallel to one another, and then it is manifeſt at Sight, that the Lines *DE*, *HK*, will be parallel to one another, becauſe thoſe Tangents are equal; and the reſt of the Demonſtration is the ſame as above.

## PROPOSITION IX.

### Theorem.

274. *IF there be two Right Lines* MN, AB, *terminating in a Conick Section, or the oppoſite Sections meeting one another in the Point* P; *and if theſe Lines be parallel to two other Lines* SE, SD, *given in Poſition: I ſay the Rectangle* MP × PN *to the Rectangle* AP × PB, *will be in a given Ratio; that is, the Ratio of theſe two Rectangles will be always the ſame, let the Lines* MN, AB, *be any where drawn.* FIG. 141, 142.

Draw two Planes thro' the Parallels *SE*, *MN*, and *SD*, *AB*, then theſe Planes will form two Right Lines *Enm*, *Dba*, in the Plane of the Baſe, and the Sides of the Cone *SMm*, *SNn*, *SAa*, *SBb*, and their common Section will be the Line *SPp*, which meets the Plane of the Baſe in the Point *p*, wherein the two Right Lines *Em*, *Da*, cut one another; through which Point draw the Right Line *HK* in the Plane *SMN*, parallel to *MN*, and the Right Line *FG* in the Plane *SAB*, parallel to *AB*. This being done.

Becauſe the Triangles $SPM$, $SpH$; $SPN$, $SpK$; $SPA$, $SpF$ $SPB$, $SpG$, are ſimilar, therefore $MP \times PN : Hp \times pK :: \overline{SP}^2 : \overline{Sp}^2 : AP \times PB : Fp \times pG$. And therefore we have $MP \times PN : AP \times PB :: Hp \times pK : Fp \times pG$. And the Ratio of $Hp \times pK$, to $Fp \times pG$, is compounded of the Ratio's of $Hp \times pK$ to $mp \times pn$, and of $mp \times pn$, or, by the Property of the Circle, of $ap \times pb$ to $Fp \times pG$. But becauſe the Triangles $Hpm$, $SEm$, and $Kpn$, $SEn$, are ſimilar, therefore $Hp : mp :: SE : mE$. And $pK : pn :: SE : En$. And multiplying the Antecedents and Conſequents of theſe two Ratio's, we have $Hp \times pK : mp \times pn :: \overline{SE}^2 : mE \times En$: In like manner, becauſe the Triangles $Fpa$, $SDa$, and $Gpb$, $SDb$, are ſimilar, therefore $ap \times pb : Fp \times pG :: aD \times Db : \overline{SD}^2$. Whence it is manifeſt, that the Ratio of $MP \times PN$ to $AP \times PB$, is compounded of the Ratio of $\overline{SE}^2$ to $mE \times En$, and of $aD \times Db$ to $\overline{SD}^2$. Which two laſt Ratio's, by the Property of the Circular Baſe of the Cone, always are the ſame whereſoever the Lines $MN$, $AB$ are drawn, becauſe the Points $E$, $D$ do not vary. Therefore the Rectangle $MP \times PN$ to the Rectangle $AP \times PB$ is in a given Ratio. *W. W. D.*

### COROLLARY.

Fig. 143, 144. 275. HENCE, in any Conick Section, or the oppoſite Sections, if there be any two parallel right Lines $MN$, $OR$, meeting a third right Line $AB$ bounded by the Section, in the Points $P$, $Q$; I ſay, $MP \times PN : OQ \times QR :: AP \times PB :: AQ \times QB$.

## PROPOSITION X.

### Theorem.

Fig. 145. 276. *IF the right Line* AB *be drawn through any Point* A *in a Parabola, or Hyperbola* MAN, *parallel to the Side* SD *of the Cone, which, if the Section be a Parabola, paſſes through the Point* D, *wherein the Directrix touches the Baſe; or, if an Hyperbola, through one of the Points wherein the ſame cuts the Baſe; and if the Line* MN *be drawn through any Point* P *in the ſaid Line* AB, *parallel to the Line* SE *given in Poſition, and terminating in the Section, or oppoſite Sections, as alſo another Line* FG, *parallel to the Line* Da, *the common Section of the Plane* SAB, *and the Plane of the Baſe, and terminating in the Sides* Sa, SD; *I ſay, the Rectangle* MP × PN *to the Rectangle* FP × PG *is given, that is, it will be always the ſame in whatſoever Part of the Line* AB *the Point* P *be taken.*

If

If a Plane be drawn through the Parallels $SE$, $MN$, this will form the right Line $Enm$ in the Plane of the Baſe; the Sides $SMm$, $SNn$, in the Conick Superficies; and the Line $SPp$ in the Plane $SDa$, which Line meets the Baſe in the Point $p$, wherein the Lines $Em$, $Da$, interſect one another; and through this Point draw the Line $HK$ in the Plane $SMN$, parallel to $MN$. This being laid down, the ſimilar Triangles $SPM$, $SpH$; $SPN$, $SPK$; $SPF$, $Spa$; $SPG$, $SPD$, will give the following Proportions, $MP \times PN : Hp \times pK :: \overline{SP}^2 : \overline{Sp}^2 :: FP \times PG : ap \times pD$, or $mp \times pn$ (by the Property of the Circle.) And therefore we have $MP \times PN : FP \times PG :: Hp \times pK : mp \times pn$. But the Ratio of $Hp \times pK$ to $mp \times pn$ is compounded of the Ratio of $Hp$ to $pm$, and of $pK$ to $pn$, that is, (by the Similarity of the Triangles $Hpm$, $SEm$, and $Kpn$, $SEn$) of the Ratio of $SE$ to $Em$, and of $SE$ to $En$; and conſequently $Hp \times pK : mp \times pn$, or $MP \times PN :: FP \times PG : \overline{SE}^2 : mE \times En$. Whence becauſe the Poſition of the Point $E$ is the ſame, let the Point $P$ be taken at pleaſure, and all the Rectangles $Em \times En$, by the Nature of the Circle are equal to one another; therefore $MP \times PN$ is to $FP \times PG$ in a given Ratio. *W.W.D.*

### Corollary.

277. HENCE, in a Parabola or Hyperbola ($MAN$) if a Diameter ($AB$) be drawn through any Point $A$ in the Parabola, or the Line $AB$ parallel to one of the Aſymptotes in the Hyperbola; and if the two Parallels $MN$, $OR$, are drawn from any two Points $P$, $Q$, in the Line $AB$, terminating in the Section or oppoſite Sections; then we ſhall have this Proportion, $MP \times PN : OQ \times QR :: AP : AQ$. Fig. 146.

For draw the Plane $SAB$ forming the Sides of the Cone $SD$, $Sa$, ſuch, that $SD$ may paſs through the Point wherein the Directrix touches the Baſe, when the Section is a Parabola, and through one Point wherein the Directrix cuts the Baſe, when the Section is an Hyperbola: Alſo thro' the Points $P$, $Q$, draw the right Lines $FG$, $TV$, in the Plane $SDA$ parallel to $Da$; then it is evident, (by the laſt Propoſition) that $MP \times PN : FP \times PG :: OQ \times QR : TQ \times QV$. And ſo $MP \times PN : OQ \times QR :: FP \times PG : TQ \times QV$. But $PG$, $QV$, the Parts of the Lines $FG$, $TV$, are equal to one another; becauſe the Lines $AB$, $SD$, are parallel. And therefore $MP \times PN : OQ \times QR :: FP : TQ :: AP : AQ$ (by the Similarity of the Triangles $APF$, $AQT$. Whence, *&c.*

## CHAP. II.

### *Of the Ellipſis only.*

#### DEFINITIONS.

17.

FIG. 147. IF any indefinite right Line *SZ*, which is without the Plane of the Circle *VXY*, moves about the Circumference of that Circle always parallel to it ſelf, until it be returned to the ſame Place from which it went; then the Convex Superficies deſcrib'd by the Motion of the Line *SZ*, is call'd a *Cylindrick Superficies*, or the *Superficies of a Cylinder*.

18.

That Line *SZ* being in any different Poſition, is called always a *Side* of the Cylindrick Superficies.

19.

The Circle *VXY* is the *Baſe*.

20.

The indefinite right Line *CO*, drawn from *C* the Centre of the Baſe parallel to the Sides, is the *Axis* thereof.

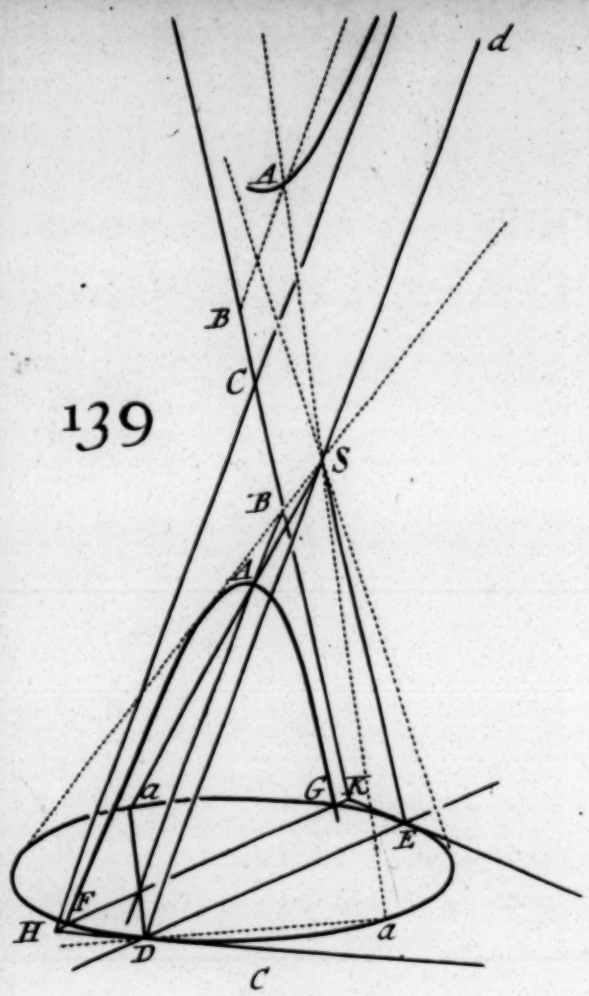
21.

The indefinite Solid comprehended under the Baſe *VXY*, and the Cylindrick Superficies, is call'd a *Cylinder*.

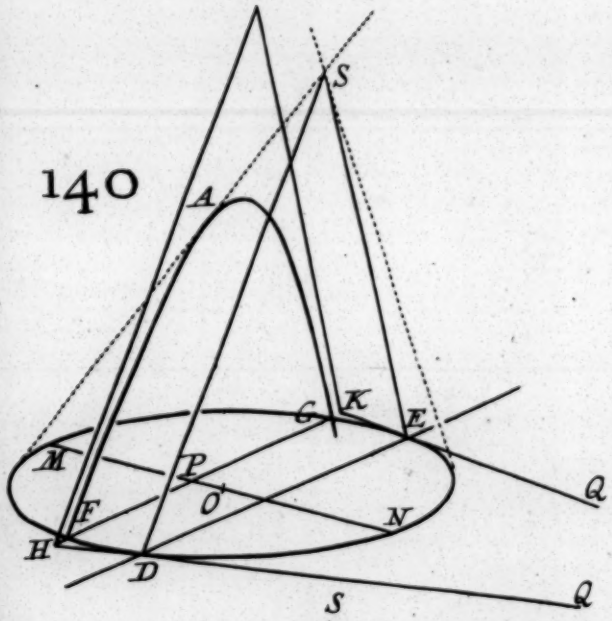
22.

If a Cylinder be cut by any Plane not parallel to its Baſe, or the Sides thereof; then the Curve *AMBN* formed thereby in the Conick Superficies, is called a *Cylindrick Section*.

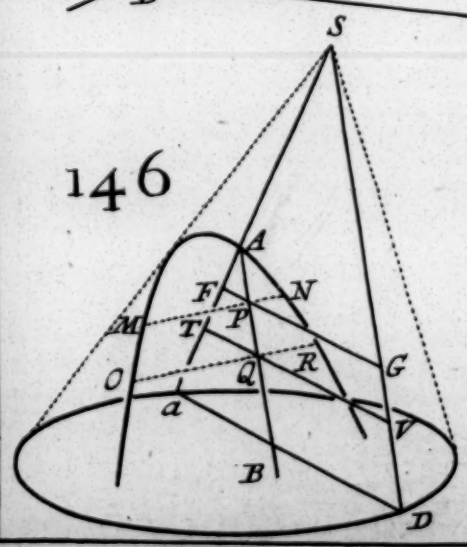
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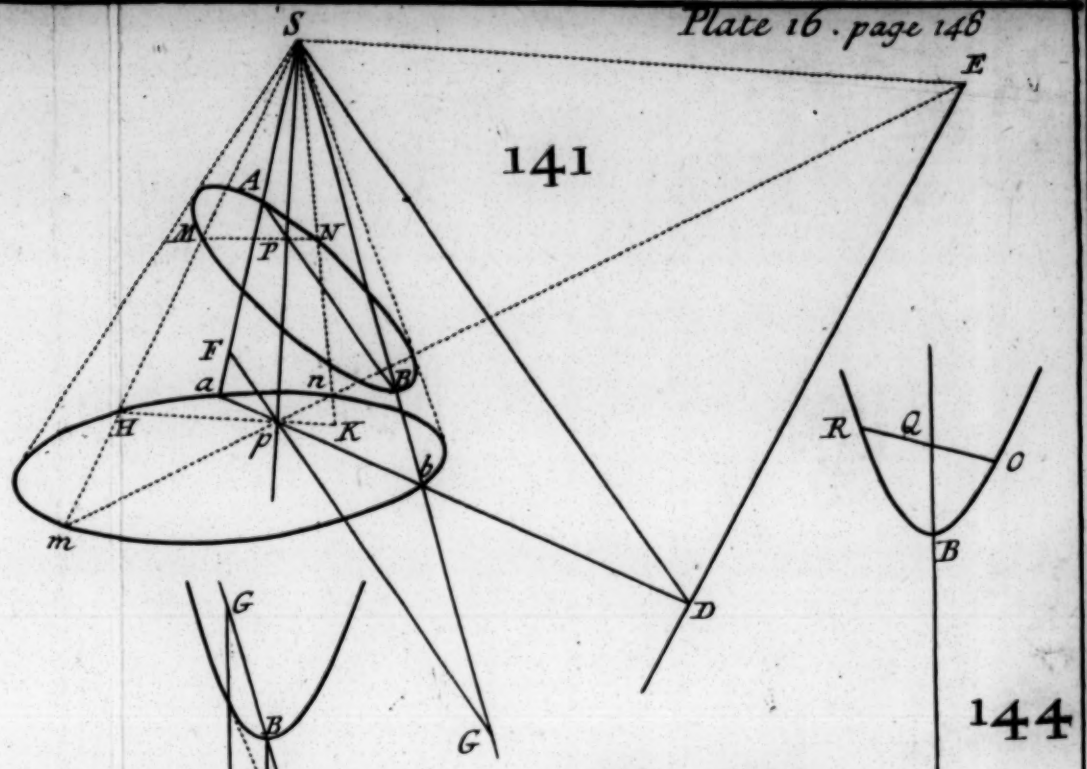

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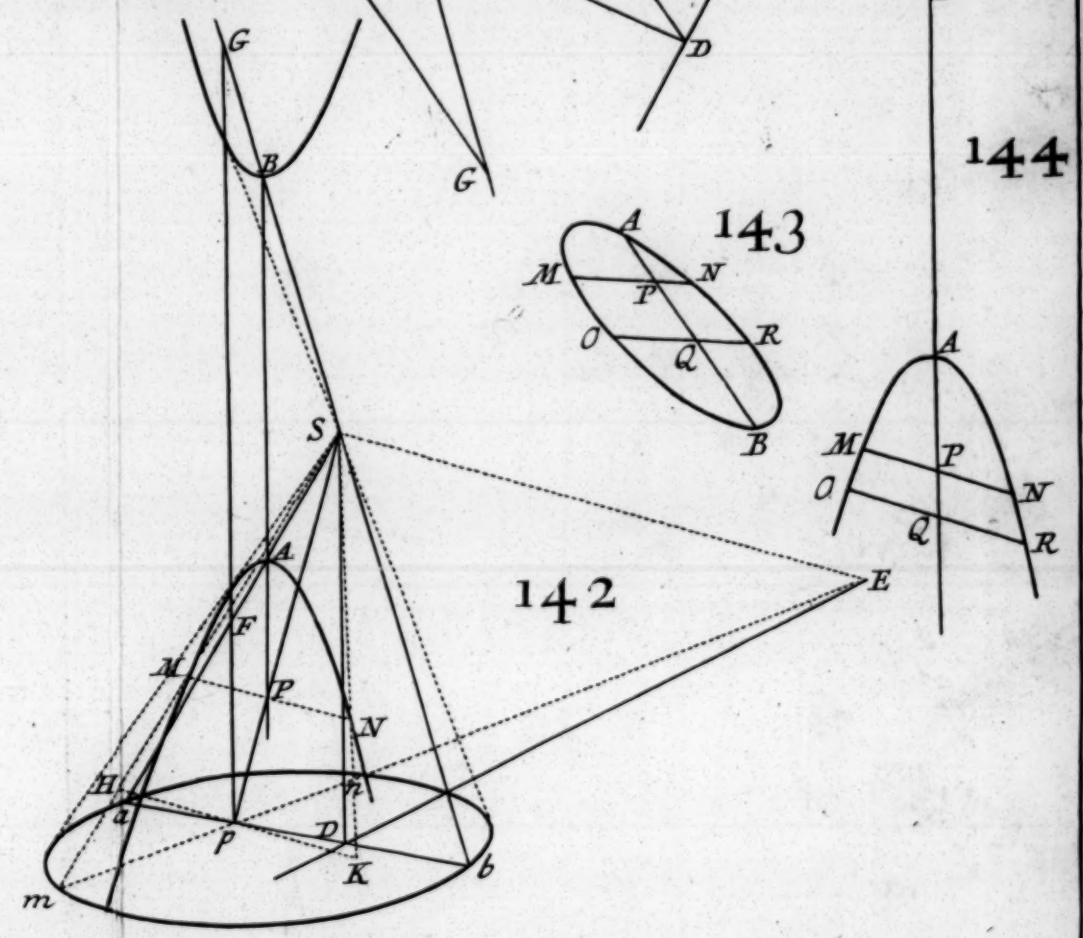

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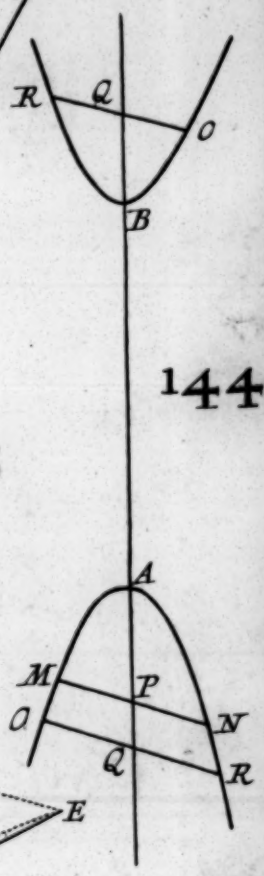

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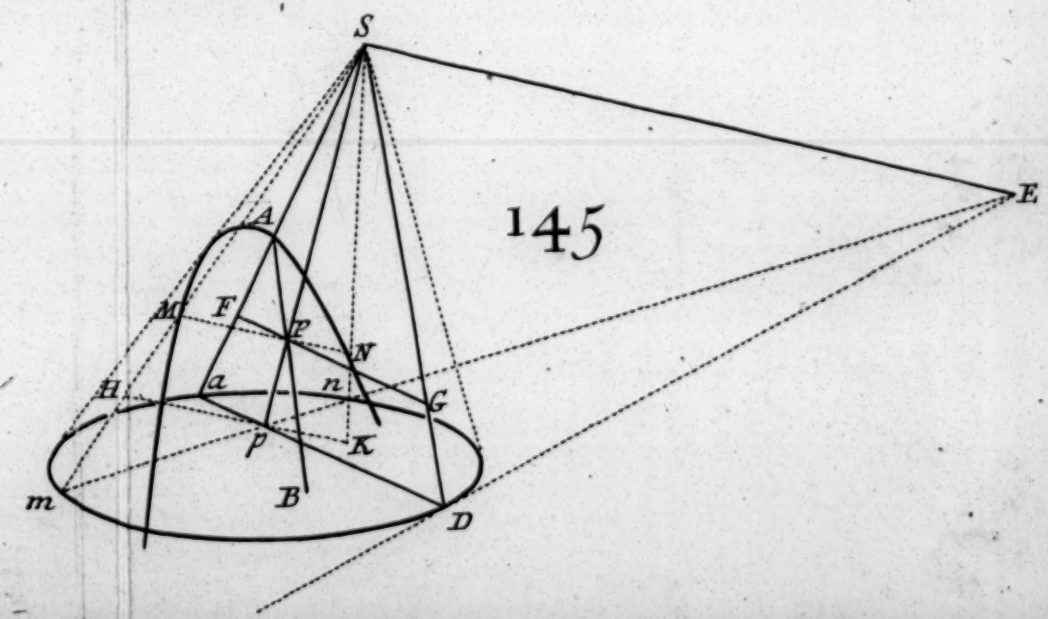

143




144




145



## PROPOSITION XI.

### Theorem.

278. *IF any Cylinder be cut by a Plane* (v x y) *parallel to the Plane of the Baſe* (V X Y); *then the Section* v x y *will be a Circle, the Point* c *wherein that Plane meets the Axis, the Centre; and the Line* c x *equal to the Radius of the Baſe* C X, *the Radius thereof.* FIG. 147.

For if the Side of the Cylindrick Superficies *x X* be drawn thro' any Point *x* in the Section *v x y*, this Side will be * parallel to the Axis *Cc*; therefore a Plane may be drawn through thoſe two Lines forming the two right Lines *C X*, *c x*, in the parallel Planes *CVXY*, *cvxy*, parallel between themſelves; which moreover will be equal to one another, as being included between the Parallels *Cc*, *Xx*. And becauſe this is ſo always, let the Point *x* be taken any whereſoever in the Section *v x y*; therefore all the Lines *c x*, drawn from the Point *c* to the Points *x* in the Section *v x y*, are equal to the Radii *C X* of the Baſe; that is, the Section *v x y* will be the Circumference of a Circle, the Point *c*, wherein the Plane *v x y* meets the Axis of the Cylinder, the Centre, and the Line *C x* equal to the Radius (*CX*) of the Baſe, the Radius thereof. *W. W. D.* * *Def.* 20.

## PROPOSITION XII.

### Theorem.

279. *EVery Ellipſis may be conſidered as the Section of a Cylinder.* FIG. 148.

Draw the Diameter *a b* in the Baſe of a Cone, wherein is produced any Ellipſis, meeting the Directrix *D E* at right Angles in the Point *D*; alſo draw the Sides *S a*, *S b*, of the Cone meeting the Plane of the Ellipſis in the Points *A*, *B*; and draw the right Lines *A B*, *S D*, in the parallel Planes *A M B*, *S D E*. This being done, aſſume (*D F*) a mean Proportional between *a D*, *D b*, draw *A G*, *B H*, parallel to *S F*, and deſcribe a Circle upon the Plane of the Cone's Baſe, with the Line *G H* for a Diameter, as alſo a Cylindrick Superficies with that Circle as a Baſe; and the right Lines *A G*, *B H*, as Sides. This being premis'd.

If a right Line be drawn through any Point *P* in the Line *A B*, parallel to the Directrix *D E*, meeting the Conick Superficies in *M*, and the Cylindrick Superficies in *O*; then, I ſay, the Points *M* and *O* will coincide, or fall both in one Point.

For, if a Plane be drawn through that parallel Line, parallel to the Planes of the Baſe of the Cone and Cylinder; this Plane will form * * *Art.* 259.

the

the Circle $KML$ in the Conick Superficies, whoſe Centre will be the common Section of that Plane, and the Axis of the Cone, together with the Circle* $QMR$ in the Cylindrick Superficies, whoſe Centre will be the common Section of the aforeſaid Plane, and the Axis of the Cylinder. But the Plane $Sab$ does paſs through * the Axis of the Cone, and the Plane $AGHB$ (coinciding with the Plane of the Triangle $Sab$) through * the Axis of the Cylinder; and conſequently the Lines $KL$, $QR$, being the common Sections of thoſe two Planes, and the Plane parallel to the Baſe paſſing through the Line $POM$, will be Diameters of the ſaid two Circles; and the Line $POM$ will be perpendicular to thoſe Diameters; becauſe the ſame is * parallel to $DE$, and $DE$ is * perpendicular to $ab$ and $GH$ (both being in the ſame * ſtrait Line) and the Diameters $KL$, $QR$, (being both in the ſame ſtrait Line) are parallel to $ab$ and $GH$. Moreover, the Lines $AB$, $SD$, being the common Sections of two parallel Planes, and the Plane $Sba$, *viz.* the Plane $SDE$, and the Plane of the Ellipſis, will be equal to one another. This being well underſtood,

*Art. 278.
* Def. 6.
* Def. 20.
* Hyp.
* Hyp.
* Hyp.

1. In the Cone, we have $\overline{PM}^2 = KP \times PL$, by the Nature of the Circle $KML$; and becauſe the Triangles $APK$, $SDa$, and $PBL$, $SDb$, are ſimilar, we ſhall get this Proportion, $AP : KP :: SD : aD$. And $PB : PL :: SD : Db$. Therefore $AP \times PB : KP \times PL$ or $\overline{PM}^2 :: \overline{SD}^2 : aD \times Db$.

2. In the Cylinder, we have $\overline{PO}^2 = QP \times PR$, by the Nature of the Circle $QOR$; and becauſe the Triangles $APQ$, $SDF$, and $PBR$, $SDF$, are ſimilar, we ſhall get the two following Proportions, *viz.* $AP : PQ :: SD : DF$; and $PB : PR :: SD : DF$; therefore $AP \times PB : QP \times QR$ or $\overline{PO}^2 :: \overline{SD}^2 : \overline{DF}^2$ or $aD \times Db$; and conſequently $\overline{PM}^2 = \overline{PO}^2$, and $PM = PO$. Whence the Points $M$, $O$, coincide, or both fall in one. And becauſe this happens always, let the Point $P$ be taken any whereſoever in the Line $AB$, therefore the Plane of the Ellipſis meets both the Conick and Cylindrick Superficies in the ſame Points, and ſo any Ellipſis may be conſider'd as a Cylindrick Section.

## ADVERTISEMENT.

Becauſe a Cylinder is more ſimple than a Cone, in having all the Sides thereof parallel to one another, whereas the Sides of a Cone do all terminate in the Vertex; therefore we ſhall conſider an Ellipſis in this Chapter, as being the Section of a Cylinder, and demonſtrate the Properties of its Diameters from the Cylinder, which may be very eaſily done; and afterwards, in the next Chapter, we ſhall ſhew the extreme

extreme Facility in proving the ſame Properties of the Diameters of the Parabola and Hyperbola, by ſuppoſing Cones to have Elliptick Baſes inſtead of Circular ones.

## PROPOSITION XIII.

### Theorem.

280. *ALL Diameters of an Ellipſis do cut one another in one Point only,* viz. *that Point wherein the Plane of the Ellipſis meets the Axis of the Cylinder, and are biſected in that Point.* Fig. 149.

*And contrariwiſe, all Lines drawn through that Point, and terminating both ways in the Ellipſis, are biſected in that Point, and are Diameters of the Ellipſis.*

Note, The aforeſaid Point is call'd the *Centre* of the Ellipſis.

1. Let *AB* be any Diameter, and *C* the Point wherein the Plane of the Ellipſis meets the Axis of the Cylinder. Now if the Lines *Aa*, *Bb*, be drawn parallel to the Axis *Cc*, then it is manifeſt, * that the ſame will be Sides of the Cylindrick Superficies, and the two Planes *FAa*, *GBb*, paſſing through theſe two Lines, and the two Tangents *AF*, *BG*, (which by the Definition of Diameters muſt be parallel to one another) will be parallel between themſelves, and touch the Cylindrick Superficies in the Sides *Aa*, *Bb*; whence thoſe two Planes will form the Lines *af*, *bg*, in the Plane of the Baſe, parallel to one another, and touching the Baſe in the Points *a*, *b*, wherein the Sides *Aa*, *Bb*, meet it. But it is demonſtrated, in the Elements of Geometry, that the Line *ab* joining the Points of Contact of two parallel Tangents (*af*, *bg*) to a Circle, paſſes through the Centre *c*; therefore the Plane *AabB* will paſs through *Cc* the Axis of the Cylinder; and the Line *AB*, which is the common Section of that Plane, and the Plane of the Ellipſis, will paſs through the Point *C*, wherein the Axis meets the Plane of the Ellipſis. Moreover, becauſe the Lines *Aa*, *Bb*, *Cc*, are parallel; it is manifeſt, that the Diameter *AB* of the Ellipſis is biſected in the Point *C*, becauſe the Diameter *ab* of the Circle is biſected in the Centre (*c*) of the ſame: Which was to be demonſtrated in the firſt place. * *Def.* 20.

2. If the Lines *Aa*, *Bb*, be drawn through *A*, *B*, the Extremities of any Line *AB*, (paſſing through the Centre *C*, wherein the Plane of the Ellipſis meets the Axis (*Cc*) of the Cylinder) parallel to that Axis; then it is manifeſt (by *Def.* 17.) that theſe Lines wil be Sides of the Cylinder, and the Plane *AabB* will paſs through the Axis *Cc*. Therefore the Line *ab* being the common Section of that Plane, and the Plane of the Baſe, will paſs through *c* the Centre of the Baſe; and ſo ſince the ſame is biſected in *c*, the Line *AB* will be biſected alſo

in.

in *C*. Moreover, becauſe the Tangents *a f*, *b g*, paſſing through the Extremities of the Diameter *a b*, are parallel to one another; therefore the touching Planes *f a A*, *g b B*, will be parallel to one another, and form two parallel Lines *A F*, *B G*, in the Plane of the Ellipſis, which will touch the ſame in *A*, *B*, the Extremities of the Line *A B*, and ſo *A B* will be a Diameter of the Ellipſis. *Which was to be demonſtrated in the ſecond place.*

### COROLLARY.

281. HENCE there can but one Diameter be drawn through a given Point (beſides the Centre) in the Plane of an Ellipſis.

## PROPOSITION XIV.

### Theorem.

Fig. 149. 282. *EVery Line* M P N *being an Ordinate on both Sides to any Diameter* A B, *is biſected by that Diameter in the Point* P.

*And contrariwiſe, if any right Line* M P N, *terminating in an Ellipſis, and not paſſing through the Centre* C, *be biſected by the Diameter* A B *in the Point* P; *then that Line will be an Ordinate on both Sides to that Diameter.*

Draw the Sides *A a*, *B b*, *M m*, *N n*, through the Points *A*, *B*, *M*, *N*, parallel to *C c* the Axis of the Cylinder, and meeting the Plane of the Baſe in the Points *a*, *b*, *m*, *n*; then the Line *P p*, being the common Section of the Planes *A a b B*, *M m n N*, will be parallel to the Sides of the Cylinder, becauſe all the Sides are parallel to one another. Moreover, the Plane *A a b B* will paſs through *C c* the Axis of the Cylinder, becauſe the Diameter *A B* paſſes through the Point *C*, wherein that Axis meets the Plane of the Ellipſis; and conſequently does form a Line *a b* in the Plane of the Baſe, which paſſes through (*c*) the Centre, that is, a Diameter. This being laid down,

Becauſe the Line *M P N*, (by Suppoſition) is an Ordinate both ways to the Diameter *A B*, the ſame will be parallel to the Tangents *A F*, *B G*, paſſing through the Extremities of that Diameter; and conſequently the touching Planes *F A a*, *G B b*, will be parallel to the Plane *M m n N*. Therefore the Lines that thoſe three Planes form in the Plane of the Baſe, *viz.* the two Tangents *a f*, *b g*, and the Line *m n*, will be parallel to one another; and ſo the Line *m n* will be perpendicular to the Diameter *a b*, which conſequently divides it into two equal Parts in the Point *p*. Therefore, becauſe *M m*, *P p*, *N n*, are parallel, the Line *M N* will be biſected likewiſe in the Point *P*.

Now for proving the Converſe, draw two Tangents *A F*, *B G*, in
* *Art.* 267. the Plane of the Ellipſis, parallel * to *M N*; then if the Diameter

*A B*

*AB* be drawn through the Points of Contact; it is manifeſt (by the 13th and 14th Definitions) that the Line *MN* will be an Ordinate both ways to that Diameter; and conſequently, by what has been already demonſtrated, is biſected in *P* by the ſame. And becauſe there can be drawn * but one Diameter through *P*, therefore if any Line *MN*, terminating in an Ellipſis, and not paſſing through the Centre *C*, be biſected by a Diameter *AB* in the Point *P*, then that Line *MN* will be an Ordinate both ways to *AB*.

* *Art.* 281.

## PROPOSITION XV.

### Theorem.

283. *IN any Ellipſis, if there be two Diameters* AB, DE; *and if one of them, as* DE *be parallel to the Tangents* AF, BG, *paſſing thro' the Extremities of the other* AB; *then reciprocally, I ſay, the Diameter* AB *will be parallel to the Tangents paſſing through the Extremities of the Diameter* DE. FIG. 149.

The two Diameters *AB*, *DE*, are called *Conjugates* to one another.

Draw the Sides (*Aa*, *Bb*, *Dd*, *Ee*) of the Cylinder through the Points *A*, *B*, *D*, *E*, which meet the Plane of the Baſe in the Points *a*, *b*, *d*, *e*; then the Planes *AabB*, *DdeE*, ſhall paſs through *Cc* the Axis of the Cylinder, becauſe the Lines *AB*, *DE*, are Diameters of the Ellipſis; and conſequently will form two Diameters *ab*, *de*, in the Plane of the Baſe. But the touching Plane *FAa*, being parallel to the Plane *DdeE*, will form a Tangent *af* in the Plane of the Baſe parallel to the Diameter *de*, which Diameter will be conſequently perpendicular to the Diameter *ab*. Whence, if the Tangent *db* be drawn to the Circle, it will be parallel to *ab*, and the Plane *bdD* parallel to the Plane *AabB*; therefore the common Sections of thoſe two Planes, and the Plane of the Ellipſis, *viz.* the Tangent *DH*, and the Diameter *AB*, are parallel to one another. We prove the ſame with regard to the Tangent paſſing thro' the other Extremity (*E*) of the Diameter *DE*. Therefore, *&c.*

### COROLLARY I.

284. HENCE, in an Ellipſis, if *AB*, *DE*, be two Conjugate Diameters; then the two Planes paſſing through thoſe Diameters, and the Axis *Cc* of the Cylinder, ſhall form two Diameters *ab*, *de*, in the Plane of the Baſe; which will be perpendicular to one another: And contrariwiſe.

### COROLLARY II.

285. IT is manifeſt moreover by this Propoſition, if the double Ordinate *MPN* be drawn to the Diameter *AB*, through any Point *P* in the ſame; that *MPN* will be parallel to the Diameter *DE*,

*Art. 275. which is a Conjugate to $AB$; and ſo we ſhall have * this Proportion, $MP \times PN$, or $\overline{PM}^2 : DC \times CE$, or $\overline{DC}^2 :: AP \times PB : AC \times CB$ or $\overline{AC}^2$. From whence we get $\overline{PM}^2 : AP \times PB :: \overline{DC}^2 : \overline{AC}^2 :: 4\overline{DC}^2$ or $\overline{DE}^2 : 4\overline{AC}^2$ or $\overline{AB}^2$; that is, the Square of any Ordinate ($MP$) to a Diameter ($AB$) to the Rectangle ($AP \times PB$) under the Parts of that Diameter, is as the Square of the Diameter $DE$, being a Conjugate thereto, to the Square of the Diameter $AB$.

## PROPOSITION XVI.

### Theorem.

FIG. 150. 286. *IF through any Point* M *in the Ellipſis* A M B, *there be drawn a Tangent* F M G *meeting two other parallel Tangents* A F, B G, *in the Points* F, G; *I ſay,* F M : M G : : A F : B G.

Draw the Sides of the Cylinder $Aa$, $Bb$, $Mm$, through the Points of Contact $A$, $B$, $M$; and let three Planes $FAa$, $GBb$, $FMm$, or $GMm$ paſs through thoſe Sides, and the Tangents $AF$, $BG$, $FG$; then it is plain, that the common Sections $Ff$, $Gg$, of the two firſt Planes, and the third, will be parallel between themſelves, and to the Sides of the Cylinder; for ſince the two Planes $FMm$, $FAa$, paſs through the Sides $Mm$, $Aa$, which are parallel to one another; their common Section $Ff$, will be parallel to thoſe Sides; and by the ſame Reaſon $Gg$, the common Section of the two Planes $GBb$, $GMm$, will be parallel to the Sides $Bb$, $Mm$. Moreover, the Lines $af$, $bg$, which the parallel touching Planes $FAa$, $GBb$, form in the Plane of the Baſe, will be parallel Tangents to the Baſe; and $fm$, $mg$, the Parts of the third Tangent form'd in the Plane of the Baſe by the third touching Plane $FMm$, or $GMm$, ſhall be equal (by the Nature of the Circle) to the Tangents $af$, $bg$; viz. $fm = fa$, and $mg = gb$. This being premis'd:

Becauſe the Lines $Aa$, $Ff$, $Mm$, $Gg$, $Bb$; as alſo $AF$, $BG$; and $af$, $bg$, are parallel, therefore $FM : MG :: fm$, or $fa : mg$, or $gb :: FA : GB$. *W. W. D.*

### COROLLARY I.

287. IF a Diameter $AB$ be drawn through $A$, $B$, the Points of Contact of two parallel Tangents $AF$, $BG$, meeting the Tangent $FMG$ in $T$; and if the Ordinate $MP$ be drawn to that Diameter; then it is manifeſt, that $AP : PB :: FM : MG :: AF : BG :: AT : BT$. And ſo $PB - AP : PB :: BT - AT$, or $AB : BT$.

COROL.

### Corollary II.

288. HENCE ariſes the following Way of drawing a Tangent $MT$ to touch an Ellipſis in the given Point $M$, by having a Diameter $AB$, and the Poſition of the Ordinates to the ſame given.

From $B$, one End of the Diameter $AB$, draw the right Line $BM$ to the given Point $M$; then having drawn the Ordinate $MP$ to the Diameter $AB$, and taken $PH=PA$ in the Diameter $AB$ towards $B$, draw $HK$ parallel to $PM$, meeting the Line $BM$ in $K$, through which, and the other Extremity $A$ draw $AK$. Finally, draw $MT$ parallel to $AK$, and the ſame will be the Tangent requir'd.

For becauſe $MP$, $HK$, and $AK$, $MT$, are parallel; therefore $BP : PH$, or $PA :: PM : MK :: BT : TA$.

### Corollary III.

289. IN an Ellipſis, if there be two Tangents $MT$, $NT$, meeting one another in the Point $T$; I ſay, the Diameter $AB$ paſſing through $P$ the Middle of the Line $MN$ joining the two Points of Contact, will paſs likewiſe through the Point $T$. For $PN$, $PM$ are each Ordinates to the Diameter $AB$; and conſequently * the Tangents $MT$, $NT$ will each meet that Diameter in one Point $T$, being ſuch, that $PB-AP : PB :: AB : BT$; that is, in the ſame Point. **Art.* 287.

### Corollary IV.

290. IN an Ellipſis, if $M$, $N$, the Points of Contact of two Tangents $MF$, $NL$, be join'd by a right Line $MN$; and if ſome third Tangent $FAL$ be parallel to $MN$; I ſay, $FA$, $AL$, the Parts of this laſt Tangent taken between the Point of Contact $A$, and the two firſt Tangents, ſhall be equal to one another. For draw the Diameter $AB$ through the Point of Contact $A$. then it is manifeſt that the Line $MN$ is an Ordinate both ways to that Diameter, becauſe the ſame is parallel to the Tangent $FL$, paſſing through $A$ the Extremity thereof; and ſo $AB$ biſects $MN$ in $P$, and conſequently paſſes * thro' $T$, the Point of Concurrence of the Tangents $MF$, $NL$; or elſe will be parallel to them, if the Line $MN$ be * a Diameter. And in both Caſes it is evident, that $FL$ will be biſected by the Diameter $AB$ in the Point $A$; becauſe $MN$ is biſected by the ſame Diameter in the Point $P$. **Art.* 289. **Art.* 283.

## CHAP. III.

### *Of the Parabola and Hyperbola only.*

### PROPOSITION XVII.

Theorem.

Fig. 151. 291. *IN the Parabola, every Ordinate* (MPN) *both ways to the Diameter* AB, *is biſected by that Diameter in the Point* P. *And contrariwiſe.*

Draw an Elliptick Plane through the Line *MN*, and this will form a Tangent *DE* in the touching Plane *SDE* (parallel to the Plane of the Parabola) parallel to *MN*. Farther, the Plane *SAF* drawn through *S* the Vertex of the Cone, and the Tangent *AF*, paſſing thro' *A* the Origin of the Diameter *AB*, will form the Tangent *af*, in the Elliptick Plane; and the Line *Da* joining the Points of Contact of the two Tangents *DE*, *af*, ſhall paſs through the Point *P*; becauſe the Diameter *AB* is parallel to the touching Side *SD*. This being laid down:

* *Def.* 14. Becauſe the two Lines *AF*, *MN*, by Suppoſition, * are parollel to each other, therefore the Tangent *af*, being the common Section of the two Planes which paſs through *AF*, *MN*, will be parallel to *MN*;
* *Def.* 13. and conſequently parallel to *DE*. From whence * it appears, that the Line *Da*, which joins the Points of Contact of the two parallel Tangents *DE*, *af*, is a Diameter of the Ellipſis; and ſo the Line *MN*, which is parallel to thoſe Tangents, and bounded by the Ellip-
* *Art.* 282. ſis, will be biſected * in the Point *P*.

Now for proving the Converſe: In the Plane of the Parabola, draw
* *Art.* 267. * the Tangent *AF* parallel to the Line *MN*, and the Diameter *AB*
* *Def.* 14. thro' the Point of Contact *A*; then the Line *MN* will be * an Ordinate both ways to the ſaid Diameter, and will be biſected thereby in the Point *P*, as we have demonſtrated. And becauſe there is but one
* *Art.* 268. Diameter only, that can * paſs thro' *P* the Middle of the Line *MN*; therefore, *&c.*

#### COROLLARY.

292. HENCE, if through any two Points *P*, *Q*, in the Diameter *AB*, there be drawn two Ordinates *MPN*, *OQR*, on both
* *Art.* 277. Sides thereof; it is manifeſt, that we ſhall have always * this

Proportion, *viz.* $MP \times PN$ or $\overline{PM}^2 : OQ \times QR$ or $\overline{QO}^2 :: AP : AQ$. That is, the Squares of any two Ordinates $PM$, $QO$, to the Diameter $AB$, will be always to one another as the Parts $AP$, $AQ$, of that Diameter taken from the Origin $A$ to the Points of Concurrence of the ſame Ordinates.

## PROPOSITION XVIII.

### Theorem.

293. *IF through any Point* M *in a Parabola, the Ordinate* M P *be drawn to any Diameter* AB, *as alſo the Tangent* M T *meeting that Diameter (produced beyond* A*) in the Point* T; *then, I ſay, the Parts* (A P, A T) *of the ſaid Diameter, will be equal to one another.* FIG. 152.

The ſame Conſtruction remaining as in the laſt Propoſition, thro' the Vertex of the Cone $S$, and the Tangent $MT$, draw the touching Plane $STM$, which will form the Tangent $MH$ in the Elliptick Plane, and meet the Diameter $Da$ of the Ellipſis in the Point $H$ thro' which the Line $ST$ will paſs; alſo draw the right Line $TG$ parallel to $SA$. This being well underſtood, we ſhall have * this Proportion, $DH : Ha :: DP : Pa$, and (*alternando*) $DH : DP :: Ha : Pa$. But becauſe $AB$, $SD$; and $SA$, $TG$, are parallel; therefore $DH : DP :: SH : ST :: Ha : Ga$. And ſo $Ha : Pa :: Ha : Ga$. Conſequently $Pa = Ga$, and $AP = AT$. *W. W. D.* *Art. 287.

## PROPOSITION XIX.

### Theorem.

294. *IN the oppoſite Sections, every Diameter* AB *paſſes through* C, *the Point wherein the two Aſymptotes cut each other, and is biſected by that Point: And contrariwiſe.* FIG. 152.

This Point is called the *Centre*.

Let $HSh$ be one of the two common Sections of a Plane parallel to the Plane of the Hyperbola, and the oppoſite Superficies; and let $FG$ be an Aſymptote formed by the Concurrence of the Plane of the Hyperbola, and that Plane which touches the oppoſite Superficies in the Line $HSh$. Alſo through the parallel Tangents $AF$, $BG$, (paſſing through the Ends of the Diameter $AB$, and meeting the Aſymptote $FG$ in the Points $FG$) let there be drawn two parallel Elliptick Planes; then theſe Planes will form the parallel Tangents $FH$, $Gbf$, in the touching Plane paſſing thro' the Side $HSh$, and the parallel Tangents $AF$, $af$, in the touching Plane $SAF$.

This being premis'd, the Parallels $FH$, $Gb$, being included between the two Parallels $FG$, $Hb$, will be equal to one another; and the

the ſimilar Triangles $SHF$, $Sbf$, and $SFA$, $Sfa$, will give this Proportion, *viz.* $HF : bf :: SF : sf :: FA : fa$. And therefore $HF : FA :: bf : fa ::$ * $bG : GB$. And ſo ſince $HF = bG$, it follows, that $AF = BG$, and $AC = CB$, becauſe the Triangles $ACF$, $BCG$, are ſimilar; that is, the Aſymptote $FG$ paſſes through $C$ the Middle of the Diameter $AB$. After the ſame way we prove, that the other Aſymptote will paſs through $C$. Whence it appears, that the Diameter $AB$ paſſes thro' $C$ the Interſection of the Aſymptotes, and is biſected by the ſame.

* *Art.* 268.

Now let the Line $AB$, paſſing through the Interſection ($C$) of the Aſymptotes, meet the oppoſite Sections in the Points $A$, $B$. And if the Tangent $AF$ be drawn through the Point $A$, and the Tangent $DG$, be drawn * to the oppoſite Hyperbola parallel to $AF$; then it is evident (as has been demonſtrated before) that the Line $AD$, which joins the Points of Contact of the Tangents $AF$, $DG$, being a Diameter, will paſs through $C$ the Interſection of the Aſymptotes. And therefore the ſame will coincide with the Line $AB$, which paſſes * likewiſe through the ſame two Points $A$, $C$; that is, the Point $D$ will coincide with the Point $B$. Whence the Line $AB$ will be a Diameter, and conſequently will be biſected in the Point $C$.

* *Art.* 267.

* *Hyp.*

## COROLLARY.

295. HENCE it appears, that there can be drawn but one Diameter through a given Point within an Hyperbola; becauſe there can but one Line only be drawn through that Point and the Centre.

## PROPOSITION XX.

### Theorem.

FIG. 153. 296. *IN the oppoſite Sections, every Ordinate* (MPN) *both ways to the Diameter* AB, *is biſected by that Diameter in the Point* P. *And contrariwiſe.*

Draw an Elliptick Plane through the Line $MN$, which will form two Tangents $af$, $bg$, in the touching Planes $SAF$, $SBG$; and the Line $ab$, joining the Points of Contact of thoſe two Tangents, being the common Section of the Elliptick Plane, and the Plane $SAB$, will paſs through the Point $P$. But becauſe the two Lines $AF$, $MN$ are parallel to one another (by Suppoſition) therefore the Line $af$, which is the common Section of the two Planes paſſing through $AF$, $MN$, will be parallel to $MN$. By the ſame Reaſon the Tangent $bg$ being the common Section of the Elliptick Plane, and the touching Plane $SBG$, which paſs through the two Parallels $MN$, $BG$, will be parallel to $MN$. Therefore the two Tangents $af$, $bg$ will be parallel to one another: Whence it follows, that the Line $ab$ * is a Diameter

* *Def.* 13.

meter of the Ellipſis; and ſo the Line $MN$* is biſected in the Point $P$. *Art. 282.

Now for proving the Converſe, draw* two Tangents $AF$, $BG$, in the Plane of the Hyperbola's, parallel to the Line $MN$ bounded by the Hyperbola; and draw the Diameter $AB$ through their Points of Contact; then it is manifeſt (by *Def.* 14.) that $MN$ will be an Ordinate both ways to that Diameter, and will be biſected by the ſame in $P$, as we have proved already; and becauſe there is* only one Diameter that can paſs through that Point, therefore if the Line $MN$, bounded by an Hyperbola, be biſected in $P$ by the Diameter $AB$, the ſaid Line $MN$ will be an Ordinate both ways to that Diameter. *Art. 267. *Art. 295.

### Corollary.

297. HENCE, if two Ordinates ($MPN$, $OQR$) be drawn (on both Sides) to the Diameter $AB$, we ſhall have always* $MP \times PN$ or $\overline{PM}^2 : OQ \times QR$ or $\overline{QO}^2 :: AP \times PB : AQ \times QB$. That is, *&c.* *Art. 275.

## Proposition XXI.

### Theorem.

298. *IF through any Point* M *in an Hyperbola, there be drawn a Tangent* MFG, *meeting two other parallel Tangents* AF, BG, *in the Points* F, G; *I ſay,* MF : MG :: AF : BG. Fig. 154.

Draw two parallel Elliptick Planes thro' the Tangents $AF$, $BG$; theſe will form two parallel Tangents $HF$, $bG$, in the touching Plane $SMG$; and the Elliptick Plane paſſing through $BG$, will form the Tangent $af$, in the touching Plane $SAF$, which ſhall meet the Tangent $bG$ in the Point $f$, wherein the Line $FS$ meets that Elliptick Plane. This being laid down, the Tangents $af$, $BG$, ſhall be parallel to one another, becauſe they are each parallel to the Tangent $AF$; and therefore* we ſhall have $BG : Gb :: af : fb :: AF : FH$; (becauſe the Triangles $Sbf$, $SHF$, and $Saf$, $SAF$, are ſimilar) whence $BG : AF :: Gb : FH :: MG : MF$ (becauſe the Triangles $MGb$, $MFH$, are ſimilar.) *W. W. D.* *Art. 268.

It is manifeſt, that the ſame Corollaries may be drawn from this Propoſition as in the Ellipſis in the 287th, 288th, 289th, and 290th Articles, and ſo I ſhall here omit them.

## Proposition XXII.

### Theorem.

299. *IF any right Line* FG, *terminating in the Aſymptotes of an Hyperbola, touches the ſame in the Point* A; *I ſay, that Line will be biſected by the Point* A. Fig. 155.

Draw

Draw two Planes through *S* the Vertex of the Cone, and the two
* *Def.* 16. Aſymptotes *CF*, *CG*, which ſhall touch * the Conick Superficies in the Sides *SM*, *SN*, wherein the Plane *MSN* parallel to the Hyperbolic Plane meets the ſame. Alſo draw an Elliptic Plane through the Right Line *FG*; which Plane will form two Tangents *MF*, *NG*, in the touching Planes, and in the Plane *MSN*, the Right Line *MN* parallel to *FG*, which joins the Points of Contact of the two Tangents *ME*, *NG*. This being done, it is manifeſt that the Line *FG*
* *Art.* 290. is * biſected in the Point *A*; becauſe the ſame touches the Ellipſis as well as the Hyperbola in that Point.

### COROLLARY I.

300. BEcauſe there is but one Line *FG* only, which paſſing through a Point *A* given within the Angle *FCG*, and being bounded by the Sides thereof, can be biſected by that Point *A*; therefore if a Right Line *FG* terminating in the Aſymptotes of an Hyperbola, meets the Hyperbola in the Point *A*, dividing that Line *FG* into two equal Parts, the ſame ſhall touch the Hyperbola in that Point.

### COROLLARY II.

301. HENCE if it be requir'd to draw a Tangent *FAG* from a Point *A* given in an Hyperbola, whoſe Aſymptotes *CF*, *CG*, are given; then you need only draw the Line *AD* parallel to one of the Aſymptotes *CG*, and terminating in the other: For if *DF* be taken equal to *CD*, and the Line *FAG* be drawn, the ſame will be the Tangent ſought. For becauſe the Triangles *FCG*, *FDA*,
* *Hyp.* are ſimilar, the Line *FG*, ſhall be biſected in *A*; ſince *CF* is * in *D*.

### COROLLARY III.

FIG. 156. 302. IF any two Points *M*, *N*, of an Hyperbola *MAN* be joyned by a Right Line meeting the Aſymptotes in the Points *H*, *K*; then the two Parts of that Line *MH*, *NK*, contained between the Hyperbola and the Aſymptotes, are equal between themſelves. For if the Diameter *CP* be drawn through *P* the middle of *MN*, and the Line *FG* through the Point *A*, wherein that Diameter meets the Hyperbola, parallel to *MN*, and terminating in the Aſymptotes:
* *Art.* 296. Then it is * manifeſt, that *FG* will be a Tangent in the Point *A*; and
* *Art.* 299. ſo will be * biſected in the ſame. Whence becauſe the Triangles *CAF*, *CPH*, and *CAG*, *CPK* are ſimilar, *PH* is = *PK*; and ſo *MH* = *NK*.

COROL-

### COROLLARY IV.

303. IF through the Point $A$, given in an Hyperbola, there be drawn two right Lines $AF$, $AG$, terminating in the Aſymptotes; and if from any other Point $M$ of the ſame Hyperbola, or the oppoſite one, there be drawn two other right Lines $MH$, $MK$, likewiſe terminating in the Aſymptotes, and parallel to the two former Lines $AF$, $AG$; I ſay $FA \times AG = HM \times MK$. Fig. 157.

For 1. When the two Points $A$, $M$, be in the ſame Hyperbola; then join them by a right Line meeting the Aſymptotes in $P$ and $Q$, and the ſimilar Triangles $PAF$, $PMH$, and $QMK$, $QAG$, give the following Proportion, $AF : MH :: AP : MP$ * $:: MQ : AQ :: MK : AG$. And ſo by multiplying the Means and Extremes, there ariſes $FA \times AG = HM \times MK$. * *Art.* 302.

2. When the Points $A$, $M$, are one in one Section, and the other in the oppoſite Section; draw the Diameter $AB$ through the given Point $A$, and the Centre $C$, and draw the right Lines $BD$, $BE$, parallel to $AF$, $AG$, and terminating in the Aſymptotes; then it is manifeſt, that the Triangles $CAF$, $CBD$, and $CAG$, $CBE$, will be ſimilar and alſo equal, ſince * $CA$ is $= CB$. Therefore $BD = AF$, and $BE = AG$; and ſo $DB \times BE = FA \times AG$. But (by the laſt Caſe) $KM \times MH = DB \times BE$. Whence alſo $FA \times AG = KM \times MH$. * *Art.* 294.

### ADVERTISEMENT.

I ſhall omit the other Properties of the Aſymptotes and Conjugate Diameters, becauſe theſe ariſe from thoſe on a Plane, as is ſhewn in the third Book: My Deſign here having been only to ſhew the Uſefulneſs of conſidering the Conick Sections in the Solid, and demonſtrating immediately without any Calculus, thoſe Properties of the Diameters, Tangents, and Aſymptotes, from which all the other Properties may be taken: Which I have done (in my Opinion) after a very eaſy and new manner; in not having us'd Lines harmonically divided, as the modern Geometricians after M. *Paſchal* and *Deſargues* have: For this obliges them to have recourſe to a great Number of Lemmata, whoſe Demonſtrations alone (ſeem to me) to take up more room than this whole 6th Book.

*The End of the Sixth Book.*

# BOOK VII.

## *Of Geometrick* LOCI.

### DEFINITIONS.

I.

FIG. 158, 159. IF there be two unknown and indeterminate right Lines, $AP$, $PM$, making any Angle ($APM$) with each other at pleaſure; and if the Beginning of one of them, *viz.* $AP$, (which I call always $x$) be fixed in the Point $A$; and the ſaid $AP$ indefinitely extends it ſelf along a right Line given in Poſition; and the other $PM$ (which I call $y$) continually alters its Poſition, and is always parallel to it ſelf. (That is, all the $PM$ being parallel to one another.) Then if there be an Equation, wherein are both thoſe unknown Quantities $x$ and $y$ mix'd with known ones, which expreſſes the Relation of every $AP$ ($x$) to its Correſpondent $PM$ ($y$), the Curve paſſing thro' the Extremities of all the Values of $y$, that is, through all the Points $M$, is called in general a *Geometrick Locus*; and in particular, the *Locus* of that Equation.

FIG. 158. For Example: Let us ſuppoſe that the Equation $y = \frac{bx}{a}$ expreſſes always the Relation of the Line $AP$ ($x$) to $PM$ ($y$), which make any Angle $APM$ at pleaſure with one another: In the Line $AP$ aſſume $AB = a$, and from $B$ draw $BE = b$ parallel to $PM$, and on the ſame Side; then the indefinite Line $AE$ is called in general a *Geometrick Locus*; and in particular, the *Locus* of the Equation $y = \frac{bx}{a}$. For if the right Line $MP$ be drawn from any one of its Points $M$ parallel to $BE$, the ſimilar Triangles $ABE$, $APM$, will give always this Proportion, *viz.* $AB\ (a) : BE\ (b) :: AP\ (x) : PM\ (y) = \frac{bx}{a}$. And therefore the right Line $AE$ is the Locus of all the Points $M$.

FIG. 159. Moreover, if $yy = aa - xx$ expreſſes the Relation of $AP$ to $PM$, and the Angle $APM$ be a right Angle; then the Circumference of a Circle, whoſe Radius is the right Line $AB = a$ taken in $AP$, is called in general a *Geometrick Locus*, and, in particular, the Locus of the

the Equation $yy = aa - xx$. For if the Perpendicular M $P$ ($y$) be drawn from any Point $M$ of the Circumference; then, by the Nature of the Circle, we ſhall have always $\overline{PM}^2$ ($yy$) $= DP \times PB$ ($aa - xx$) ſuppoſing $BD$ the Diameter of the Circle. Therefore the Locus of all the Points $M$ is the Circumference of a Circle.

## SCHOLIUM.

304. IF all the $PM$s be ſuppos'd to tend from one Side of the Line $AB$, as towards $Q$, and then they be ſuppos'd to tend from the other Side of the ſaid Line, as towards $G$; then it muſt be obſerv'd, that their Values from Poſitives (which they are ſuppos'd to be when tending towards $Q$) will become Negative, and ſo we ſhall have $PM = -y$; moreover, if the Points $P$ be ſuppos'd to fall from $A$ towards $B$, and afterwards the contrary way, as from $A$ towards $D$; then all the $AP$s on this Side $A$, will become Negative, and conſequently we have $AP = -x$. And a Geometrick Locus muſt paſs through the Extremities of all the Values (as well poſitive as negative) of one of the unknown Quantities $y$, which anſwer to the Values both poſitive and negative of the other unknown Quantity $x$. Therefore, if the right Line $QAG$ be drawn parallel to $PM$, a Geometrick Locus may be found in the four Angles $BAQ$, $BAG$, $GAD$, $DAQ$, as in the ſecond Example (*Fig.* 159.) or only in ſome of the Angles, as in the firſt Caſe (*Fig.* 158.) For in the ſecond Example, ſuppoſe that $AP$ be $= x$, and $PM = y$, the Point $M$ being firſt taken on the Quadrant $QB$; then if the Point $M$ be taken afterwards in the Quadrant $GB$, we ſhall have $AP = x$, and $PM = -y$; if $M$ be taken on $DG$, we ſhall have $AP = -x$, and $PM = -y$; and finally, if $M$ be taken on $DQ$, we ſhall have $AP = -x$, and $PM = y$; and in all theſe Caſes (by the Nature of the Circle) there will come out the ſame Equation $yy = aa - xx$; becauſe the Squares of $\pm y$, and $\pm x$, are the ſame in all Caſes, *viz.* $yy$ and $xx$. Moreover, in the firſt Example, if you make $AP = x$, and $PM = y$, in firſt taking the Point $M$ (on the ſame Side as $E$) upon $AE$, in the Angle $QAP$; and then if the Point $M$ be afterwards taken on $EA$ (produced towards $A$) in the Angle $GAD$, we ſhall have $AP = -x$, and $PM = -y$; and ſince the Triangles $ABE$, $APM$, are ſimilar, the following Proportion will be formed, *viz.* $AB$, ($a$) $BE$ ($b$) : : $AP$ ($-x$) : $PM$ ($-y$) $= -\frac{bx}{a}$; and therefore $y = \frac{bx}{a}$. Which is the ſame Equation as was form'd by ſuppoſing the Point $M$ to fall in the Angle $BAQ$.

Note, When we are hereafter to conſtruct the Locus of a given Equation, we always ſuppoſe $AP$ ($x$) and $PM$ ($y$) poſitive,

that is, all the Points *M* to fall in the ſame Angle *B A Q*. And that Part of the Locus contain'd in the Angle *B A Q*, we take for the Locus of the given Equation.

2.

The ancient Geometricians did call *plain Loci* ſuch that are right Lines or Circles; and *ſolid Loci*, thoſe that are Parabola's, Ellipſes, or Hyperbola's. But the Moderns do diſtinguiſh Geometrick Loci into different Kinds or Degrees: For under the firſt Degree are comprehended all the Loci, wherein the unknown Quantities $x$, $y$, are found in Equations only of one Dimenſion; under the ſecond, all thoſe wherein thoſe unknown Quantities have two Dimenſions; under the third, all thoſe wherein the unknown Quantities have three Dimenſions, and ſo on. Where you muſt obſerve, that there muſt be no Rectangle, or Product of the unknown Quantities $x$ and $y$ in the Equations for the Loci of the firſt Kind or Degree; and in Equations for the ſecond, thoſe Quantities muſt form a Product as $x\,y$ of no more than two Dimenſions; and in Equations for the third, a Product $x\,x\,y$, or $x\,y\,y$ of three Dimenſions, *&c.*

3.

The Terms of the Equation of a Locus are ſaid to be different, when either of the unknown Quantities $x$ and $y$, or both of them together, are found therein of different Dimenſions: So in the firſt Degree, if this Equation be propos'd $y - \frac{b\,x}{a} + c = o$, the Terms $y$, $-\frac{b\,x}{a}$, $c$, will be different. Moreover, in the ſecond Degree, if you ſuppoſe $y\,y + \frac{2\,b\,x\,y}{a} - 2\,c\,y - \frac{f\,x\,x}{a} + g\,x + h\,x - b\,b + l\,l = o$, then the Terms $y\,y$, $\frac{2\,b\,x\,y}{a}$, $-2\,c\,y$, $-\frac{f\,x\,x}{a}$, $g\,x + h\,x$, $-b\,b + l\,l$, ſhall be every one of them different.

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I ſhall here only explain particularly the Loci of the firſt and ſecond Degrees; but what I ſhall ſay thereof will give a great Inſight into the Conſtruction of more compound Loci in particular Caſes that may occur; ſome Examples of which will be found hereafter: Therefore my Deſign in this Book, is to give a general Method for conſtructing the Loci of given Equations of the firſt and ſecond Degrees; and to ſhew that the Loci of the firſt Degree are ſtrait Lines only; and thoſe of the ſecond, either Parabolas, Ellipſes, Circles, Hyperbolas, or the oppoſite Sections.

POSTU-

### POSTULATE.

305. GRant that any given litteral Quantity, be it never ſo compounded, may be reduced to a ſimple Fraction in its leaſt Terms.

For Example: 1. Grant that one may take the ſimple Fraction $\frac{b}{a} = \frac{cc+ff}{af+fc} + \frac{a\,a}{g\,g}$, wherein the Letters $a, c, f, g$, denote given right Lines 2. That one can find a right Line $s = \frac{age-bce}{bb+af}$, wherein the right Lines $a, b, c, e, f, g$, are given. 3. That one can find a Square $t\,t = s\,s - \frac{ccee-eehh}{bb+af}$, wherein the Lines $a, b, c, e, f, b, s$, are given; ſo that its Side $t$ be $= \sqrt{s\,s - \frac{ccee-eehh}{bb+af}}$. We ſhall ſhew the manner of doing this, in the Beginning of the Eighth Book.

## PROPOSITION I.
### Problem.

306. *AN Equation of any Locus of the firſt Degree being given, to conſtruct the Locus.*

When the unknown Quantities $x$ and $y$, have but one Dimenſion in a given Equation, and their Product $x\,y$ is not in the ſame; then the Locus of that Equation will be always a ſtrait Line, and it may be reduced to ſome one of the four following Formula's.

1. $y = \frac{b\,x}{a}$, 2. $y = \frac{b\,x}{a} + c$, 3. $y = \frac{b\,x}{a} - c$, 4. $y = c - \frac{b\,x}{a}$,

in all which we ſuppoſe the unknown Quantity $y$ to be freed from Fractions, and alſo that the Fraction multiplying the other unknown Quantity $x$ be * reduced to this Expreſſion $\frac{b}{a}$, and all the known Terms to this, *viz.* $c$. * *Art.* 305.

The ſame Things being premis'd as in the firſt Definition, the Loci of the three laſt Forms may be conſtructed in the following manner; for the Locus of the firſt Form has been conſtructed in that Definition already.

Now to conſtruct the Locus of the ſecond Formula $y = \frac{b\,x}{a} + c$. In the Line $AP$ aſſume $AB = a$; and draw the right Lines $BE = b$, $AD = c$, parallel to $PM$, and on the ſame Side; then if the indefinite right Line $AE$ be drawn, as alſo the right Line $DM$ parallel to $AF$. I ſay, this Line $DM$, contain'd in the Angle $PAQ$, (made FIG. 160.

by

by the Line $AQ$, drawn parallel to $PM$, and on the ſame Side) will be the Locus of the ſaid Equation or Formula. For if through any one of its Points $M$, the Line $MP$ be drawn parallel to $AQ$, meeting $AE$ in $F$; then the ſimilar Triangles $ABE$, $APF$, will give this Proportion, $AB(a) : BE(b) :: AP(x) : PF = \frac{bx}{a}$. And therefore $PM(y) = PF\left(\frac{bx}{a}\right) + FM(c)$.

FIG. 161. The Locus of the third Formula $y = \frac{bx}{a} - c$, may be conſtructed after the following manner: Aſſume $AB = a$, and draw the right Lines $BE = b$, $AD = c$, parallel to $PM$; *viz.* $BE$ on the ſame Side as $AQ$, and $AD$ on the contrary Side; alſo thro' the Points $A$, $E$, draw the Line $AE$ of an indefinite length towards $E$, and thro' the Point $D$ the Line $DM$ parallel to $AE$, meeting the Line $AP$ in $G$. I ſay, the indefinite right Line $GM$, contain'd in the Angle $PAQ$, will be the Locus ſought. For we have always $PM(y) = PF\left(\frac{bx}{a}\right) - FM(c.)$

FIG. 162. Laſtly, To conſtruct the Locus of the fourth Formula $y = c - \frac{bx}{a}$. In $AP$ aſſume $AB = a$, and draw the right Lines $BE = b$, $AD = c$, parallel to $PM$; *viz.* $BE$ on the contrary Side that $AQ$ is, and $AD$ on the ſame Side; then through the Points $A$, $E$, draw the Line $AE$ of an indefinite Length towards $E$, and through the Point $D$ the Line $DM$ parallel to $AE$, meeting the Line $AP$ in $G$. I ſay, the right Line $DG$, contain'd in the Angle $PAQ$, will be the Locus ſought. For if the Line $MP$ be drawn from any Point $M$ thereof parallel to $AQ$, and meeting $AE$ in $F$, we ſhall have always $PM(y) = FM(c) - PF\left(\frac{bx}{a}\right)$.

If the unknown Quantity $x$ be not multiplied by a Fraction; then the four foregoing Formula's will be changed into theſe here:

1. $y = x$, 2. $y = x + c$, 3. $y = x - c$, 4. $y = c - x$, all of which may be conſtructed after the ſame manner as before, only obſerving to take the right Line $BE$ equal to $AB$, which before was taken of any Length at pleaſure.

## SCHOLIUM.

307. IT may happen that the Locus of an Equation may be a ſtraight Line, although there be but one of the unknown Quantities $x$, $y$, contain'd therein; and from hence ariſes theſe two new Formula's, $y = c$, and $x = c$.

Now to conſtruct the firſt Formula $y=c$, the ſame things being always premiſed, as in Def. 1; through the fixed Point $A$ draw the Right Line $AD = c$ parallel to $PM$ and towards the ſame Parts, and then draw the indefinite Right Line $DM$ parallel to $AP$: I ſay that Line $DM$, will be the Locus of the propos'd Equation. For if the Right Line $MP$ be drawn from any Point $M$ of the ſame, parallel to $AD$; then it is manifeſt that we ſhall have always $PM(y) = AD(c)$. FIG. 163.

Again, to conſtruct the ſecond Formula $x = c$. Aſſume $AP = c$, and draw the indefinite Right Line $PM$ making the Angle $APM$ with $AP$, which is either given or taken at pleaſure: Then I ſay, $PM$ will be the Locus of all the Points $M$. For if the Right Line $MQ$ be drawn from any Point $M$, thereof parallel to $AP$, and meeting the indefinite Line $PQ$ (parallel to $PM$) in the Point $Q$; then it is manifeſt that we ſhall have always $MQ$ or $AP(x) = c$, let $PM(y)$ be aſſum'd of any Length whatſoever. FIG. 164.

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Here it will not be amiſs to give the Learner an Idea of the Method I am going to uſe, in conſtructing the Loci of the ſecond Degree. And this Method conſiſts firſt in conſtructing a Parabola, and afterwards an Ellipſis and Hyperbola being ſuch, that the Equation expreſſing the Nature thereof, with regard to the Diameters and Aſymptotes, be the moſt compounded poſſible; and this furniſhes general Equations or Formula's. Then I examine the particular Parts of theſe general Equations, that ſo an Equation being propos'd, I may know which of theſe general Formula's to compare the ſame with; this being known, and all the Terms of the Equation compar'd with thoſe of the general Formula, I gather from thence the Conſtruction of the Locus of that Equation, in obſerving certain Remarks ſerving for all the Formula's. But all this will be much better underſtood in the following Lemmata and Propoſitions.

## *The* FUNDAMENTAL LEMMA *for the Conſtruction of Loci, which are Parabola's.*

308. LET there be two unknown and indeterminate Right Lines $AP, (x), PM(y)$ (as in Def. 1.); alſo let there be given Right Lines, as $m$, $n$, $p$, $r$, $s$. This being premiſed. FIG. 165, 166.

1. In the Line $AP$, aſſume $AB = m$; and draw the Right Lines $BE = n$, $AD = r$, parallel to $PM$, and towards the ſame Parts, alſo through the Point $A$ draw the Right Line $AE$ which I call $e$, and through the Point $D$ the indefinite Right Line $DG$ parallel to $AE$; and in $DG$ aſſume $DC = s$ on the ſame Side as $PM$, and with FIG. 165.

with the Diameter $CG$, having its Ordinates parallel to $PM$, and Parameter $CH=p$, deſcribe * a Parabola $CM$ tending towards the ſame Parts as $AP$. Now I ſay the Portion of this Parabola contain'd in the Angle $PAD$, (form'd by the Line $AP$, and a Line $AD$ drawn thro' the fixed Point $A$, parallel to $PM$, and towards the ſame Parts) is the Locus of the following Equation or Formula.

*Art. 161.

$$yy-\frac{2n}{m}xy+\frac{nn}{mm}xx-2ry+\frac{2mr}{m}x+rr=0.$$
$$-\frac{ep}{m}x+ps.$$

For through any Point ($M$) of that Portion of the Parabola, draw the Line $MP$, making the Angle $APM$ (either given or taken at pleaſure) with $AP$, and meeting the Parallels $AE$, $DG$, in the Points $F$, $G$; then the ſimilar Triangles $ABE$, $APF$, will give theſe two Proportions; $AB\ (m):AE\ (e)::AP\ (x):AF$ or $DG=\frac{ex}{m}$. And $AB\ (m):BE\ (n)::AP\ (x):PF=\frac{nx}{m}$. And conſequently $GM$ or $PM-PF-FG=y-\frac{nx}{m}-r$, and $CG$, or $DG-DC=\frac{ex}{m}-s$. But the Parabola gives * $GM^2=CG\times CH$, which Equation will be the ſame as the propos'd one, by putting for thoſe Lines their analytick Values. Therefore, &c.

* *Art.* 19.

Fig. 166. 2. From the fix'd Point $A$, draw the indefinite Line $AQ$, parallel to $PM$, and towards the ſame Parts, in which aſſume $AB=m$, and draw $BE=n$, parallel to $AP$, and towards the ſame Parts as $PM$, and through the determinate Points $A$, $E$, draw the Line $AE$, which I call $e$; then if in $AP$ you take $AD=r$ on the ſame Side as $PM$, and draw the indefinite Right Line $DG$ parallel to $AE$, and if in the ſame you take $DC=s$, likewiſe on the ſame Side as $PM$, and afterwards with the Diameter $CG$, whoſe Ordinates are parallel to $AP$, and the Parameter $CH=p$, there be deſcrib'd * a Parabola $CM$, tending the ſame way as $AQ$. I ſay the Portion of this Parabola contain'd in the Angle $BAP$, will be the Locus of this ſecond Equation.

*Art. 161.

$$xx-\frac{2n}{m}yx+\frac{nn}{mm}yy-2rx+\frac{2nr}{m}y+rr=0.$$
$$-\frac{ep}{m}y+ps.$$

For if the Line $MQ$ be drawn from any Point $M$ thereof, parallel to $AP$, and meeting the Parallels $AE$, $DG$, in the Points $F$, $G$; then the ſimilar Triangles $ABE$, $AQF$, will give theſe two Proportions

portions, $AB\ (m) : AE\ (e) : : AQ$, or $PM\ (y) : AF$ or $DG = \frac{ey}{m}$. And $AB\ (m) : BE\ (n) :: AQ\ (y) : QF = \frac{ny}{m}$. And consequently $GM$ or $QM - QF - FG = x - \frac{ny}{m} - r$, and $CG$ or $DG - DC = \frac{ey}{m} - s$. But the Parabola gives $\overline{GM}^2 = CG \times CH$, which Equation will be the ſame as the propoſed one, by putting for thoſe Lines their Analytick Values; therefore, *&c.*

### COROLLARY.

309. IT is manifeſt, (1.) That in the former of the two Equations, or Formula's, the Square $yy$ is found without a Fraction, and in the 2d the Square $xx$. 2. In both the Formulas the two Squares $xx$ and $yy$ are found with the ſame Signs; ſo that the Square $\left(\frac{nn}{mm}\right)$ of half the Fraction $\frac{2n}{m}$, which multiplies the Plane $xy$, multiplies $xx$ and $yy$. And ſo if the Plane $xy$ be in neither of thoſe two Equations, then the Square $\frac{nnxx}{mm}$, or $\frac{nnyy}{mm}$ will alſo be not in them, becauſe the given Fraction $\frac{2n}{m}$ will then be equal to nothing.

## PROPOSITION II.

### Problem.

310. *TO conſtruct the Locus of a given Equation, wherein if the Plane* xy *be not, there will likewiſe be but one of the Squares* xx *and* yy; *or elſe if the Plane* xy *be found in the ſame, the two Squares* xx *and* yy *are alſo therein both with the ſame Signs; ſo that the Square of half the Fraction multiply'd by* xy, *be equal to that which multiplies the Square of one of the unknown Quantities: Suppoſing always one of the Squares* xx *or* yy *in the given Equation to be free from Fractions.*

Compare each Term of the given Equation with that Term anſwering to it in the firſt Formula of the foregoing Lemma, if the Square $yy$ happens to be without a Fraction; or with that Term anſwering to it in the ſecond Formula, when the Square $xx$ is without a Fraction. Then by comparing theſe Terms, get the Values of the Quantities $m$, $n$, $p$, $r$, $s$; by means of which, if a Parabola be deſcrib'd according as is directed in the Lemma (uſing the two following Obſervations) the ſame will be the Locus requir'd.

### REMARK I.

311. 1. THE Line $AB\ (m)$ may be taken of any positive Magnitude at pleasure. 2. The Lines $AB\ (m)$, $BE\ (n)$ being given, the Line $AE\ (e)$ is given also, because the Angle $ABE$ is given. 3. When $n = o$, the Line $AE$ falls in $AB$, that is, in $AP$ in the Construction of the first Formula, and in $AQ$ in that of the second; then we shall have $AB\ (m) = AE\ (e)$; because the Points $B, E$, will coincide. 4. When the Value of one of the Quantities. $n, r, s$, is negative, then the Line that the same expresses must be taken or drawn on the contrary Side $AP$ with regard to $PM$; whereas when it is positive, it must be drawn on the same Side, as it is in the Lemma.

### REMARK II.

312. IF the Value of the Parameter $CH\ (p)$ happens to be negative, then the Parabola must be drawn the contrary way to that in the Lemma; that is, on the opposite Parts to which the indeterminate Line $AP$, in the Construction of the first Formula, and the indeterminate Line $AQ$ in the second, tends. All this will be manifest by the following Examples.

### EXAMPLE I.

313. LET there be a given Equation $yy - 2ay - bx + cc = o$; it is requir'd to construct the Locus of the same.

Because the Square $yy$ is here without a Fraction, therefore I chuse * the first Formula of the Lemma; and comparing each Term there-
* *Art.* 308. *n.* 1. of answering it in the propos'd Equation, (1.) I have $\frac{2n}{m} = o$, because the Plane $xy$ being not in the proposed Equation, that Plane may be esteem'd as multiply'd by $o$; from whence I get $n = o$, and consequent-
* *Art.* 311. ly * $m = e$: Therefore, if all the Terms in the Equation affected with $\frac{n}{m}$ be struck out, and $m$ be put for $e$ the Value thereof, there will arise $yy - 2ry - px + rr + ps = o$. 2. By comparing the correspondent Terms $-2ry$ and $-2ay$, as also $-px$ and $-bx$, I get $r = a$, and $p = b$. 3. By comparing the Terms not affected with the unknown Quantities $x$ and $y$, I have $rr + ps = cc$; and so if $a$ and $b$ be put for their Values $r$ and $p$, there arises $s = \frac{cc - aa}{b}$, which is a negative Value when $a$ exceeds $c$, as here suppos'd. There is no need of comparing the first Terms $yy$, because they are exactly the same. Now the
* *Art.* 308. *n.* 2. Values of $n, r, p, s$, being thus determin'd, I construct the Locus by using the Construction of the Equation *, and observing the first * Re-
* *Art.* 311. mark, after the following manner. Because

Becauſe $BE(n) = o$, therefore the Points $B$, $E$, do coincide, and the Line $AE$ falls * in $AP$; whence thro' the fix'd Point $A$, I draw firſt the Line $AD(r) = a$ parallel to $PM$, and on the ſame Side $AB$, as $PM$, becauſe the Value thereof is poſitive: Then I draw $DG$ parallel to $AP$, and in the ſame aſſume $DC = \frac{aa-cc}{b} = -s$ on the contrary Side to $PM$; becauſe $s$ is $= \frac{cc-aa}{b}$, which is a negative Value. And laſtly, if a Parabola * be deſcribed with the Diameter $CG$, (whoſe Ordinates are parallel to $PM$) and Parameter $CH(p) = b$: I ſay the two Portions $OMM$, $RMS$, contain'd in the Angle $PAO$, (made by $AP$, and the Line $AO$ drawn parallel to $PM$, and towards the ſame Parts) will be the Locus of the given Equation. * *Art.* 311. FIG. 167. * *Art.* 161.

For if the Line $MP$ be drawn from any Point $M$ of theſe two Portions, making with $AP$ the Angle $APM$ either given or taken at pleaſure, and meeting $DG$ in the Point $G$; then we have $GM = y - a$, or $GM = a - y$, according as the Point $M$ be taken above or below the Diameter $CG$; and $CG$ or $DG + CD = x + \frac{aa-cc}{b}$; and therefore, by * the Property of the Parabola $\overline{GM}^2$ $(yy - 2ay + aa) =$ $CG \times CH$ $(bx + aa - cc)$ that is, $yy - 2ay - bx + cc = o$, which is the Equation given. Therefore, *&c.* * *Art.* 19.

## SCHOLIUM.

314. IF $AO$ be produced on the other Side of $A$ towards $X$, then obſerve,

1. That the indefinite Portion $SM$ of the Parabola, contain'd in the Angle $SAX$, will be the Locus of all the negative Values of the unknown Quantity $y$, anſwering to the poſitive Values of the other unknown Quantity $x$ in the given Equation. For if $AP$ be aſſum'd greater than $AS$, and $PM$ be drawn parallel to $AX$, and towards the ſame Parts, meeting the Portion $SM$ in $M$; then we ſhall have * $PM = -y$, and ſo the right Line $GM$ or $GP + PM = a - y$, and by the Property of the Parabola we ſhall get the given Equation again, as above. * *Art.* 304.

2. The Portion $RCO$ of that Parabola, which falls in the Angle $TAO$ vertically oppoſite to the Angle $SAX$, will be the Locus of all the poſitive Values of the unknown Quantity $y$ in the given Equation, anſwering to the negative Values of the other unknown Quantity $x$; for if you make * $AP = -x$, we ſhall get the given Equation again. * *Art.* 304.

3. If a Portion of the ſaid Parabola falls in the Angle $TAX$ vertically oppoſite to the Angle $PAO$, the ſame will be the Locus of all

the negative Values of the unknown Quantity $y$, anſwering to the negative Values of the other unknown Quantity $x$. So that that Parabola is the compleat Locus of all the poſitive and negative Values of $y$, anſwering to all the poſitive and negative Values of $x$, in the given Equation $yy - 2ay - bx + cc = o$.

Hence it appears, that in this Example there are two poſitive Values ($PM$, $PM$,) of the unknown Quantity $y$, which anſwer to the ſame poſitive Value ($AP$) of the other unknown Quantity $x$, when the Line $AP$ is leſs than $AS$; that there is one poſitive Value $PM$, and a negative one $-PM$, when $AP$ exceeds $AS$; that there is but one poſitive Value ($SV$) of $y$, the other being $= o$, when $AP = AS$; that there are two poſitive Values ($PM$, $PM$) of $y$, anſwering to the ſame negative Value ($-AP$) of $x$, when $AP$ is leſſer than $AT$; that thoſe two Values will become each equal to the Tangent $TC$, when $AP = AT$; and finally, if $AP(-x)$ be taken greater than $AT$, then $PM$ apply'd to $AP$, will not meet the Parabola at all, and ſo in this Caſe there can be had neither a poſitive or negative Value of $y$, that can anſwer to the negative Value ($-AP$) of $x$; that is, the Values of $y$ will then become imaginary.

All this muſt be underſtood after the ſame manner in all the following Examples, as well in the other Conick Sections, as the Parabola: So that the Conick Section deſcrib'd, will not be only the Locus of all the poſitive Values of $y$, with regard to the poſitive Values of $x$; but the ſame will be likewiſe the Locus of all the poſitive and negative Values of $y$, with regard to all the poſitive and negative Values of $x$.

## EXAMPLE II.

315. LET $yy + \frac{2b}{a}xy + \frac{bb}{aa}xx + 2cy - bx + cc = o$, be an Equation, whoſe Locus is requir'd to be conſtructed.

Becauſe the Square $yy$ is here found without a Fraction, therefore I chuſe, (as before) the firſt Formula * of the Lemma; then by comparing the Terms thereof with thoſe anſwering to them in the propos'd Equation, I have 1. $\frac{2n}{m} = -\frac{2b}{a}$; from whence making * $m = a$, there ariſes $n = -b$. 2. $\frac{nn}{mm} = \frac{bb}{aa}$; from whence there comes out, as above, $n = -b$. 3. $r = -c$. 4. $\frac{2nr - ep}{m} = -b$; and therefore $p = \frac{ab + 2bc}{e}$, by putting $a$, $-b$, $-c$, for $m$, $n$, $r$. 5. $rr + ps = cc$, and ſo $s = o$, by ſubſtituting $cc$ for $rr$. Now the Values of $m$, $n$, $r$, $p$, $s$, being thus determin'd, I conſtruct the Locus of the Equation (by uſing the Conſtruction of the firſt * Formula) after the following manner.

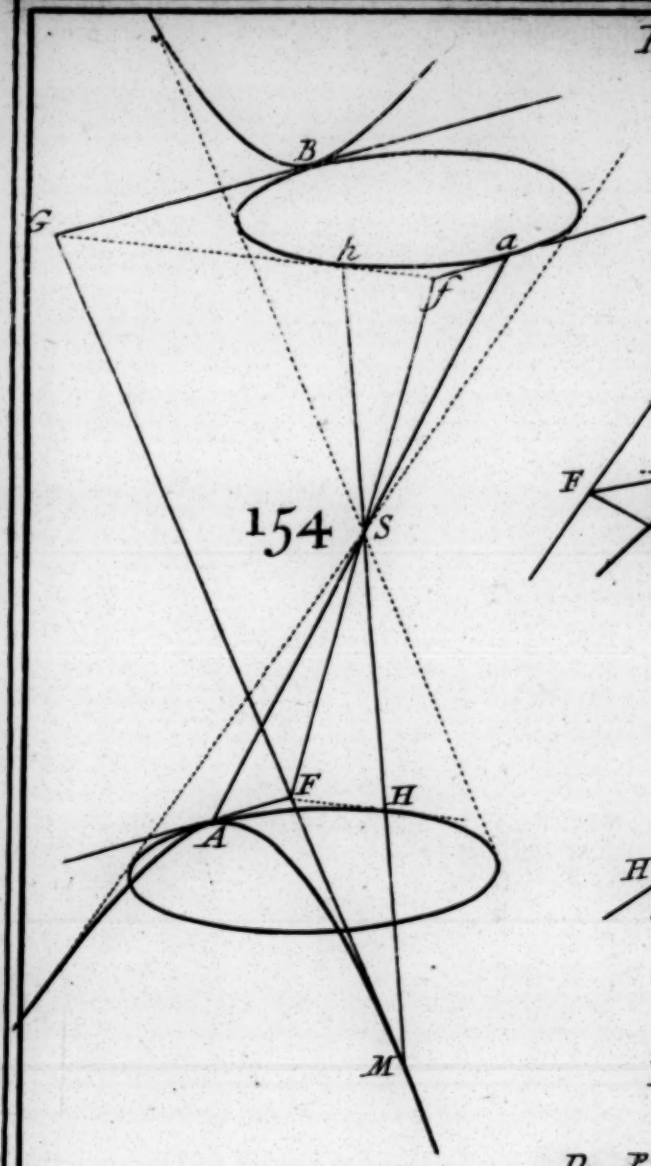
*Art. 308. n. 1.

*Art. 311.

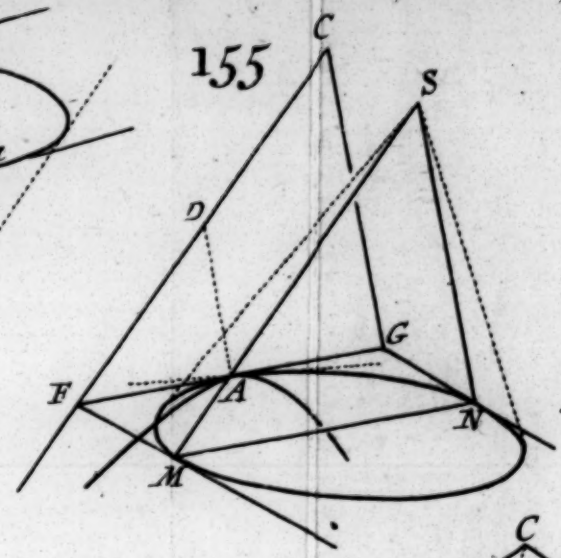
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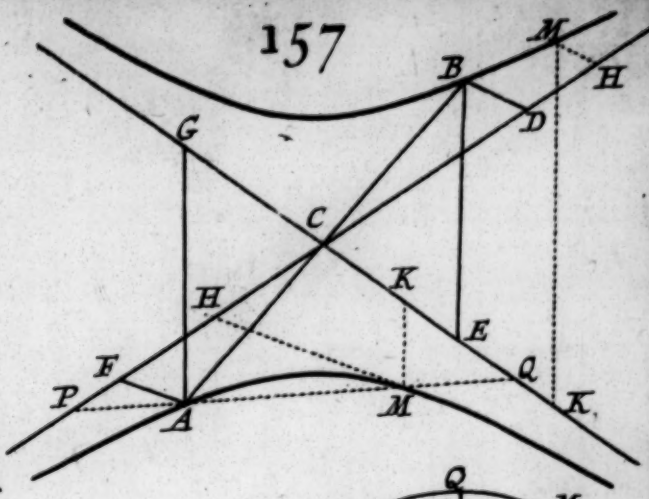
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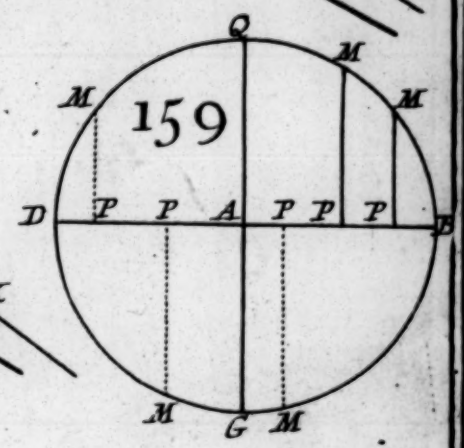

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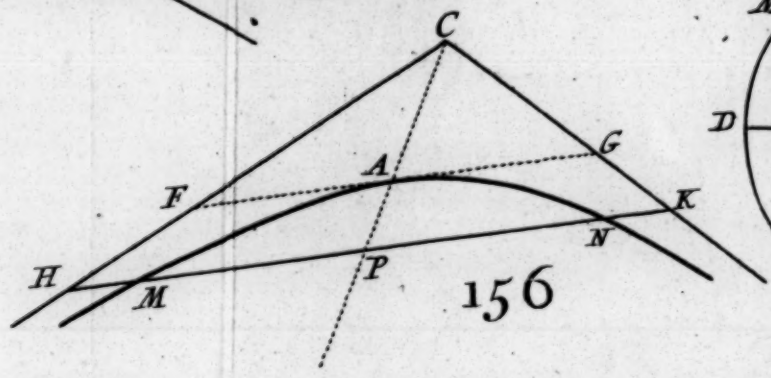

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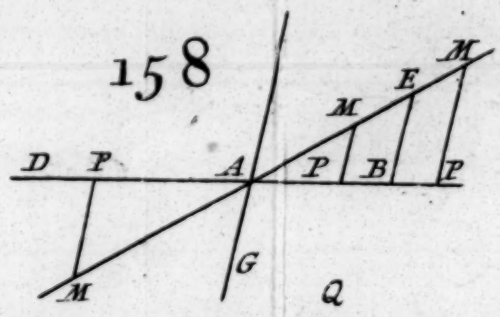

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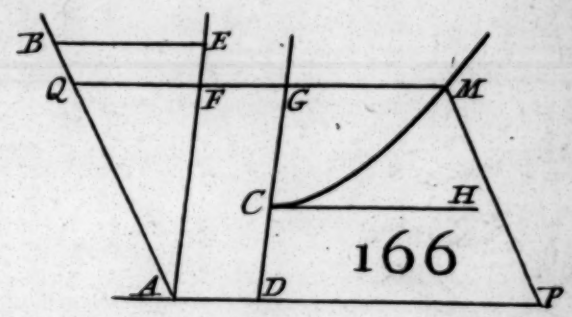

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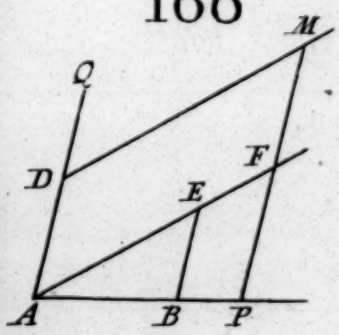

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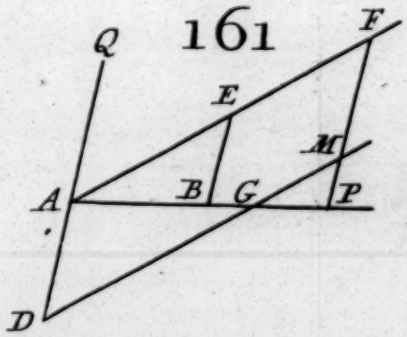

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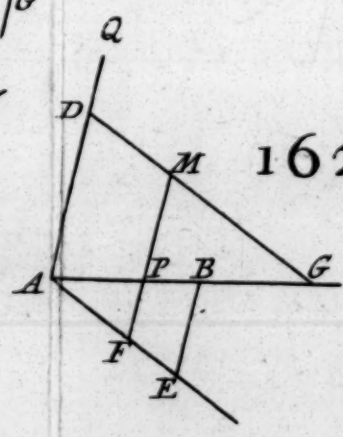

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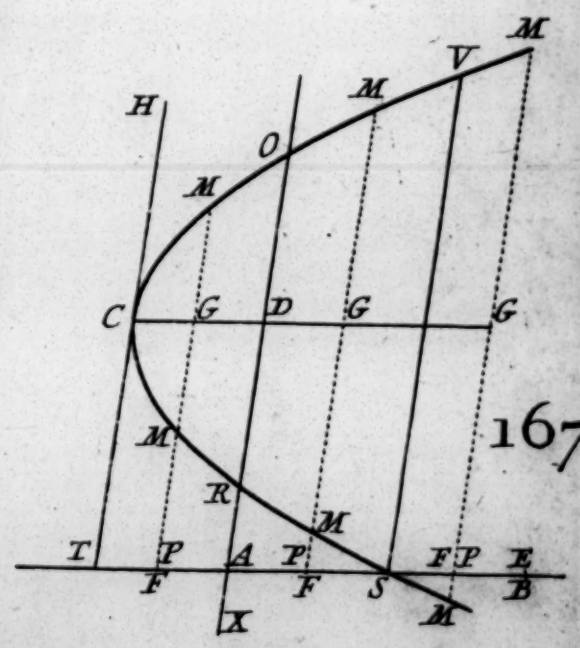

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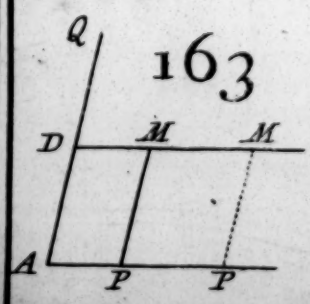

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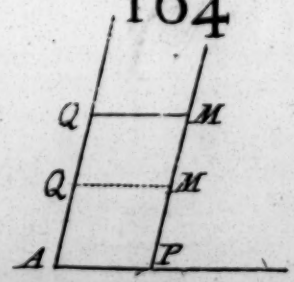

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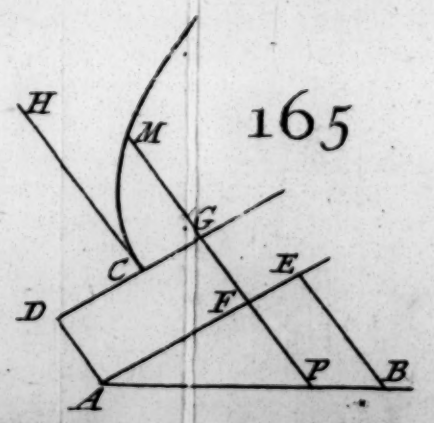

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In the Line $AP$ aſſume $AB\ (m) = a$, and draw the right Lines $BE = b = -n$, $AD = c = -r$ parallel to $PM$, and on the contrary Side, becauſe $n = -b$, and $r = -c$, which are negative Values. Then through the determinate Points $A$, $E$, draw the Line $AE$ $(e)$ which is given, and through the Point $D$, the Line $DG$ parallel to $AE$. This being done, ſince $DC\ (s)$ is $= o$, the Point $C$ falls on $D$; therefore if a Parabola be deſcrib'd * with the Diameter $DG$, whoſe Ordinates are parallel to $PM$, and the Parameter $DH\ (p) = \frac{ab+2bc}{e}$; I ſay, the Portion $OM$ thereof contain'd in the Angle $PAH$, wherein all the Points $M$ are ſuppos'd to fall, will be the Locus of the Equation given. Fig. 168. *Art. 161.

For if the Line $MP$ be drawn through any Point $M$ thereof, making the Angle $APM$ either given or taken at pleaſure, and meeting the Parallels $AE$, $DG$, in the Points $F$, $G$; then the ſimilar Triangles $ABE$, $APF$, will give theſe two Proportions, $AB\ (a) : AE\ (e) :: AP\ (x) : AF$ or $DG = \frac{ex}{a}$. And $AB\ (a) : BE\ (b) :: AP\ (x) : PF = \frac{bx}{a}$. And conſequently $GM$ or $PM + PF + FG = y + \frac{bx}{a} + c$. But by the Property of the * Parabola $\overline{GM}^2 = GD \times DH$; that is, by putting the Analytick Values, $yy + \frac{2b}{a} xy + \frac{bb}{aa} xx + 2cy - bx + cc = o$. Therefore, *&c.* *Art. 19.

## SCHOLIUM I.

316. IF the Line $AP$ ſhould not cut the Parabola, but touch or fall quite without it; then not one of the ſought Points $M$ would fall in the Angle $PAH$, as we have ſuppos'd in the Conſtruction; and ſo there could be no poſitive Value of $x$, that would anſwer to the poſitive Value of $y$. Fig. 168.

This Obſervation is general for all Examples of the like Nature, not only in the Paarabola, but alſo in the other Sections.

## SCHOLIUM II.

317. HERE it is neceſſary to take notice, that if $AB\ (m)$ had been aſſum'd of any other Length beſides $a$; then the Values of $BE\ (n)$, and $AE\ (e)$ would indeed vary: But the Ratio's of $\frac{n}{m}$, $\frac{e}{m}$, will remain the ſame always; becauſe in the Triangle $ABE$, the Angle $ABE$ is given, as alſo the Ratio of the Sides $AB$, $BE$, *viz.* $\frac{n}{m} = \frac{b}{a}$ in this Example. Now, becauſe there

are

are only thoſe two Ratio's of $\frac{n}{m}$, $\frac{e}{m}$, that can be found in the Values of $p, r, s$; therefore theſe Values do remain the ſame always, let the Line $AB$ $(m)$ be taken of any poſitive Magnitude at pleaſure: So that $m$ was taken equal to $a$ only, for rendring the Conſtruction more ſimple; which muſt be always obſerv'd hereafter.

### EXAMPLE III.

318. IT is requir'd to find the Locus of this given Equation, $xx + \frac{2b}{a}yx + \frac{bb}{aa}yy - 2cx + by - \frac{2bc}{a}y = 0$.

Becauſe the Square $xx$ is here free from Fractions, I chuſe the ſecond Formula * of the Lemma; then in comparing the correſpondent Terms, I have 1. $\frac{2n}{m} = -\frac{2b}{1a}$; from whence making $m = a$, I get $n = -b$. 2. $\frac{nn}{mm} = \frac{bb}{aa}$; and ſo (ſince $m = a$) we have $n = -b$, as before. 3. $r = c$. 4. $\frac{2nr - ep}{m} = b - \frac{2bc}{a}$; from whence $p = -\frac{ab}{e}$; by ſubſtituting $a, -b, c$, for their Values $m, n, r$. 5. $rr + ps = 0$, becauſe there is no Term in the given Equation entirely known, which can be compar'd with the Term $rr + ps$ of the Formula; and ſo we have $s = -\frac{rr}{p} = \frac{ecc}{ab}$, by ſubſtituting $c$ and $-\frac{ab}{e}$, for $r$ and $p$. Now theſe Values being thus determin'd, I conſtruct the Locus requir'd after the following manner, by uſing the Conſtruction of the ſecond Formula * of the Lemma, and exactly obſerving the 311th and 312th Articles.

*Art. 308. n. 2.

FIG. 169. Through the fixed Point $A$, draw the indefinite right Line $AQ$ parallel to $PM$, and on the ſame aſſume $AB$ $(m) = a$; and from the Point $B$ draw $BE = b = -n$ parallel to $AP$, on the contrary Side to $PM$, becauſe the Value of $n$ is negative; moreover, through the determinate Points $A, E$, draw the Line $AE$, $(e)$ which is given. This being done, in $AP$ take $AD$ $(r) = c$ in from $A$ towards $PM$, and draw the indefinite right Line $DG$ parallel to $AE$, in which aſſume $DC$ $(s) = \frac{ecc}{ab}$ on the ſame Side $AP$ as $PM$. Then deſcribe * a Parabola with the Diameter $CG$, whoſe Ordinates are parallel to $AP$, and parameter the Line $CH = \frac{ab}{e} = -p$, tending * the contrary way to what which $AQ$ tends, becauſe $p = -\frac{ab}{e}$ which is a negative Value

*Art. 161.

*Art. 312.

lue. I ſay the Portion *O M R* of that Parabola, contained in the Angle *P A B*, will be the Locus ſought.

For if the Line *M Q*, be drawn through any Point *M* thereof, parallel to *A P*, and meeting the Parallels *A E*, *D G*, in the Points *F*, *G*; then the ſimilar Triangles *A B E*, *A Q F*, will give theſe two Proportions, $AB\ (a) : AE\ (e) :: AQ \text{ or } PM\ (y) : AF \text{ or } DG = \frac{ey}{a}$. And $AB\ (a) : BE\ (b) :: AQ\ (y) : QF = \frac{by}{a}$. And conſequently $GM\ (QM + QF - FG) = x + \frac{by}{a} - c$, or $GM\ (FG - FQ - QM) = c - \frac{by}{a} - x$, according as, on which Side of the Diameter *C D* the Point *M* falls; and $CG \text{ or } CD - DG = \frac{ecc}{ab} - \frac{ey}{a}$. But by the Property * of the Parabola, $\overline{GM}^2 = CG \times CH$; that is, by ſubſtituting the Analytick Values of thoſe Lines, $xx + \frac{2b}{a}yx + \frac{bb}{aa}yy - 2cx + by - \frac{2bc}{a}y = o$, which is the given Equation. Therefore, *&c.* * *Art.* 19.

SCHOLIUM.

319. IF it ſhould happen, in comparing the Terms of the given Equation with thoſe of the Formula, that $p$ is $= o$; then it is manifeſt, that the Conſtruction of the Parabola, which ought to be the Locus of the Equation, would be impoſſible; and that Equation may always be brought lower, ſo that the Locus thereof will be a ſtrait Line; as will appear by the Formula's * of the Lemma. For Example; if all the Terms affected with $p$ in the firſt Formula be ſtruck out, there will ariſe $yy - \frac{2n}{m}xy + \frac{nn}{mm}xx - 2ry + \frac{2nr}{m}x + rr = o$, the ſquare Root of which being extracted, will be $y - \frac{nx}{m} - r = o$, or $y = \frac{nx}{m} + r$, whoſe Locus is a ſtrait Line, and may be conſtructed by *Art.* 306. The ſame thing will happen alſo in the ſecond Formula of *Art.* 308. * *Art.* 308.

EXAMPLE IV.

320. LET there be an Equation $xx - ay = o$, it is requir'd to find the Locus of the ſame.

Becauſe the Square $xx$ is free from Fractions, I chuſe the ſecond Formula * of the Lemma; then by comparing the correſpondent Terms, * *Art.* 308. n. 2.

Terms, I have 1. $\frac{2n}{m} = o$, becauſe $xy$ is not in that Equation; whence I get $n = o$, and conſequently $m$ * $= e$. 2. $\frac{nn}{mm} = o$, becauſe the Square $yy$ is not in the Equation; from whence I get again $n = o$. 3. $r = o$, becauſe the unknown Quantity $x$ is not found, in the propoſed Equation, of one Dimenſion: Therefore ſtriking out all the Terms in the Formula wherein are $\frac{n}{m}$ and $r$, and ſubſtituting $m$ for $e$; then there ariſes $x\,x - p\,y + p\,s = o$, whoſe Terms remain to be compared with thoſe anſwering to them in the propoſed Equation. 4. By comparing the Terms $-p\,y$ and $-ay$, I get $p = a$. 5. Becauſe there is no Term in the propoſed Equation entirely known to compare with the Term $p\,s$; therefore $p\,s = o$, and ſo $s$ is $= o$. Now the Values of $n$, $r$, $p$, $s$, being thus determin'd, the Locus required may be conſtructed in the following Manner, regard being had to the Conſtruction of the ſecond Formula of *Art.* 308, and *Art.* 311.

*Art.* 311.

Fig. 170. Becauſe $BE\,(n) = o$, the Line $AE$ falls * in $A\,\mathcal{Q}$ drawn parallel to $PM$ towards the ſame Parts; as alſo $DG$, becauſe $AD\,(r) = o$. And becauſe $CD\,(s) = o$ the Point $C$ falls on $D$, and $D$ on $A$. Then if a Parabola be deſcribed * with the Diameter $A\,\mathcal{Q}$, whoſe Ordinates are Right Lines $M\,\mathcal{Q}$ parallel to $AP$, and Parameter $AH\,(p) = a$: I ſay the indefinite Portion ($AM$) thereof, contain'd in the Angle $PA\,\mathcal{Q}$, is the Locus ſought.

*Art.* 311.

*Art.* 161.

For if through any Point $M$ thereof, you draw the Right Lines $MP$, $M\,\mathcal{Q}$, parallel to $A\,\mathcal{Q}$ and $AP$; then by * the Property of the Parabola, we ſhall have $\overline{\mathcal{Q}M}^2\,(xx) = A\,\mathcal{Q} \times AH\,(ay)$; and therefore $xx - ay = o$, which is the Equation propoſed. *W. W. D.*

*Art.* 19.

*The Demonſtration of the* PROBLEM.

321. IF inſtead of $m$, $n$, $r$, $s$, $p$, in the general Formula, * you ſubſtitute the Values found by comparing the Terms thereof with the Terms of a propoſed Equation, be it what it will, provided the ſame to have the Conditions denoted in the Problem; then it is manifeſt that that general Formula will be changed into the propoſed Equation: And therefore if thoſe Values be taken alſo in the Conſtruction of the * Lemma, then the Locus of the general Formula will be changed into that of the propoſed Equation. And this is what hath been taught in the Problem, as appears fully in the foregoing Examples. Therefore, *&c.*

*Art.* 308.

*Art.* 308.

The

*The* FUNDAMENTAL LEMMA, *for the Conſtruction of Loci, which are Ellipſes or Circles.*

322. LET $AP(x)$, $PM(y)$, be two unknown and indeterminite Right Lines (as in the firſt Definition) and let $m, n, p, r, s, t$, be given Lines. This being premiſed. FIG. 171.

In the Line $AP$, aſſume $AB = m$, and draw the Right Lines $BE = n$, $AD = r$, parallel to $PM$, and on the ſame Side $AP$ as $PM$, alſo thro' the Point $A$ draw the Right Line $AE$ $(e)$ which is given, and thro' the Point $D$, the indefinite Right Line $DG$ parallel to $AE$, in which aſſume $DC = s$ towards $PM$; and the Parts $CK$, $CL$, on both Sides $C$, each equal to $t$. Then if an Ellipſis be deſcrib'd * with the Diameter $LK$ $(2t)$ whoſe Ordinates are parallel to $PM$, and having $HK = p$ for the Parameter; I ſay the Portion $OMR$ thereof contain'd in the Angle $PAD$, made by the Line $AP$ and the Line $AD$ (drawn thro' the fixed Point $A$ parallel to $PM$, and the ſame way) will be the Locus of the following general Formula. * *Art.* 161.

$$yy - \frac{2n}{m}xy + \frac{nn}{mm}xx - 2ry + \frac{2nr}{m}x + rr = 0.$$
$$+ \frac{eep}{2mmt} - \frac{2eps}{2mt} - \frac{ptt}{2t}$$
$$+ \frac{pss}{2t}$$

For if from any Point $M$ of that Portion of the Ellipſis, you draw the Line $MP$, making (with $AP$) the Angle $APM$ either given or taken at pleaſure, and meeting the Parallels $AE$, $DG$, in the Points $F$, $G$; then by means of the ſimilar Triangles $ABE$, $APF$, we ſhall have $AF$ or $DG = \frac{ex}{m}$, and $PF = \frac{nx}{m}$. And therefore $GM = y - \frac{nx}{m} - r$, and $CG = \frac{ex}{m} - s$. And by the Property * of the Ellipſis, $KL(2t) : KH(p) :: LG \times GK$ or $\overline{CK}^2 - \overline{CG}^2 \left( tt - ss + \frac{2esx}{m} - \frac{eexx}{mm} \right) : \overline{GM}^2 \left( yy - \frac{2n}{m}xy - 2ry + \frac{nn}{mm}xx + \frac{2nr}{m}x + rr \right) = \frac{ptt - pss}{2t} + \frac{2epsx}{2mt} - \frac{eepxx}{2mmt}$. Therefore, &c. * *Art.* 55. *and* 41.

If it happens that the Diameter $KL$ $(2t)$ and its Parameter $KH$ $(p)$ be equal to one another, we ſhall have always $\overline{GM}^2 = LG \times CK$; from whence if the Angle $CGM$ be a right one, it is manifeſt (by the Elements) that the Ellipſis will be then changed into a Circle, whoſe Diameter is the Line $KL$.

### COROLLARY.

323. IT is evident that the two Squares $y\,y$ and $x\,x$ have always the ſame Signs in that Formula; and when the Plane $x\,y$ happens to be therein, then the Square $\left(\frac{nn}{mm}\right)$ of half the Fraction $\frac{2n}{m}$ multiplying that Plane, muſt be leſſer then the Fraction $\frac{nn}{mm}+\frac{eep}{2mmt}$ multiplying the Square $x\,x$.

## PROPOSITION III.

### Problem.

324. *TO conſtruct the Locus of a given Equation, wherein are the two Squares* y y, *and* x x, *having the ſame Signs, without the Plane* x y, *or with it, if the Square of half the Fraction multiplying the ſame be leſs than the Fraction multiplying the Square* x x. *Suppoſing the Square* y y *to be always free from Fractions.*

Compare the Terms of the given Equation with thoſe Terms anſwering to them in the general Formula * of the aforeſaid Lemma; from whence get the Values of the Quantities $m, n, p, r, s, t$, by means of which deſcribe an Ellipſis according to the Directions of the Lemma, (obſerving exactly the 311th Article) and the ſame will be the Locus ſought.

* *Art.* 322.

### EXAMPLE I.

325. IT is requir'd to find the Locus of this Equation, $y\,y+x\,y+\frac{1}{2}x\,x-2a\,y+b\,x+c\,c=o$, in which the Square of $\frac{1}{2}$ the Fraction $\frac{1}{1}$ or 1, which multiply's $x\,y$, is leſs then the Fraction $\frac{1}{2}$ which multiplies $x\,x$.

Now by comparing each Term of the general Formula of the Lemma * with that anſwering to it in the Equation, we ſhall have 1. $\frac{2n}{m}=-1$, for ſince the Plane $x\,y$ is not multiply'd by any litteral Fraction, the ſame may be conſider'd as being multiply'd by unity or 1: And conſequently if you make $m=a$, we ſhall have $n=-\frac{1}{2}a$. 2. $\frac{nn}{mm}+\frac{eep}{2mmt}=\frac{1}{2}$; from whence we get $\frac{p}{t}=\frac{mm-2nn}{ee}=\frac{aa}{2ee}$, by ſubſtituting $a, -\frac{1}{2}a$, for $m, n$; and conſequently $p=\frac{aat}{2ee}$. 3. $r=a$. 4. $\frac{2nr}{m}-\frac{2eps}{2mt}=b$; whence putting $a, -\frac{1}{2}a, a, \frac{aa}{2ee}$, for their Values $m, n, r, \frac{p}{t}$, and there ariſes $s=\frac{-2ae-2eb}{a}$. 5. $rr-\frac{ptt}{2t}+\frac{pss}{2t}=cc$: And therefore $t\,t=$

* *Art.* 322.

$s\,s+$

$ss + \frac{2trr}{p} - \frac{2tcc}{p} = ss + 4ee - \frac{4cceee}{aa}$, by putting $\frac{aa}{2ee}$ for $\frac{p}{t}$, $r$. Now the Values of $m$, $n$, $r$, $s$, $t$ $p$, being thus found, the Ellipſis ſought may be deſcrib'd after the following Manner, uſing the Conſtruction of the Lemma * and the 311th Article. *Art. 322.

In the Line $AP$ aſſume $AB\,(m) = a$, and draw the right Line $AD\,(r) = a$ parallel to $PM$, and on the ſame Side, as alſo the right Line $BE = \frac{1}{2}a = -n$ on the contrary Side, becauſe $n = -\frac{1}{2}a$ is a negative Value; moreover, through the Point $A$ draw the right Line $AE\,(e)$ which is given, and through the Point $D$ the right Line $DG$ parallel to $AE$; in which aſſume $DC = \frac{2ae + 2be}{a} = -s$ on the contrary Side to $PM$, as alſo $CK$, $CL$, on both Sides $C$, each equal to $t = \sqrt{ss + 4ee - \frac{4cceee}{aa}}$. Then if an Ellipſis be * deſcrib'd with the Diameter $LK$, whoſe Ordinates are parallel to $PM$, and Parameter the Line $KH\,(p) = \frac{aat}{2ee}$. I ſay, the Portion $(OMR)$ thereof, contain'd in the Angle $PAD$, will be the Locus of the given Equation. FIG. 172. *Art. 161.

For if thro' any Point $M$ thereof, you draw the Line $MP$, making an Angle $(APM)$ with $AP$, either given or taken at pleaſure, and meeting the Parallels $AE$, $DG$, in the Points $F$, $G$; then the ſimilar Triangles $ABE$, $APF$, will give theſe Proportions, $AB\,(a) : AE\,(e) :: AP\,(x) : AF$ or $DG = \frac{ex}{a}$. And $AB\,(a) : BE\,(\frac{1}{2}a) :: AP\,(x) : PF = \frac{1}{2}x$. Therefore we have $GM = y + \frac{1}{2}x - a$; and $CG$ or $DG + DC = \frac{ex}{a} - s$, becauſe $DC = -s$. But by the Property * of the Ellipſis $KL\,(2t) : KH\left(\frac{aat}{2ee}\right) :: LG \times GK$ or $\overline{CK}^2 - \overline{CG}^2\,(tt - ss + \frac{2esx}{a} - \frac{eexx}{aa}) : \overline{GM}^2\,(yy + xy - 2ay + \frac{1}{4}xx - ax + aa.)$ From whence ſubſtituting $4ee - \frac{4cceee}{aa}$ and $-\frac{2ae - 2be}{a}$ inſtead of $tt - ss$ and $s$, and afterwards multiplying the Extremes and Means, and dividing both Sides by $2t$, we ſhall get again the propos'd Equation. Therefore, *&c.* * *Art.* 55, *and* 41.

### Scholium.

326. If $ss + 4ee$ ſhould be equal or leſs than $\frac{4cceee}{aa}$, then it is evident, that the Value of $t$ will be nothing or imaginary; in which Caſe it will be impoſſible to conſtruct the Ellipſis that ought to be the Locus of the given Equation. And ſince this Equation neceſſarily contains Conditions, therefore it is poſſible for the ſame not to have any Line for the Locus thereof; that is, all the Values of $y$ anſwering to all the Values of $x$, may be imaginary.

*Art. 322. This will appear plain in the general Formula * of the Lemma, which, by tranſpoſing ſome of the Terms, will become $yy - \frac{2n}{m}xy - 2ry + \frac{nn}{mm}xx + \frac{2nr}{m}x + rr = \frac{ptt-pss}{2t} + \frac{2pesx}{2mt} - \frac{eepxx}{2mmt}$, in which Equation the firſt Member (or Side) is the Square of $y - \frac{n}{m}x - r$; and the ſecond, the Square of $t$ *minus* the Square of $s - \frac{ex}{m}$, multiply'd by the Fraction $\frac{p}{2t}$. Now it is evident, if the Value of the Square $tt$ be nothing or negative; then the Value of the ſecond Member of the Equation will be negative; and ſo in both Caſes we ſhall have a Square, *viz.* the firſt Member, having a negative Value, which cannot be.

## EXAMPLE II.

327. IT is requir'd to find the Locus of this Equation $yy + \frac{b}{a}xy + xx + cy + fx - ag = o$, in which I ſuppoſe (according to Article 323.) that $\frac{bb}{4aa}$ is leſs than the Fraction $\frac{1}{1}$, or 1 multiplying the Square $xx$; *viz.* that $b$ is leſs than $2a$.

*Art. 322. By comparing the Terms of the general Formula * with thoſe anſwering to them in the Equation propos'd, we have 1. $\frac{2n}{m} = -\frac{b}{a}$; whence making $m = a$, and then $n$ will be $= -\frac{1}{2}b$. 2. $\frac{nn}{mm} + \frac{eep}{2mmt} = 1$; from whence putting $a$, $-\frac{1}{2}b$ for $m$, $n$, and we ſhall have $\frac{p}{t} = \frac{4aa-bb}{2ee}$: And ſo $p = \frac{4aat-bbt}{2ee}$. 3. $r = -\frac{1}{2}c$. 4. $s = \frac{bce-2afe}{4aa-bb}$. 5. $t = \sqrt{ss + \frac{ccee+4agee}{4aa-bb}}$. From whence ariſes the following Conſtruction.

FIG. 173. In the indefinite right Line $AP$, aſſume $AB$ $(m) = a$, and draw the right Lines $BE = \frac{1}{2}b = -n$, $AD = \frac{1}{2}c = -r$, parallel to $PM$, and both on the contrary Side; alſo through the Point $A$ draw the right Line $AE$ $(e)$ which is given, and through the Point $D$ the right Line $DG$ parallel to $AE$; in which aſſume $DC$ $(s) = \frac{bce-2afe}{4aa-bb}$ from $D$ towards $PM$, if $bc$ exceeds $2af$, (as it is here ſuppos'd to do) and the contrary way, if the ſame be leſs; then on both Sides the Point $C$, aſſume

assume $CK$ and $CL$, each equal to $t = \sqrt{ss + \frac{ccee + 4agee}{4aa - bb}}$. This being done, describe * an Ellipsis with the Diameter $LK$ $(2t)$ whose Ordinates are parallel to $PM$, and Parameter the Line $KH$ $(p) = \frac{4aat - bbt}{2ee}$. I say, the Portion $OR$ of this Ellipsis will be the Locus of the proposed Equation. * *Art.* 161.

For drawing the Line $MP$ from any Point $M$ thereof, making the Angle $APM$ with $MP$ either given or assum'd at pleasure, and meeting the Parallels $AE$, $LK$, in the Points $F$, $G$; then we have $PF = \frac{bx}{2a}$, and $AF$ or $DG = \frac{ex}{a}$; from whence we get $MG$ or $MP + PF + FG = y + \frac{bx}{2a} + \frac{1}{2}c$, and $CG = \frac{ex}{a} - s$, or $s - \frac{ex}{a}$. But by the Property * of the Ellipsis $LK$ $(2t) : KH \left(\frac{4aat - bbt}{2ee}\right) :: LG \times GK \left(tt - ss + \frac{2esx}{a} - \frac{eexx}{aa}\right) : \overline{GM}^2 \left(yy + \frac{b}{a}xy + gy + \frac{bbxx}{4aa} + \frac{bc}{2a}x + \frac{1}{4}cc\right)$. * *Art.* 55. and 41.

from whence, (if $\frac{ccee + 4agee}{4aa - bb}$ and $\frac{bce - 2afe}{4aa - bb}$ be substituted for $tt - ss$, and $s$, and then you multiply the Means and Extremes, and divide by $2t$) and there will arise the proposed Equation.

Here it is necessary to observe, that if the Angle $AEB$ be a right one, then will the Angle $CGM$ be so likewise, and the Diameter $LK$ $(2t)$ equal to the Parameter $KH \left(\frac{4aat - bbt}{2ee}\right)$, because $ee = aa - \frac{1}{4}bb$, since $AEB$ is a right-angled Triangle. Therefore, in this Case, the Ellipsis will be a Circle, and the right Line $CK$ or $CL$ $(t) = \sqrt{ss + \frac{1}{4}cc + ag}$, will be a Diameter thereof, and $DC$ $(s)$ will be $= \frac{bc - 2af}{4e}$; from whence arises a much simpler Construction.

## EXAMPLE III.

328. LET it be requir'd to construct the Locus of this Equation $yy + xx - ax = o$.

By comparing the Terms of the general * Formula with those answering to them in the given Equation, we have 1. $\frac{2n}{m} = o$, because the Term $xy$ being wanting, the same must be conceiv'd as multiply'd by $o$; from whence we get $n = o$: and therefore $m = e$. 2. $\frac{nn}{mm} + \frac{eep}{2mmt} = 1$; that is, $\frac{p}{2t} = 1$, by substituting $o$ and $e$ for $n$ and $m$; * *Art.* 322.

and therefore $p = 2t$. 3. $r = o$; becauſe the unknown Quantity $y$ not being found of one Dimenſion in the given Equation, may be ſuppos'd alſo as being multiply'd by $o$; therefore, if all the Terms where-

*Art. 322. in are $\frac{n}{m}$ and $r$ in the general Formula * be ſtruck out, and $m$ and $1$ be ſubſtituted for $e$ and $\frac{p}{2t}$, then that Formula will become this, *viz.* $yy + xx - 2sx - tt + ss = o$, whoſe Terms remain to be compared with thoſe of the propos'd Equation. 4. $2s = a$; and therefore $s = \frac{1}{2}a$. 5. $ss - tt = o$; becauſe there is no known Term in the given Equation: and therefore $tt = ss = \frac{1}{4}aa$; and ſo $t = \frac{1}{2}a$, by extracting the ſquare Root of both Sides. Now the ſaid Values being thus found, the Locus may be conſtructed after the following manner.

FIG. 174. Becauſe $BE\,(n) = o$, therefore $AE$ falls in $AP$, and $AP$ in $DG$ likewiſe, ſince $AD\,(r) = o$; ſo that the Point $D$ falls in $A$. Therefore in $AP$ aſſume $AC\,(s) = \frac{1}{2}a$ towards $PM$; as alſo $CK$, $CL$, on both Sides the Point $C$, each equal to $t = \frac{1}{2}a$ (the Point $L$ here coincides with $A$;) and with the Diameter $AK$, whoſe Ordinates are pa-

*Art. 161. rallel to $PM$, and Parameter the Line $KH\,(p) = 2t = a$, deſcribe * an Ellipſis, and the ſame ſhall be the Locus ſought.

For if the right Line $MP$ be drawn through any Point $M$ thereof, making the Angle $APM$ with $AP$ either given or taken at plea-

* *Art.* 55, *and* 41. ſure; then we ſhall have * $AK\,(a) : KH\,(a) :: AP \times PK\,(ax - xx) : \overline{PM}^2\,(yy)$. From whence ariſes $yy + xx - ax = o$.

If the Angle $APM$ be a right Angle, then the Ellipſis will become a Circle, and the Line $AK = a$ will be a Diameter thereof.

## SCHOLIUM.

329. THERE may happen two Caſes, wherein the Locus of a given Equation is a Circle.

*Caſe* 1. When the Squares $yy$ and $xx$ are both found with the ſame Signs and without a Fraction, as alſo the Plane $xy$ in a given Equation; and when the Angle $AEB$ is a right one (which happens when $AF$ being drawn perpendicular on $PM$, the Ratio of $PF$ to $AP$, being the ſame as the Ratio of $BE$ to $AB$, is expreſſed by one half of the Fraction multiplying the Plane $xy$): Then the Locus of that Equation will be always a Circle, as appears already in *Art.* 324, and the Reaſon thereof is evident by the general Formula. For comparing the correſpondent Terms affected with $xx$, and we ſhall have this Equation $\frac{nn}{mm} + \frac{eep}{2mmt} = 1$; and ſo $\frac{p}{2t} = \frac{mm - nn}{ee} = 1$, ſince the Triangle $AEB$ being right-angled, the Square $mm = nn + ee$. But

becauſe the Angle $AEB$ is a right one, the Angle $CGM$ being that made by the Diameter $LK$ and the Ordinates thereof, will be a right one alſo; and ſo ſince the Diameter $LK$ is equal to its Parameter $KH$, the Ellipſis will then become a Circle.

*Caſe* 2. When the Squares $yy$ and $xx$, are both found with the ſame Signs, and without a Fraction, in an Equation, wanting the Plane $xy$, and when the Angle $APM$ is a right one: Then the Locus of that Equation will be always a Circle, as hath been obſerv'd in *Art.* 328; and this may be proved by means of the general Formula. For becauſe the Plane $xy$ is not in the propos'd Equation, the Fraction $\frac{2n}{m}$ in that Formula will be $= o$; and therefore $BE\,(n) = o$, and $m = e$. From whence it follows, 1. that the Diameter $LK$ is parallel to the Line $AP$, and ſo the Angle $CGM$ made by the ſame and its Ordinates, being equal to the Angle $APM$, will be a right Angle. 2. That the Fraction $\frac{nn}{mm} + \frac{eep}{2mmt}$ which multiplies the Square $xx$ in the Formula, becomes $\frac{p}{2t}$, from whence we have $\frac{p}{2t} = 1$: That is, the Diameter $LK$ will be equal to its Parameter $KH$. Therefore the Ellipſis which is the Locus of the given Equation, will be a Circle. And becauſe the general Formula in this Caſe, is

$$yy + xx - 2ry - 2sx + rr = o \quad -tt \quad +ss$$

You may abridge the Calculus by comparing the Terms of this Formula, with thoſe of the propoſed Equation, and finding by that means the Values of $r$, $s$, $t$, which ſerve to deſcribe the Circle that is the Locus of the propoſed Equation.

### *The* FUNDAMENTAL LEMMA *for the Conſtruction of Loci, which are Hyperbola's reſpecting their Diameters.*

330. THE ſame things being laid down as in the foregoing Lemma for the Ellipſis. With the Diameter $LK\,(2t)$, whoſe Ordinates are parallel to $PM$, and Parameter $KH\,(p)$ deſcribe * an Hyperbola or two oppoſite Sections. I ſay the Portion or Portions $(OM)$ thereof, contain'd in the Angle $PAD$ made by the Line $AP$ and the Line $AD$ drawn through the fixed Point $A$ parallel to $PM$ towards the ſame Parts, will be the Locus of the following Equation or Formula. FIG. 175, 176. * *Art.* 161.

$yy -$

$$yy-\frac{2n}{m}xy+\frac{nn}{mm}xx-2ry+\frac{2nr}{m}x+rr=o$$
$$-\frac{eep}{2mmt}+\frac{2eps}{2mt}x\pm\frac{ptt}{2t}$$
$$-\frac{pss}{2t}$$

in which you muſt obſerve that $\frac{ptt}{2t}$ is affirmative, when $LK$ is a firſt Diameter, and negative when the ſame is a ſecond.

For if the Line $MP$ be drawn through any Point $M$ of that Portion, making the Angle $APM$, with $AP$, either given or taken at pleaſure, and meeting the Parallels $AE$, $DG$, in the Points $F$, $G$; then by the Property * of the Hyperbola, we ſhall have $KL(2t):KH(p)::\overline{CG}^2\pm\overline{CK}^2\left(\frac{eexx}{mm}-\frac{2esx}{m}+ss\pm tt\right):\overline{GM}^2=\frac{peexx}{2mmt}-\frac{2epsx}{2mt}+\frac{pss}{2t}\pm\frac{ptt}{2t}$

*Art. 81, and 118.

$=yy-\frac{2n}{m}xy-2ry+\frac{nn}{mm}xx+\frac{2nr}{m}x+rr=o$. Therefore, *&c.*

If the Diameter $KL$ $(2t)$ and its Parameter $KH(p)$ be equal between themſelves; then the Hyperbola will be an equilateral one.

### COROLLARY.

331. IT is manifeſt, 1. That the two Squares $yy$ and $xx$ have always different Signs in that Formula, when the Plane $xy$ happens not to be therein; or even when it is in the ſame, if $\frac{eep}{2mmt}$ exceeds $\frac{nn}{mm}$. 2. That the ſaid Squares may have the ſame Signs, upon Condition that the Plane $xy$ be not in the Formula, and the Square $\left(\frac{nn}{mm}\right)$ of one half the Fraction multiplying $xy$, be greater than the Fraction $\frac{nn}{mm}-\frac{eep}{2mmt}$ multiplying the Square $xx$.

## PROPOSITION IV.

### Problem.

332. *TO conſtruct the Locus of a given Equation, wherein the Squares* yy *and* xx *have different Signs, or even the ſame Signs, upon Condition that the Plane* xy *does not happen therein, and the Square of one half the Fraction multiplying the ſaid Plane* xy, *be greater than the Fraction multiplying the Square* xx. *Suppoſing the Square* yy *to be free from Fractions.*

The

The Locus of the Equation being an Hyperbola, muſt be conſtructed in the ſame manner as the Ellipſis was in the laſt Problem, as ſhall appear in the following Examples.

## EXAMPLE I.

333. LET there be an Equation $yy + \frac{2b}{a}xy + \frac{f}{a}xx + 2cy - 2gx - bb = o$, whoſe Locus it is requir'd to conſtruct, ſuppoſing the Square $\frac{bb}{aa}$ therein to exceed $\frac{f}{a}$.

Compare the Terms of this Equation with thoſe anſwering them in the Formula of the Lemma, and then we have 1. $\frac{2n}{m} = -\frac{2b}{a}$, and therefore if you make $m = a$, then $n$ will be $= -b$. 2. $\frac{eep}{2mmt} - \frac{nn}{mm} = \frac{f}{a}$, and ſo $\frac{p}{2t} = \frac{bb-af}{ee}$, and $p = \frac{2bbt-2aft}{ee}$. 3. $r = c$. 4. $\frac{2nr}{m} + \frac{2eps}{2mt} = -2g$, from whence ſubſtituting for $m$, $n$, $r$, $\frac{p}{2t}$, their Values already found, and we ſhall get $s = \frac{-bce-age}{bb-af}$. 5. $\pm tt = ss - \frac{2rrt-2bbt}{p} = ss - \frac{eecc-eebb}{bb-af}$, $tt$ being affirmative when the Square $ss$ exceeds $\frac{eecc+eebb}{bb-af}$, and negative when the ſame is leſs, becauſe the Square $tt$ muſt be poſitive; from whence there are two Caſes. Now the Values of $m$, $n$, $r$, $s$, $t$, $p$, being thus determined, the Locus of the Equation may be conſtructed after the following manner, uſing the Conſtruction of the Lemma.

In $AP$, aſſume $AB = a$, and draw the right Lines $BE = b = -n$, $AD = c = -r$, parallel to $PM$, and on the coutrary Side of $AP$ in reſpect to $PM$; alſo through the Points $A$, $E$, draw the right Line $AE$ ($e$) which is given, and through the Point $D$ the indefinite right Line $DG$ parallel to $AE$, in which aſſume $DC = \frac{eag+ebc}{bb-af} = -s$ from $D$ towards $PM$, and on both Sides the Point $C$ aſſume $CL$, $CK$, each equal to $t = \sqrt{ss - \frac{eecc-eebb}{bb-af}}$ or $\sqrt{\frac{eecc+eebb}{bb-af} - ss}$, according as $ss$ is greater or leſs than $\frac{eecc+eebb}{bb-af}$. This being done, with the Diameter $LK$, whoſe Ordinates are parallel to $PM$, and Parameter the Fig. 178.

Line $KH\,(p) = \frac{2bbt-2aft}{ee}$, deſcribe an Hyperbola. obſerving that $LK$ muſt be a firſt Diameter *(Fig.* 177.*)* in the former Caſe, and a ſecond *(Fig.* 178.*)* in the latter Caſe. Then the Portion $OM$ of that Hyperla will be the Locus ſought.

For if from any one of the Points thereof, as $M$, the Line $MP$ be drawn parallel to $AD$, meeting the Lines $AB$, $AE$, $DG$, in the Points $P, F, G$; then we ſhall have $PF = \frac{bx}{a}$, and $AF$ or $DG = \frac{ex}{a}$. And conſequently $MG = y + \frac{bx}{a} + c$, and $CG$ or $DG + CD = \frac{ex}{a} - s$, becauſe $CD = -s$. But by the Property of the Hyperbola, $LK\,(2t) : KH\left(\frac{2bbt-2aft}{ee}\right) :: \overline{CG}^2 \pm \overline{CK}^2 \left(\frac{eexx}{aa} - \frac{2esx}{a} + ss \pm tt\right) : \overline{GM}^2\,(yy + \frac{2b}{a}xy + 2cy + \frac{bb}{aa}xx + \frac{2bc}{a}x + cc)$ which gives the given Equation by ſubſtituting $\frac{ccee-eebb}{bb-af}$ and $\frac{-bce-age}{bb-af}$ for their Values $ss \pm tt$ and $s$, and multiplying the Means and Extremes, and dividing by $2t$. Therefore, *&c.*

## SCHOLIUM.

334. IF $ss$ be $= \frac{ccee+eebb}{bb-af}$; then it is manifeſt, that the Value of $tt$ will be $= o$, and ſo the Conſtruction of the Hyperbola will be impoſſible. In which Caſe it muſt be obſerv'd, that the propoſed Equation may always be brought lower, ſo that the Locus thereof, which ought to be an Hyperbola, or two oppoſite ones, will become one or two right Lines: For in our Example the propoſed Equation may be reduced to this Proportion, *viz.* $ee : bb - af :: \frac{eexx}{aa} - \frac{2esx}{a} + ss \pm tt : yy + \frac{2b}{a}xy + \frac{bb}{aa}xx + 2cy + \frac{2bc}{a}x + cc$; from whence ſtriking out $tt = o$, multiplying the Extremes and Means, and extracting the ſquare Root of both Sides, and there will ariſe $ey + \frac{ebx}{a} + ec = \frac{ex}{a} - s\sqrt{bb-af}$, that is, (putting $\frac{bce+age}{bb-af}$ for $-s$, and dividing both Sides by $e$) this Equation $y + \frac{bx}{a} + c = \frac{x\sqrt{bb-af}}{a} + \frac{ag+bc}{\sqrt{bb-af}}$, or $y = \frac{-b+\sqrt{bb-af}}{a}x +$

ag

$\frac{ag+bc}{\sqrt{bb-af}} - c$, which will be chang'd into this $y = p - \frac{n}{m}x$ (by making $\frac{n}{m} = \frac{b-\sqrt{bb-af}}{a}$, and $p = \frac{ag+bc}{\sqrt{bb-af}} - c$) whoſe Locus is a ſtrait Line, and may be conſtructed by *Art.* 306.

The Reaſon of this is evident by the general Formula of the Lemma: For if the Term $\mp\frac{ptt}{2t}$, be ſtruck out in that Lemma, becauſe $tt = o$; then by tranſpoſing ſome Terms, and extracting the ſquare Root, we ſhall have this Equation $y - \frac{n}{m}x - r = \frac{ex}{m} - s\sqrt{\frac{p}{2t}}$ or $s - \frac{ex}{m}\sqrt{\frac{p}{2t}}$, wherein the unknown Quantities $x$ and $y$, are but of one Dimenſion, and conſequently the Locus thereof will be ſtrait Lines.

## EXAMPLE II.

335. THE Locus of the following given Equation is requir'd, *viz.* $yy - xx + 2ay + ax = o$.

By comparing the correſpondent given Terms, we have 1. $\frac{2n}{m} = o$, becauſe the Plane $xy$ is not in the propoſed Equation; from whence we get $n = o$, and ſo $m = e$. 2. $\frac{p}{2t} = 1$, and therefore $p = 2t$. 3. $r = -a$. 4. $\frac{2ps}{2t} = a$; whence $s = \frac{1}{2}a$. 5. $rr \mp \frac{ptt}{2t} - \frac{pss}{2t} = o$; and ſo $\mp tt = ss - \frac{2rrt}{p} = -\frac{3}{4}aa$, by putting $-a$, 1, $\frac{1}{2}a$, for their Values $r$, $\frac{2t}{p}$, $s$; from whence we know, that $-tt$ muſt be taken in the laſt Term in the Formula, and not $+tt$, that ſo the Value of $tt$ may be poſitive. Now the Locus may be thus conſtructed.

Becauſe $AD\,(r) = -a$, therefore through the Point $A$ draw the right Line $AD = a$, parallel to $PM$, and on the contrary Side of $AP$ with regard to $PM$; and becauſe $BE\,(n) = o$, therefore through the Point $D$ draw the right Line $DG$ parallel to $AP$, in which aſſume $DC\,(s) = \frac{1}{2}a$ from $D$ towards $PM$; alſo aſſume $CK$, $CL$, on both Sides the Point $C$, each equal to $t = \sqrt{\frac{3}{4}aa}$. This being done with the ſecond Diameter $LK$ (becauſe $-tt$ was taken in the laſt Term of the Formula) whoſe Ordinates are parallel to $PM$, and Parameter the right Line $KH\,(p) = 2t = LK$, deſcribe an Hyperbola. Then the Portion $OM$ thereof will be the Locus ſought. FIG. 179.

For if $MP$ be drawn thro' any one of its Points $M$ parallel to $AD$; meeting

the right Lines $AP$, $DG$, in the Points $P$,$G$; then we ſhall have $MG = y + a$, $CG$ or $DG - DC = x - \frac{1}{2}a$; and by the Property of the Hyperbola, $LK\ (2t) : KH\ (2t) :: \overline{CG}^2 + \overline{CK}^2\ (xx - ax + \frac{1}{4}aa + tt) : \overline{GM}^2\ (yy + 2ay + aa)$; which gives the propoſed Equation $yy + 2ay - xx + ax = o$, by putting $\frac{3}{4}aa$ for $tt$. Hence it appears, that the Hyperbola is an Equilateral one.

### SCHOLIUM.

336. WHEN the Squares $yy$ and $xx$ happen to have different Signs, and without Fractions in an Equation, wherein the Plane $xy$ is not; then the Locus of that Equation will be always an Equilateral Hyperbola: for the Fraction $\left(\frac{2n}{m}\right)$ of the Formula will be $= o$; and therefore $BE\ (n) = o$, and $m = e$. Conſequently the Fraction $\frac{nn}{mm} - \frac{eep}{2mmt}$, which multiplies the Square $xx$ in the Formula, will become $-\frac{p}{2t}$; and ſo we ſhall have $-\frac{p}{2t} = 1$, that is, the Diameter $LK$ will be equal to its Parameter $KH$; or, which is the ſame thing, the Hyperbola will be an Equilateral one. And becauſe the general Formula will then be changed into this;

$$\begin{array}{l} yy - xx - 2ry + 2sx + rr = o \\ \quad + tt \\ \quad - ss \end{array}$$

therefore the Values of $r$, $s$, $t$, may be firſt gotten from the ſame, and afterwards the Equilateral Hyperbola being the Locus of the propoſed Equation may be conſtructed by means of thoſe Values; which will very much ſhorten the Operation.

### *The* FUNDAMENTAL LEMMA, *for the Conſtruction of the Loci of Equations, which are Hyperbola's between their Aſymptotes.*

337. LET there be (as in *Def.* 1.) two unknown and indeterminate right Lines $AP\ (x)$, $PM$, $y$, making the Angle $APM$ with one another either given or taken at pleaſure; alſo let $m$, $n$, $p$, $r$, $s$, be given right Lines. This being ſuppos'd;

FIG. 180. 1. In the Line $AP$, aſſume $AB = m$, and draw the right Lines $BE = n$, $AD = r$, parallel to $PM$, and both on the ſame Side $AP$ with regard to $PM$; then through the Point $A$ draw the right Line $AE\ (e)$ which is given, and through the Point $D$ the indefinite right Line $DG$ parallel to $AE$; in which aſſume $DC = s$, $CK = e$, both tending

tending from $D$ the ſame way as $AP$ tends, and draw the indefinite right Line $CL$ parallel to $PM$, and on the ſame Side $AP$; and laſtly draw the Line $KH = p$. Then deſcribe * an Hyperbola between the Aſymptotes $CL$, $CK$, paſſing through the Point $H$. I ſay, the ſame will be the Locus of the following Equation or Formula, *viz.* * *Art.* 130, 131.

$$xy - \frac{n}{m}xx - \frac{ms}{e}y + \frac{ns}{e}x + \frac{mrs}{e} = o$$
$$- rx - mp$$

For $GM = y - \frac{nx}{m} - r$, $CG = \frac{ex}{m} - s$, and by the Property of the Hyperbola * $CG \times GM\left(\frac{exy}{m} - sy - \frac{enxx}{mm} + \frac{nsx}{m} - \frac{erx}{m} + rs\right) = CK \times KH\,(ep)$; from whence freeing the Term $xy$ from Fractions, and putting all the Terms in order, there will ariſe the ſame Equation as above, *viz.* $xy - \frac{n}{m}xx - \frac{ms}{e}y$ &c. * *Art.* 101.

2. Through the fixed Point $A$ draw the indefinite Line $AQ$ parallel to $PM$ and on the ſame Side $AP$, with regard to $PM$; then in $AQ$ aſſume $AB = m$, draw $BE = n$ parallel to $AP$, and towards the ſame Parts, and join the determinate Points $A$, $E$, by the Line $AE\,(e)$; moreover in $AP$ aſſume $AD = r$ from $A$ towards $PM$, and draw the indefinite Line $DG$ parallel to $AE$, in which aſſume $DC = s$, $CK = e$ tending the ſame way as $PM$ tends, and draw the indefinite Right Lines $CL$ and $KH = p$ both parallel to $AP$, (on the ſame Side thereof as $PM$). This being done between the Aſymptotes $CL$, $CK$, deſcribe * an Hyperbola which paſſes through the Point $H$. Then I ſay that the ſame will be the Locus of the following Equation or Formula, *viz.* FIG. 181. * *Art.* 130, 131.

$$xy - \frac{n}{m}yy - \frac{ms}{e}x + \frac{ns}{e}y + \frac{mrs}{e} = o$$
$$- ry - mp$$

For if the Line $MQ$ be drawn from any Point $M$ thereof, parallel to $AP$, and meeting the Parallels $AE$, $DG$, in the Points $F$, $G$; then the ſimilar Triangles $ABE$, $AQF$, will give theſe Proportions $AB\,(m) : AE\,(e) :: AQ$ or $PM\,(y) : AF$ or $DG = \frac{ey}{m}$, and $AB\,(m) : BE\,(n) :: AQ\,(y) : QF = \frac{ny}{m}$. And conſequently $GM = x - \frac{ny}{m} - r$, $CG = \frac{ey}{m} - s$. But by the Property of the Hyperbola, $CG \times GM = CK \times KH$; from whence, by ſubſtituting for theſe Lines their analytical Values, and freeing the Term $xy$ from Fractions, the ſecond Formula aforegoing will be had. Therefore, *&c.*

COROL-

### COROLLARY.

338. IT is manifeſt, 1. That the Term $x\,y$ is found always in both the Formula's; ſince the ſame being multiply'd by no Fraction, cannot be ſuppoſed equal to nothing, in order for that Term to vaniſh or be ſtruck out. 2. In either of the Formulas there is only one of the Squares $x\,x$ and $y\,y$, which will vaniſh if the Fraction $\frac{n}{m}$ multiplying the ſame be equal to nothing.

## PROPOSITION V.

### Problem.

339. *TO find the Locus of a given Equation, wherein is the Plane* x y, *without either of the Squares* x x, y y, *or only with one of the Squares* x x *or* y y.

Free the Plane $x\,y$ from Fractions, and compare the Terms of the given Equation with thoſe that anſwer to it in the firſt Formula, when the Square $x\,x$ is in the given Equation, and with thoſe of the ſecond Formula, when $y\,y$ is in the ſame; and finally either of them at pleaſure, when neither of the Squares $x\,x$ and $y\,y$ are in the given Equation. By this Means, get the Values of the Quantities $m, n, p, r, s$, and then by means of thoſe Values deſcribe an Hyperbola between its Aſymptotes, according to the Directions in the foregoing Lemma, always obſerving to draw the Lines whoſe Values are negative, on the contrary Side of $AP$, with reſpect to $PM$, and aſſume thoſe whoſe Values are ſo likewiſe, tending the contrary way to which $A\,P$ tends. The following Example will make this manifeſt.

### EXAMPLE I.

340. IT is requir'd to find the Locus of $x\,y - \frac{b}{a}\,x\,x - c\,y = o$.

Becauſe the Square $x\,x$ happens to be in the given Equation, therefore chuſe the firſt Formula, and then comparing the Terms thereof with thoſe of the propoſed Equation, and we have 1. $\frac{n}{m} = \frac{b}{a}$, whence making $m = a$, and we get $n = b$. 2. $\frac{ms}{e} = c$, and therefore $s = \frac{ec}{a}$. 3. $\frac{ns}{e} - r = o$, becauſe $x$ is not in the given Equation, and therefore $r = \frac{bc}{a}$. 4. $\frac{mrs}{e} - mp = o$, becauſe there is no Term in the propoſed Equation quite known: And therefore $p = \frac{rs}{e} = \frac{bcc}{aa}$. Now becauſe the Values

Values of $AP(m)$, $BE(n)$, $CD(s)$, $AD(r)$, $KH(p)$ are all positive, the Locus muſt be conſtructed in the very ſame manner as in the Lemma (*Fig.* 180.) obſerving to uſe the Values of the Lines as here determin'd.

For $GM = y - \frac{bx}{a} - \frac{bc}{a}$, $CG$ or $DG - DC = \frac{ex - ec}{a}$, and by the Property of the Hyperbola $CG \times GM = CK \times KH$, that is, by ſubſtituting the analytick Values, the propoſed Equation. Therefore, *&c.* FIG. 180.

## EXAMPLE II.

341. LET $xy + \frac{b}{a}yy - cy - ff = o$, be an Equation, whoſe Locus it is required to conſtruct.

Becauſe the Square $yy$ is in this Equation, therefore chuſe the ſecond Formula, and by comparing the Terms thereof with thoſe of the propoſed Equation, we have, 1. $\frac{n}{m} = -\frac{b}{a}$; and making $m = a$, we get $n = -b$. 2. $\frac{ms}{e} = o$, and therefore $s = o$. 3. $r = c$. 4. $mp = ff$, and therefore $p = \frac{ff}{a}$. From whence ariſes the following Conſtruction.

Through the fixed Point $A$ draw the indefinite Right Line $A\mathcal{Q}$ parallel to $PM$, and on the ſame Side $AP$, and in $A\mathcal{Q}$ aſſume $AB(m) = a$, and draw $BE = b = -n$ parallel to $AP$, and tending the contrary way to $AP$; alſo through the determinate Points $A$, $E$, draw the Line $AE(e)$. This being done, in $AP$, take $AD(r) = c$ from $A$ towards $PM$, and draw the indefinite Right Line $DG$ parallel to $AE$, and becauſe the Points $D$, $C$, do coincide, ſince $DC(s) = o$, therefore in $DG$ aſſume $DK = e$, tending the ſame way as $PM$, and draw the Line $KH(p) = \frac{ff}{a}$, parallel to $AP$, (and tending the ſame way) as alſo the indefinite Right Line $DL$, which here falls in $AP$; then deſcribe an Hyperbola between the Aſymptotes $DL$, $DK$, which paſſes through the Point $H$. And the ſame will be the Locus requir'd. FIG. 182.

For if the Right Line $M\mathcal{Q}$ be drawn from any Point $M$ thereof, parallel to $AP$, and meeting the Parallels $AE$, $DG$, in the Points $F$, $G$; then we ſhall have $GM$ or $M\mathcal{Q} + \mathcal{Q}F - FG = x + \frac{by}{a} - c$, $DG$ or $AF = \frac{ey}{a}$, and therefore $DG \times GM = \frac{exy}{a} + \frac{ebyy}{aa} - \frac{ecy}{a} = DK \times KH\left(\frac{eff}{a}\right)$. From whence by freeing the Term $xy$ from Fractions, there

will

will ariſe the propoſed Equation $xy + \frac{b}{a}yy - cy - ff = o$.

## SCHOLIUM.

342. IF for the arbitrary Quantity $AB$ $(m)$ be taken ſome Value greater or leſs than $a$, then the Values of $CK$ $(e)$ and $KH$ $(p)$ will alter, but the Values of the Rectangle $CK \times KH$ $(ep)$ and of the Right Lines $AD$ $(r)$, $CD$ $(s)$ will remain always the ſame: For in the Expreſſions of theſe Values are contain'd only the Ratio's $\frac{n}{m}$, $\frac{n}{e}$, $\frac{m}{e}$ which do not vary, becauſe in the Triangle $ABE$, the Angle $ABE$ is given, as alſo the Ratio $\frac{n}{m}$ (which in this Example is $\frac{b}{a}$) of the Side $AB$ $(m)$ to the Side $BE$ $(n)$. And becauſe the Hyperbola which muſt paſs through the Point $H$, ſhall be the ſame * always, let the Magnitude of $CK$ $(e)$ and $KH$ $(p)$, be what it will, provided the Rectangle $CK \times KH$ remains the ſame, therefore the ſame Hyperbola will be conſtructed, let the Magnitude of the arbitrary Quantity $AB$ $(m)$ be what it will.

*Art. 101.

## EXAMPLE III.

343. IT is requir'd to conſtruct the Locus of the given Equation $xy - ay + bx + cc = o$.

Becauſe neither of the Squares $xx$, $yy$, are found in the propoſed Equation, therefore chuſe indifferently any one of the two Formulas, for Example, the firſt, and comparing the Terms of the ſame with thoſe of the propoſed Equation, we have 1. $\frac{n}{m} = o$, and therefore $n = o$, and $m = e$, and make $m = a$. 2. $\frac{ms}{e}$ or $s = a$. 3. $r = -b$, becauſe $\frac{ns}{e} = o$. 4. $rs - mp = cc$, and therefore $p = -b - \frac{cc}{a}$. Now the Values of $m$, $n$, $r$, $s$, $p$, being thus determin'd, the Locus of the propoſed Equation may be conſtructed after the following manner

FIG. 183. Becauſe $AD$ $(r) = -b$, therefore draw the Line $AD = b$ parallel to $PM$, and on the contrary Side of $AP$ with regard to $PM$; and becauſe $BE$ $(n) = o$, therefore draw the indefinite right Line $DG$ parallel to $AP$, and in the ſame aſſume $DC$ $(s) = a$, $CK$ $(e) = m = a$ tending the ſame way as $AP$: Alſo draw the indefinite right Line

$CL$, and the Line $KH = b + \frac{cc}{a} = -p$, both parallel to $PM$, and tending the contrary way. Then deſcribe an Hyperbola oppoſite to that, whoſe Aſymptotes are $CL$, $CK$, and which paſſes thro' the Point $H$. I ſay, the indefinite Portion $OM$ thereof contain'd in the Angle $PAS$, form'd by the indefinite right Line $AP$, and the Line $AS$, drawn parallel to $PM$ on the ſame Side $AP$, ſhall be the Locus ſought.

For $GM$ or $PG + PM = y + b$, and $CG$ or $CD - DG = a - x$, and conſequently $CG \times GM = ay - xy + ab - bx = CK \times KH$ $(ab + cc)$; from whence, ſtriking out the Rectangle $ab$ from both Sides, and tranſpoſing, there ariſes $xy - ay + bx + cc = o$, which is the propoſed Equation. The Deſcription of the Hyperbola paſſing through the Point $H$ in this Example is uſeleſs, becauſe no Point thereof can fall in the Angle $PAS$, wherein the Points $M$ muſt be ſuppos'd to fall.

### SCHOLIUM.

344. IN comparing the Terms of the Formula with thoſe of a given Equation, if it ſhould happen that $p = o$; then it would be impoſſible to deſcribe the Hyperbola that ought to be the Locus thereof, becauſe its *Power*, which is equal to the Rectangle $pe$, will be $= o$. But then the Equation may be brought lower, ſo that the Locus will become a ſtrait Line. For Example; if the Term $mp$ be ſtruck out of the firſt Formula of the Lemma, then that Formula will become $xy - \frac{n}{m}xx - \frac{ms}{e}y + \frac{ns}{e}x - rx + \frac{mrs}{e} = o$, which being divided by $\frac{ex}{m} - s$ gives $y - \frac{nx}{m} - r = o$, whoſe Locus is * a ſtrait Line. *Art. 306.

## PROPOSITION VI.

### Problem.

345. *TO conſtruct the Locus of any given Equation of the ſecond Degree.*

Bring over all the Terms of the Equation to one Side, ſo that one Member thereof be $o$, then there may happen two Caſes.

*Caſe* 1. When the Plane $xy$ is not in the given Equation. 1. If there be but one of the Squares $yy$ or $xx$ therein, then the Locus will be * a Parabola. 2. If both the Squares $yy$ and $xx$ are found therein with the ſame Signs, then the Locus will be * an Ellipſis or Circle. 3. If the ſaid two Squares are found therein with different Signs, then the Locus thereof will be * an Hyperbola, or the oppoſite Sections, regarding their Diameters. *Art. 310. *Art. 324. *Art. 332.

*Caſe* 2. When the Plane $x\,y$ happens to be in a given Equation. 1. If neither of the Squares $y\,y$ and $x\,x$, or but one of them, are found in the Equation, then the Locus will be * an Hyperbola between its Aſymptotes. 2. If the Squares $y\,y$ and $x\,x$ are found therein with different Signs, then the Locus ſhall be * an Hyperbola regarding its Diameters. 3. If the ſaid two Squares have the ſame Signs, the Square $y\,y$ muſt be freed from Fractions, and then the Locus ſhall be * a Parabola, when the Square of half the Fraction multiplying $x\,y$, is equal to the Fraction multiplying $x\,x$; an Ellipſis * or Circle, when the ſame is leſs; and finally, an Hyperbola, * or two oppoſite ones, regarding their Diameters, when the ſame is greater.

*Art.* 339. *Art.* 332. *Art.* 310. *Art.* 324. *Art.* 332.

Now deſcribe the Locus, by *Art.* 310. if the ſame be a Parabola; by *Art.* 324. if the ſame be an Ellipſis or Circle; by *Art.* 332. if it be an Hyperbola, or the oppoſite Sections regarding their Diameters; and laſtly, by *Art.* 339. if the ſame be an Hyperbola between the Aſymptotes. All this is but a Continuation of thoſe four Articles.

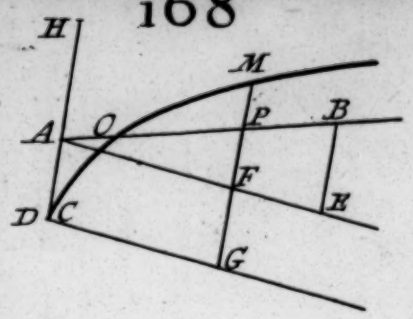
## COROLLARY.

346. BEcauſe the Conick Section found by the aforeſaid Rules, is the Locus * of all the poſitive and negative Values of the unknown Quantity $y$, anſwering to the poſitive and negative Values of $x$, in a given Equation of the ſecond Degree; therefore there is but that Section only that can be the Locus of the given Equation.

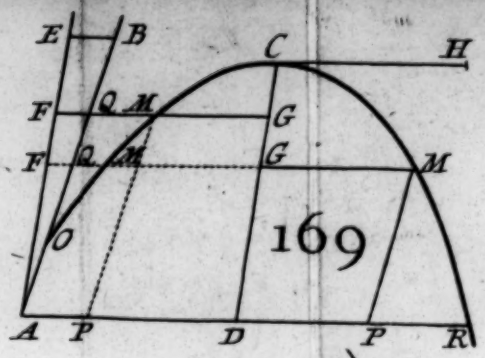
*Art.* 314.

*The End of the Seventh Book.*

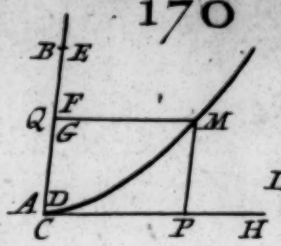
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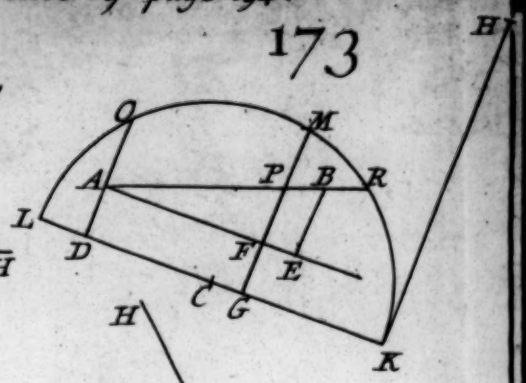

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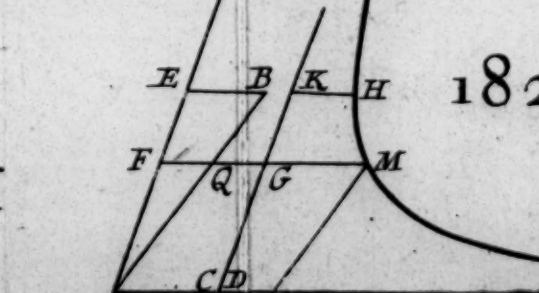

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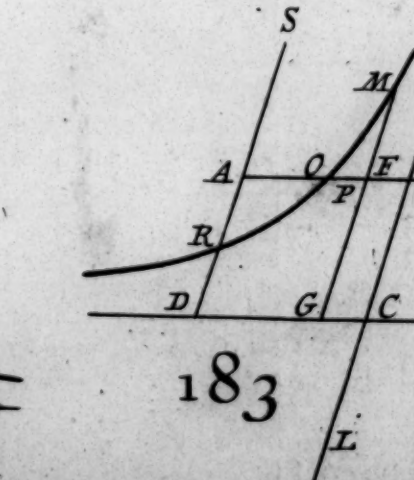

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# BOOK VIII.

## *A General* PROPOSITION.

347. *TO find the Locus of an infinite Number of Points, having all certain known Conditions; provided the ſaid Locus exceeds not the ſecond Degree.* FIG. 184.

1. Suppoſe two unknown and indeterminate Right Lines $AP(x)$ $PM(y)$ making an Angle ($APM$) with one another, either given or taken at pleaſure, as being known and determinate; and let one of them as $AP$, have a fixed and ſtable Origin in the Point $A$, and be extended along a Right Line given in Poſition; and let the other $PM$, whoſe Extremity $M$ determines one of the Points ſought, continually alter its Poſition, and be always parallel to the ſame Line. 2. Draw the other Lines that you think convenient for the Solution of the Problem, and expreſs them by Letters, *viz.* the known Lines by the former Letters of the Alphabet, and the unknown ones, by the latter Letters. 3. Suppoſe the Queſtion reſolved, and after you have gone over all the Conditions thereof, form an Equation containing only the two unknown Lines $x$ and $y$, mixed with known ones. 4. Having form'd an Equation, wherein the unknown Lines $x$ and $y$ have not more than two Dimenſions, conſtruct the Locus of that Equation according to the Directions in the laſt Book; and then the Locus thus conſtructed, will ſolve the Queſtion. All this will be manifeſt by the following Examples.

### EXAMPLE I.

348. *THE Angle $BAC$ being given, to find the Point $M$ in it; ſuch that two right Lines $MF$, $MG$, drawn from the ſame, making the given Angles $MFB$, $MGC$, with the Sides $AB$, $AC$, always towards the ſame Parts; the Right Line $MF$ may be to the Right Line $MG$ in a given Ratio, ſuppoſe as $a$ to $b$. And becauſe there are an infinite Number of ſuch Points, it is requir'd to find the Line containing them all, that is, their Locus.* FIG. 184.

Through the Point $M$, which is ſuppoſed to be one of the Points ſought, draw the Line $MP$ parallel to $AC$, and conceive the two unknown and indeterminate Lines $AP(x)$, $PM(y)$, as known and determinate. Then in the Side $AB$ aſſume $AB=a$, and draw the Right Lines $BC$, $BD$, parallel to $MF$, $MG$, meeting the other Side $AC$ ( produced if neceſſary ) in the Points $C$, $D$. This being done, call the known Lines $AC$, $c$; $BC$, $f$; $BD$, $g$; then drawing $M\mathcal{Q}$ parallel to $AB$, the Triangles $ACB$, $PMF$, and $ABD$, $\mathcal{Q}MG$, will be ſimilar, therefore $AC(c):CB(f)::MP(y):MF=\frac{fy}{c}$, and $AB(a):BD(g)::M\mathcal{Q}$ or $AP(x):MG=\frac{gx}{a}$; and this ſatisfies the firſt State of the Problem, ſince the Lines $MF$, $MG$, are ſuppoſed to be always parallel to the ſame two ſtraight Lines $BC$, $BD$, which make the given Angles with the Sides $AB$, $AC$. Now by the ſecond Condition remaining to be fulfilled, it muſt be as $MF\left(\frac{fy}{c}\right):MG\left(\frac{gx}{a}\right)::a:b$; from whence there ariſes $y=\frac{cgx}{bf}$ which includes all the Conditions of the Problem, and the Locus of the ſame will conſequently be the Locus ſought, and may be conſtructed thus *;

*Art. 306.

In the Line $AP$, aſſume $AH=b$, and draw $HE=\frac{cg}{f}$ parallel to $PM$, and on the ſame Side $AP$ with regard to $PM$; then if the indefinite Right Line $AE$ be drawn, I ſay the ſame will be the Locus of all the ſought Points $M$.

For through any Point $M$ thereof draw the Right Lines $MP$, $M\mathcal{Q}$, parallel to the two Sides $AC$, $AB$, and the right Lines $MF$, $MG$, parallel to $BC$, $BD$, which conſequently ſhall make the given Angles with the Sides $AB$, $AC$; then becauſe the Triangles $AHE$, $APM$, are ſimilar, therefore $AH(b):HE\left(\frac{cg}{f}\right)::AP(x):PM(y)=\frac{cgx}{bf}$, and becauſe the Triangles $ACB$, $PMF$, and $ABD$, $\mathcal{Q}MG$, are ſimilar, therefore $AC(c):CB(f)::MP\left(\frac{cgx}{bf}\right):MF=\frac{gx}{b}$; and $AB(a):BD(g)::M\mathcal{Q}$ or $AP(x)\ MG=\frac{gx}{a}$. And conſequently $MF\left(\frac{gx}{b}\right):MG\left(\frac{gx}{a}\right)::a:b$. Which is the Proportion propoſed.

We have ſolv'd this Problem by Calculation, only to explain the general Propoſition, and ſhew the Application thereof in beginning firſt with ſimple and eaſy Examples; for the ſame may be ſolv'd in a much eaſier manner, without any Calculus, thus.

Draw

Draw the right Lines $AK$, $AL$, making the given Angles $KAB$, $LAC$, with $AB$, $AC$, being to one another in the given Ratio of $a$ to $b$. Alſo draw the right Lines $KM$, $LM$, parallel to the Sides $AB$, $AC$, meeting each other in the Point $M$; through which, and the Vertex $A$ of the given Angle $BAC$, draw the Line $AM$; and the ſame will be the Locus ſought. Fig. 185.

For drawing the right Lines $ER$, $ES$, from any Point $E$ thereof parallel to $AK$, $AL$; then becauſe the Triangles $AER$, $MAK$, and $AES$, $MAL$, are ſimilar; therefore $ER : AK :: AE : AM :: ES : AL$. And therefore $ER : ES :: AK : AL :: a : b$.

## EXAMPLE II.

349. THE Parallels $AB$, $CD$, being given in Poſition; to find the Locus of all the Points $M$ ſo ſituate between thoſe Parallels, that the right Lines $MP$, $MG$, being drawn, making always given Angles $MPB$, $MGD$, with them towards the ſame Part; the ſaid Lines $MP$, $MG$ may be to one another always in the given Ratio of $a$ to $b$. Fig. 186.

Aſſume any Point ($A$) of the Line $AB$, for the fix'd Origin of the indeterminate Line $AP(x)$, and the two unknown and indeterminate Lines $AP(x)$, $PM(y)$ being ſuppos'd known and determinate, draw the Lines $AC$, $AE$, parallel to $MP$, $MG$; then call the known Lines $AC$, $c$; $AE$, $f$; this being done, produce $PM$ till it meets $CD$ in $F$; and the ſimilar Triangles $CAE$, $FMG$, ſhall give $AC(c) : AE(f) :: MF(c-y) : MG = \frac{cf-fy}{c}$. But according to the Condition of the Problem remaining to be fulfilled, it muſt be as $MP(y) : MG \left(\frac{cf-fy}{c}\right) :: a : b$; from whence ariſes this Equation $y = \frac{acf}{bc+af}$, which takes in all the Conditions of the Problem, and the Locus thereof, which is * an indefinite right Line $HM$ drawn parallel to $AB$, ſo that $AH$ be $= \frac{acf}{bc+af}$, will be the Locus ſought. * *Art.* 307.

## EXAMPLE III.

350. TWO Points $A$ and $B$ being given: To find a third Point $M$ ſuch, that if two right Lines $MA$, $MB$ be drawn to the ſame, they may be always to one another in the given Ratio of $a$ to $b$. And becauſe there are an infinite Number of ſuch Points, it is requir'd to deſcribe the Locus of them all. Fig. 187.

Here there may happen three different Caſes, according as $a$ is leſs, greater, or equal to $b$.

Caſe

*Case* 1. From the Point *M*, which is ſuppos'd to be one of the Points ſought, draw the Line *MP* perpendicular to *AB* (for ſince no Angle is given in the Problem, we chuſe a right Angle, as being more ſimple than any other) and the two unknown and indeterminate right Lines *AP* (*x*), *PM* (*y*), being ſuppos'd known and determinate, call the given Line *AB*, *c*; then becauſe the Triangles *APM*, *BPM*, are right-angled, therefore $\overline{AM}^2 = xx + yy$, and $\overline{BM}^2 = cc - 2cx + xx + yy$. But according to the Condition of the Problem, it muſt be as $\overline{AM}^2$ ($xx + yy$) : $\overline{BM}^2$ ($cc - 2cx + xx + yy$) : : $aa$ : $bb$. From whence (multiplying the Extremes and Means; and afterwards dividing by $bb - aa$) we ſhall get this Equation, *viz.* $yy + xx + \frac{2aac}{bb-aa}x - \frac{aacc}{bb-aa} = 0$, the Locus of which will be that ſought; and it may be conſtructed by means of *Art.* 322. after the following manner.

FIG. 187. In the Line *AP* aſſume $AC = \frac{aac}{bb-aa}$ from *A* the contrary way to *P*; and about the Centre *C*, with the Radius *CD* or $CE = \frac{abc}{bb-aa}$ deſcribe the Circumference of a Circle. I ſay, the Portion (*DMO*) thereof contain'd in the Angle *PAO*, made by the Line *AP*, and the right Line *AO*, drawn parallel to *PM*, and on the ſame Side *AP* with regard to *PM*, will be the Locus of the Equation found as above.

For if *MP* be drawn from any Point *M* thereof perpendicular on *AB*; then, by the Nature of the Circle, we ſhall have $\overline{CD}^2 - \overline{CP}^2$ or $EP \times PD = \overline{PM}^2$; that is, the precedent Equation, by putting for theſe Squares their analytick Values.

Now, if the Points *M* be ſuppoſed to fall in the Angle *EAR*, oppoſite to the Angle *BAO*, in which they were ſuppoſed to be ſituate in the foreſaid Proceſs; then if you make * $AP = -x$, and $PM = -y$, there will ariſe the ſame Equation as above, from the Conditions of the Problem and the Property of the Portion (*RME*) of the Circumference already deſcrib'd; and therefore that Portion is the Locus of all the ſought Points *M*, ſituate in the Angle *EAR*. And laſtly, if the Points *M* be ſuppoſed to fall in the Angle *BAR*, and afterwards in the Angle *EAO*, you will find, in like manner, that the Portions *DR*, *EO*, of the ſame Circumference, will be the Loci of thoſe Points (obſerving to make $PM = -y$, when the ſame falls on the other Side the Line *AB*; and $AP = -x$, when the Point *P* falls on the other Side of the fixed Point *A*); therefore the whole Circumference, whoſe Diameter is the Line *DE*, is the compleat Locus of all the ſought Points *M*.

* *Art.* 304.

*Caſe*

*Caſe* 2. In reaſoning after the ſame manner as in *Caſe* 1. there will ariſe this Equation, *viz.* $yy + xx - \frac{2aacx}{aa-bb} + \frac{aacc}{aa-bb} = o$, whoſe Locus may be thus conſtructed.

In $AP$ aſſume $AC = \frac{aac}{aa-bb}$ from $A$ towards $PM$; and with the Centre $C$, and Radius $CD$, or $CE = \frac{abc}{aa-bb}$ deſcribe a Circle. I ſay, the Circumference thereof will be the Locus of all the ſought Points $M$. We prove this after the ſame manner as in *Caſe* 1. FIG. 188.

The Conſtructions in the two laſt Caſes may be much ſhorten'd, if you conſider that the Circumference (whoſe Diameter is $DE$) which is the Locus of all the ſought Points $M$, muſt cut the Line $AB$ in two Points $D$, $E$, ſuch that $AD : DB :: a : b$, and $AE : EB :: a : b$; becauſe the Point $M$ coinciding with $D$, the right Line $AM$ does become $AD$; and $BM$, $BD$; and moreover, when the Point $M$ coincides with $E$, the right Line $AM$ does become $AE$, and $BM$, $BE$. For if the Line $AB$, produced on that Side (as is neceſſary) be divided in the Points $D$, $E$, ſo that $AD : DB :: a : b$, and $AE : EB :: a : b$; then it is evident, that the Line $DE$ in both Caſes will be the Diameter of the Circumference, being the Locus ſought.

*Caſe* 3. Becauſe $a$ is $= b$ in this Caſe, the aforeſaid Equation will be changed into this, $x = \frac{1}{2}c$; from whence if $AP$ be taken equal to $\frac{1}{2}AB$, and the right Line $PM$ be drawn perpendicular on $AB$; the Line $PM$ both ways indefinitely produced, will be * the Locus of all the ſought Points $M$; as is otherwiſe manifeſt by the Elements of Geometry. FIG. 189. *Art. 307.

## EXAMPLE IV.

351. IF there be two Right Lines $DE$, $DN$, (indefinitely produced both ways from the Point $D$) given in poſition on a Plane, together with the Point $C$ without thoſe Lines; and if a given Angle as $CEM$ be ſuppoſed to move along ſo that its Vertex $E$ be always in the Line $DE$, and the Side $EC$ thereof (meeting $DN$ in $N$) paſſes always through the Point $C$, and its other Side $EM$ be always a third proportional to $NC$, $CE$; It is required to find the Locus made by all the Points $M$ in that Motion. FIG. 190.

Draw $CA$ parallel to $DN$; and $CB$ making an Angle at the Point $B$ with $DE$, equal to the given Angle $CEM$, on that Side as is neceſſary, ſo that when $CE$ coincides with $CB$, $EM$ likewiſe coincides with $DE$. Now this Problem may be diſtinguiſh'd into three Caſes: For either the Vertex ($E$) of the given Angle $CEM$ moves along the right Line $DE$ on the other Side of the Point $B$ with reſpect to the

Point $A$; or between the Points $B$, $A$; or finally on the other Side of the Point $A$, with reſpect to the Point $B$.

*Caſe* 1. When the Vertex $E$ moves along the Line $DE$ on the other Side of the Point $B$ with reſpect to $A$; draw the Line $A\mathcal{Q}$ towards the Point C, making the Angle $BA\mathcal{Q}$ with $DE$, equal to the Angle $ABC$; and through one of the Points ſought, as $M$ (which muſt be ſuppoſed as given) draw the Line $MP$ parallel to $A\mathcal{Q}$, meeting $DE$ in $P$; then we ſhall have two ſimilar Triangles $CBE$, $EPM$, for the Angles $CBE$, $EPM$, are each equal to the given Angle $CEM$; and moreover the Angles $BCE$, $PEM$, are alſo equal to one another; becauſe in the Triangle $CBE$, the external Angle $CEP$ or $CEM + PEM$, is equal to the two internal and oppoſite Angles $BCE$, $CBE$ or $CEM$. Now if you call the given Lines $AD$, $a$; $AB$, $b$; $BC$, $c$; and the unknown aud indeterminate ones $AP$, $x$; $PM$, $y$; $AE$, $z$; then according to the Conditions of the Problem, and becauſe $DN$, $AC$, are parallel, we ſhall have theſe Proportions, $AD\,(a) : AE\,(z) :: CN : CE :: CE : EM :: CB\,(c) : EP\,(x-z) :: BE\,(z-b) : PM\,(y)$; from whence (by multiplying the Extremes and Means) we get theſe two Equations, *viz.* $ax - az = cz$ and $ay = zz - bz$, which, taking $f = a + c$ (for brevities ſake), and getting out $z$, will be brought to this here $xx - \frac{bf}{a}x - \frac{ff}{a}y = o$, containing only the two unknown Quantities $x$ and $y$, with their Coefficients, and the Locus of this Equation, which is the Locus ſought, may be * conſtructed after the following manner.

* *Art.* 310.

In $AP$ aſſume the right Line $AF = \frac{bf}{2a}$ on the ſame Side $AP$ as $PM$, and draw $FL$ parallel to $PM$, and in the ſame aſſume $FG = \frac{bb}{4a}$, on the contrary Side $AP$ with regard to $PM$. Then with the Diameter $GL$, whoſe Origin is the Point $G$, Parameter the Line $GH = \frac{ff}{a}$, and Ordinates, are parallel to $AP$, deſcribe a Parabola tending the ſame way as $PM$. I ſay the indefinite Portion ($OM$) thereof contained in the Angle $PA\mathcal{Q}$, will be the Locus of all the ſought Points $M$.

For if the Line $M\mathcal{Q}$ be drawn from any Point $M$ of the ſame parallel to $AP$, and meeting the Diameter $CL$ in $L$; then we ſhall have $ML$ or $PF = x - \frac{bf}{2a}$, and $GL = y + \frac{bb}{4a}$, and by the Nature of the Parabola $\overline{ML}^2 \left(xx - \frac{bf}{a}x + \frac{bbff}{4aa}\right) = LG \times GH \left(\frac{ff}{a}y + \frac{bbff}{4aa}\right)$

from whence by Tranſpoſition there ariſes the given Equation $xx - \frac{bf}{a}x - \frac{ff}{a}y = o$, which was requir'd to be conſtructed.

*Caſe* 2. When the Vertex $E$ moves along $BA$, in this Caſe it is manifeſt, that the Points $M$ ſhall fall on the other Side of $DE$, becauſe the given Angle $CEM$ will be always greater than the Angle $CEP$, which continually diminiſhes; therefore $PM = -y$; and ſince the ſame Equation as above, is found by the like way of Argument as before; therefore the Portion ($AGO$) of the Parabola already deſcrib'd ſhall be the Locus of all the Points $M$, becauſe the aboveſaid Equation ſhall be had likewiſe by the Property of that Portion.

*Caſe* 3. When the Vertex moves on the other Side of the Point $A$, with reſpect to the Points $B$; in this Caſe it is manifeſt moreover, that all the ſought Points $M$ muſt fall below the Line $DE$; and as in the firſt Caſe, we ſhall have $AD : AE :: CN : CE :: CE : EM :: CB : EP$. And therefore $AD : CB :: AE : EP$. From whence it appears, that $EP$ is greater, leſs, or equal to $AE$, according as $CB$ is greater, leſs, or equal to $AD$; and ſo producing $AQ$ below $DE$ towards $K$, all the ſought Points $M$, in the firſt Caſe, do fall in the Angle $BAK$; in the ſecond Caſe, in the Angle $DAK$ the Complement of $BAK$; and laſtly, in the third, on the right Line $AK$. Now ſuppoſe $CB$ to be greater than $AD$; and becauſe if you make $PM = -y$ (ſince the ſame falls on the other Side of $AP$) there will not ariſe the ſame Equation as in the firſt Caſe; and ſo the Conſtruction of that Caſe will be uſeleſs here; therefore calling, as before, $AP$, $x$; $PM$, $y$; and we ſhall get this Equation, *viz.* $xx + \frac{bg}{a}x - \frac{gg}{a}y = o$, (wherein $g = c - a$) whoſe Locus being that requir'd, will be the indefinite Portion ($AM$) of a Parabola different from the former one, and tending the oppoſite way to that; and the ſame may be conſtructed * after the following manner.

* *Art.* 310.

In $AP$ aſſume $AS = \frac{bg}{2a}$ from $A$ the contrary way to $PM$, and draw $ST = \frac{bb}{4a}$ parallel to $AQ$; and on the contrary Side, $AP$ with regard to $PM$; then with the Diameter $TS$, whoſe Origin is the Point $T$, Parameter a Line $= \frac{gg}{a}$, and whoſe Ordinates are parallel to $AP$, deſcribe a Parabola tending from $P$ towards $M$. And the indefinite Portion ($AM$) thereof contain'd in the Angle $PAK$, ſhall be the Locus of all the Points $M$ in this laſt Caſe, wherein $CB$ is ſuppos'd to be greater than $AD$. Therefore it is evident, that the ſought Locus of all the Points $M$, conſiſts of two indefinite Portions of different Parabola's; one of which, as $AGOM$, tends towards $C$, and the other $AM$ the contrary way, and both of them proceed from the Point $A$; for when $CE$

the Side of the given Angle $CEM$ falls in $CA$, which is parallel to $DN$; then it is manifeſt, that $CN$ will become infinite, and ſo $EM$ is $= o$, ſince we have always $NC : CE :: CE : EM$; that is, the Point $M$ does coincide with the Points $E$ and $D$: Therefore it appears, that $AF$ is an Ordinate to the Diameter $FG$, and $AS$ to the Diameter $ST$; and from hence ariſes the following general Conſtruction.

In the indefinite right Line $AP$ aſſume $BO$, $BR$, on both Sides the Point $B$, each equal to a fourth proportional to the three Lines $DA$, $AB$, $BC$; and through $F$, $S$, the middle Points of $AO$, $AR$, draw the Right Lines $FG$, $ST$, parallel to $AQ$, and each equal to a third Proportion to $4AD$, and $AB$, *viz.* $FG$ on the contrary Side of $AP$, with regard to the Point $C$, and $ST$ on the ſame Side. This being done, deſcribe two Parabola's; one with $GF$, as a Diameter, and $FA$ an Ordinate thereto; and the other with $TS$ as a Diameter, and $SA$ an Ordinate to the ſame: I ſay the indefinite Portions $MAGOM$ of thoſe Parabola's, will be the compleat Locus of all the ſought Points $M$.

For $BO$ or $BR = \frac{bc}{a}$, and therefore $AF$ or $\frac{1}{2}AO = \frac{1}{2}b + \frac{bc}{2a} = \frac{bf}{2a}$; and alſo $AS$ or $\frac{1}{2}AR = \frac{bc}{2a} - \frac{1}{2}b = \frac{bg}{2a}$. Therefore, *&c.*

Here it may be obſerved (by the by) that if the given Angle whoſe Vertex moves along the Line $DE$, ſhould be equal to the Complement of the Angle $CEM$ to two Right Angles, every thing elſe remaining the ſame; that is, if the Points $M$ ſhould fall on the Line $EM$ produced on the other Side of the Point $E$: Then the Locus of all the Points $M$ would be the remaining Portions of the two Parabola's above deſcrib'd.

If the Points $A$, $B$, $C$, ſhould have a different Situation from what they are ſuppoſed to have in the Figure; two Equations would ſtill be had, only differing from the former ones in ſome Signs; and ſo their Loci will conſequently be Portions of Parabola's, that may be deſcrib'd with the ſame eaſe, as before.

This Problem was propoſed in the Journal of *Parma*, for *April*, in the Year, 1693. by *Count Vintimille*, which gave Occaſion for Father *Saquerius* to publiſh a ſmall Tract at *Milan*, wherein he acknowledges that he cou'd not ſolve the Problem, though by the Solutions of others it ſufficiently appears that he is a good Geometrician.

## EXAMPLE V.

FIG. 191. 352. AN indefinite Right Line $AP$ being given in Poſition, together with two fixed Points $A$, $C$, one being in that Line, and the other without the ſame; let there be deſcrib'd a Parabola $AM$, with

with $A P$ as an Axis, whoſe Origin is in $A$, and any Line whatſoſoever for the Parameter; and from the given Point $C$ let $CM$ be drawn perpendicular to the Parabola. Now it is requir'd to find the Locus of all the Points $M$, whereof it is manifeſt that there are an infinite Number; becauſe, the Parameters being indeterminate, there may be deſcribed an infinite Number of different Parabola's, to the ſame Axis $AP$, whoſe Origin is always in $A$.

Through the given Point $C$ draw $CB$ perpendicular to $AP$, and through one of the ſought Points, as $M$, (ſuppoſed to be given) draw the Right Lines $MP$, $MK$, parallel to $BC$, $AP$, and the Tangent $MT$; then call the given Lines $AB$, $a$; $BC$, $b$; and the unknown and indeterminate ones $AP$, $x$; $PM$, $y$; from whence we have $CK = b - y$, and $MK = a + x$. But according to the Conditions of the Problem, the Angle $CMT$ is a Right Angle; and conſequently the Right-angled Triangles $TPM$, $CKM$, ſhall be ſimilar; for if the ſame Angle $KMT$ be taken from the Right Angles $CMT$, $KMP$, there will remain $CMK$, $TMP$, equal to one another: Therefore * $TP(2x) : PM(y) :: CK(b-y) : KM(a+x)$ and ſo (multiplying the Means and Extrems) we ſhall have this Equation $yy - by + 2xx + 2ax = o$, whoſe Locus being that ſought, will be * an Ellipſis, and may be * conſtructed after the following Manner. * *Art.* 22. *and* 23. *Art.* 322 *Art.* 324.

Draw $AD = \frac{1}{2}b$ perpendicular to $AP$, and on the ſame Side as $PM$, and draw the indefinite Right Line $DL$, parallel to $AP$, and in the ſame aſſume $DE = \frac{1}{2}a$ from $D$ the contrary way to $PM$; on both Sides the Point $E$, take $EF$, $EG$, each equal to $\sqrt{\frac{1}{4}aa + \frac{1}{8}bb}$. Then deſcribe an Ellipſis with the Axis $FG$, whoſe Parameter is a Line $GH$, being double to $FG$. I ſay the Portion of this Ellipſis $AMO$ contained in the Angle $PAD$, is the Locus of the aforeſaid Equation; and conſequently of all the ſought Points $M$, when they fall in that Angle.

For producing $PM$ (if neceſſary) until it meets the Axis $FG$ in $L$, we ſhall have the Ordinate $ML = \frac{1}{2}b - y$, and $EL = \frac{1}{2}a + x$, and by the Property of the Ellipſis, $FL \times LG$ or $\overline{EF}^2 - \overline{EL}^2 (\frac{1}{8}bb - ax - xx) : \overline{LM}^2 (\frac{1}{4}bb - by + yy) :: FG : GH :: 1 : 2$; from whence by multiplying the Means and Extrems, and there ariſes $\frac{1}{4}bb - 2ax - 2xx = \frac{1}{4}bb - by + yy$. Therefore, *&c.*

Now if the Point $M$ falls in the Angles $BAD$, $BAR$, you will find always the ſame Equation as above, from the Conditions of the Problem, and the Nature of the Ellipſis; obſerving to make $AP = -x$, and $PM = -y$, when the Point $P$ falls on the other Side of the Origin $A$, and $PM$, on the other Side of the Line $AP$. From whence it follows, that the Portions of the Ellipſis (deſcribed as above) con-

tain'd in the Angles $B\,A\,D$, $B\,A\,R$, are the Loci of the Points $M$.

Here it muſt be obſerved, that not one of the ſought Points $M$ does fall in the Angle $P\,A\,R$, oppoſite to the Angle $B\,A\,D$, wherein the given Point $C$ is ſituate. For if Right Lines as $M\,P$, $M\,T$, be drawn from any Point, taken in the ſaid Angle $P\,A\,R$, perpendicular to $A\,P$, $C\,M$, then it is plain that the Points $P$, $T$, ſhall fall on the ſame Side the Point $A$; and conſequently that Line cannot be a Tangent in $M$, as the Queſtion requires.

If $A\,P\,(x)$ be ſuppoſed $= o$, then the foregoing Equation, *viz.* $y\,y - b\,y + 2\,x\,x + a\,x = o$ will be changed into this $y\,y - b\,y = o$, whoſe two Roots are $y = o$, and $y = b$: From whence it may be gathered, in drawing $A\,O$ parallel and equal to $B\,C$, that the Locus of the ſought Points $M$, ſhall paſs through the Points $A$ and $O$. After the ſame manner, if the Point $P$ be ſuppoſed to fall on the other Side of the Origin $A$, and you make $A\,P\,(-x) = A\,B\,(a)$; then the ſame Locus ſhall paſs through the Points $B$, $C$; ſo that the Ellipſis muſt be deſcrib'd about the Rectangle $A\,B\,C\,O$: From whence ariſes the following new Conſtruction.

**Art.* 176. Form the Rectangle $A\,B\,C\,O$, and about * the ſame deſcribe an Ellipſis, whoſe Axis $F\,G$ being parallel to the Sides $A\,B$, $O\,C$, let be to its Parameter $G\,H$, as 1 to 2. Then it is manifeſt that the Ellipſis will be the Locus ſought.

## SCHOLIUM I.

353. IF the Nature of any Curve as $A\,M$ be expreſſed by the general Equation $y^n = x^m\,a^{n-m}$ for * Parabola's of all Degrees (where the Letters $m$, $n$, denote the Index's of the Powers of $y$ and $x$); then

**Art.* 229. **Art.* 237. we ſhall have $TP^{*}\left(\frac{n}{m}\,x\right) : P\,M\,(y) :: C\,K\,(b-y) : K\,M\,(a+x)$:

and therefore $y\,y - b\,y + \frac{n}{m}\,x\,x + \frac{n}{m}\,a\,x = o$, and the Locus thereof beiug that ſought, will be an Ellipſis, which may be drawn acccording to the 322d, or elſe the 176th Articles, if you obſerve that the ſaid Ellipſis muſt circumſcribe the given Rectangle $A\,B\,C\,O$, and that its Axis $F\,G$ being parallel to the Sides $A\,B$, $O\,C$, be to the Parameter $G\,H$ in the given Ratio of $m$ to $n$.

## SCHOLIUM II.

FIG. 191. 354. IF $E$ the Centre of the Ellipſis ſhould fall on the Origin $(A)$ of the common Axis $(A\,P)$ of all the Parabola's $A\,M$; and the Axis $F\,G$ of the Ellipſis, in the Axis $A\,P$ of the Parabola's: Then that Ellipſis would cut all the different Parabola's at right Angles. This Theorem may be expreſſed after the following Manner.

If

If there be an infinite Number of Parabola's as $AM$, of any Degree whatſoever, and the Line $AP$, whoſe Origin is always at the ſame Point $A$, be the common Axis of them all; and if there be an Ellipſis, whoſe Centre is the Point $A$, and whoſe Axis $FG$ ſituate in $AP$, is to the Parameter thereof, as $m$ the Index of the Power of $AP(x)$ to $n$, the Index of the Power of $PM(y)$ in the general Equation $y^n = x^m a^{n-m}$ expreſſing the Nature of the Parabola's $AM$: I ſay that Ellipſis ſhall cut all the Parabola's at right Angles. FIG. 192.

Through the Point $M$, in which the Ellipſis cuts any one of the Parabola's, draw the Line $MT$ to that Parabola, and $MS$ perpendicular to that Tangent. Then we are to prove, that $MS$ touches the Ellipſis in the Point $M$. For doing of which, draw $MP$ perpendicular to the Axis, and calling the indeterminate Quantities $AP$, $x$; $PM$, $y$; and the given Quantity $FG$, $2t$; then by the Property of the Ellipſis we ſhall have this Proportion $FP \times PG\ (tt - xx) : \overline{PM}^2\ (yy) :: m : n$. And therefore $myy = ntt - nxx$. But becauſe $TPM$, $TMS$, are right Angles, therefore * $TP\ \left(\frac{n}{m}x\right) : PM\ (y) :: PM\ (y) : PS = \frac{myy}{nx}$, * *Art.* 237. and conſequently $AS$ or $AP + PS = \frac{nxx + myy}{nx} = \frac{tt}{x}$ by putting $ntt - nxx$ for $myy$. Whence it appears, that $AP : AF :: AE : AS$, and ſo * the Line $MS$ does touch the Ellipſis in the Point $M$. *W. W. D.* * *Art.* 57.

## EXAMPLE VI.

355. LET there be ſuppos'd an infinite Number of Hyperbola's, which have all the ſame right Lines $AP$, $AO$, given in Poſition, and being at right Angles to one another, for their Aſymptotes; and conceive an infinite Number of Perpendiculars, as $CM$, to be drawn from a given Point $C$ to the ſaid Hyperbola's. It is requir'd to find the Locus of all the Points $M$, wherein every of the right Lines $CM$ meets the Hyperbola to which it is perpendicular. FIG. 193.

Draw the ſame Lines as in the laſt Example, and calling them by the ſame Names, we ſhall get * this Proportion, *viz.* $TP(x) : PM(y)$ $:: CK(b-y) : KM(a-x)$; from whence ariſes $yy - bx - xx + ax = o$, whoſe Locus may be * thus conſtructed. * *Art.* 107. * *Art.* 330, or 335.

In the Aſymptote $AO$ parallel to $PM$, aſſume $AD = \frac{1}{2}b$, and draw $LL$ parallel to $AP$; in which aſſume $DE = \frac{1}{2}a$ from $D$ towards $PM$, and on both Sides the Point $E$, the Parts $EF$, $EG$, each equal to $\sqrt{\frac{1}{4}aa - \frac{1}{4}bb}$, or $\sqrt{\frac{1}{4}bb - \frac{1}{4}aa}$, according as $a$ is greater or leſs than $b$. This being done, deſcribe two oppoſite equilateral Hyperbola's to the Line $FG$, as a firſt Axis in the former Caſe, and as a ſecond in the latter. I ſay, the Portions of theſe Hyperbola's contain'd

tain'd in the Angle $PAO$, will be the Locus of the aboveſaid Equation, and conſequently the Locus of all the ſought Points $M$.

For producing $PM$ (if neceſſary) to meet the Axis $FG$ in $L$; then the Ordinate $ML$ will be $= \frac{1}{2}b - y$, and $EL = x - \frac{1}{2}a$; and by * the Property of the Equilateral Hyperbola's $\overline{EL}^2 \pm \overline{EF}^2 (xx - ax + \frac{1}{4}bb) = LM (\frac{1}{4}bb - by + yy)$. Therefore, *&c.*

*Art. 127.

If $a$ be $= b$, the aforeſaid Conſtruction will be impoſſible, becauſe the Value of the Semi-axis $EF$ or $EG$ will become $= o$. And becauſe the aboveſaid Equation will become this $yy - ay - xx + ax = o$, or $yy - ay + \frac{1}{4}aa = xx - ax + \frac{1}{4}aa$; therefore extracting the ſquare Root of both Sides, and there ariſes $y - \frac{1}{2}a = x - \frac{1}{2}a$, or $y = x$, and $\frac{1}{2}a - y = x - \frac{1}{2}a$, or $y = a - x$: And ſo if the Rectangle $ABCO$ be compleated, and the Diagonals $AC$, $BO$, be drawn, theſe Diagonals will be the Loci of all the ſought Points $M$; for the Diagonal $AC$, is the Locus of the firſt Equation $y = x$, and the other Diagonal $BO$ the Locus of the ſecond $y = a - x$.

FIG. 194.

## SCHOLIUM I.

356. IF the Nature of the Hyperbola's, whoſe Aſymptotes are the right Lines $AB$, $AO$, be expreſs'd by the general * Equation $x^m y^n = a^{m+n}$,

FIG. 193. *Art. 229.

then we ſhall have * $TP\left(\frac{n}{m}x\right) : PM\,(y) :: CK\,(b - y) : KM\,(a - x)$

*Art. 237.

and therefore $yy - by - \frac{n}{m}xx + \frac{n}{m}ax = o$, and the Locus of this Equation may be * thus conſtructed.

*Art. 330.

Find the Point $E$, as in the Example, and in $DL$ both ways from $E$ aſſume $EF$, $EG$, each equal to $\sqrt{\frac{1}{4}aa - \frac{m}{4n}bb}$, or $\sqrt{\frac{m}{4n}bb - \frac{1}{4}aa}$; according as $naa$ is greater or leſs than $mbb$. Then deſcribe two oppoſite Sections with the Line $FG$, as a firſt Axis in the former Caſe, and as a ſecond in the latter, that may be to its Parameter in the given Ratio of $M$ to $N$; and the Portions of theſe Sections contain'd in the Angle $OAB$, will be the Locus ſought.

If $a : b :: \sqrt{m} : \sqrt{n}$, then the Equation $yy - by - \frac{n}{m}xx + \frac{n}{m}ax = o$, will be changed into this $yy - ay\sqrt{\frac{n}{m} - \frac{n}{m}}xx + \frac{n}{m}ax = o$,

FIG. 194.

or $yy - ay\sqrt{\frac{n}{m} + \frac{naa}{4m}} = \frac{n}{m}xx - \frac{n}{m}ax + \frac{naa}{4m}$, and extracting the ſquare Root of both Sides, and there comes out $y - \frac{1}{2}a\sqrt{\frac{n}{m}} = x\sqrt{\frac{n}{m}}$

$\frac{n}{m} - \frac{1}{2}a\sqrt{\frac{n}{m}}$, or $y = x\sqrt{\frac{n}{m}}$; and $\frac{1}{2}a\sqrt{\frac{n}{m}} - y = x\sqrt{\frac{n}{m}} - \frac{1}{2}a\sqrt{\frac{n}{m}}$, or $y = a\sqrt{\frac{n}{m}} - x\sqrt{\frac{n}{m}}$. Whence, if the Rectangle $AECO$ be compleated, and the Diagonals $BO$, $AC$, are drawn; theſe Diagonals will be the Loci of all the ſought Points $M$. For the Diagonal $AC$ is the Locus of the firſt Equation $y = x\sqrt{\frac{n}{m}}$, and the other Diagonal $BO$ the Locus of the ſecond $y = a\sqrt{\frac{a}{m}} - x\sqrt{\frac{n}{m}}$.

After the ſame manner, as in the Ellipſis, we prove, that the oppoſite Sections ſought muſt be deſcrib'd about the given Rectangle $ABCO$; and becauſe the Axis $FG$, parallel to the Sides $AE$, $OC$, is to its Parameter in the given Ratio of $m$ to $n$; therefore you may deſcribe the oppoſite Sections (if you pleaſe) by means of the 176th Article. FIG. 193.

## SCHOLIUM II.

357. IF the Centre $E$ of the Hyperbola $BFC$ ſhould fall on the Point $A$, and its Axis $FG$ in the Line $AP$; then that Hyperbola would cut all thoſe whoſe Aſymptotes are $AP$, $AO$, at right Angles; which may be thus expreſs'd. FIG. 193.

If there be an infinite Number of Hyperbola's of any Degree whatſoever, having the ſame ſtrait Lines $AP$, $AO$, at right Angles to one another, for their Aſymptotes; and if there be a common Hyperbola $FM$, whoſe Centre is $A$, and whoſe firſt Axis $FG$ ſituate in $AP$, be to its Parameter as $m$ the Index of the Power of $AP$ $(x)$ to $n$, the Index of the Power of $PM$ $(y)$ in the general Equation $x^m y^n = a^{m+n}$, which expreſſes the Nature of the Hyperbola's $MAM$. I ſay, the Hyperbola $FM$ does cut all thoſe different Hyperbola's at right Angles. FIG. 195.

Through the Point $M$, wherein the Hyperbola $FM$ cuts any one of the different ones, as $MAM$, draw the Tangent $MT$ to the ſame; and the right Line $MS$ perpendicular to that Tangent: Now we are to prove, that the Angle $TMS$ is a right Angle. To do this, draw $MP$ perpendicular to the Aſymptote $AP$; and calling the indeterminate Quantities $AP$ $(x)$; $PM$, $y$; and the given Quantity $FG$, $2t$; then by the Property of the Hyperbola $FM$, we have this Proportion, *viz.* $FP \times PG\ (xx - tt) : \overline{PM}^2\ (yy) :: m : n$, and therefore $m\,y\,y = n\,x\,x - n\,t\,t$. But ſince the Angles $TPM$, $TMS$, are right Angles, there ariſes $TP \ast \left(\frac{n}{m}\,x\right) : PM\,(y) :: PM\,(y) : PS = \frac{myy}{nx}$. And conſequently $AS$ or $AP - PS = \frac{nxx - myy}{nx} = \frac{tt}{x}$, by ſubſtituting $n\,x\,x - n\,t\,t$ for *Art. 237.

$myy$. From whence it appears, that $AS$ is a third Proportional to
*Art. 121. $AP$, $AF$; and ſo * the Line $MS$ does touch the Hyperbola $FM$ in the Point $M$. *W. W. D.*

## EXAMPLE VII.

FIG. 196. 358. THE Parabola $BAC$ being given, it is required to find the Locus of all the Points $M$, being ſuch, that two Tangents $MB$, $MC$, to the Parabola being drawn from them; the Angle $BMC$ contained by theſe Tangents, may be equal always to a given Angle.

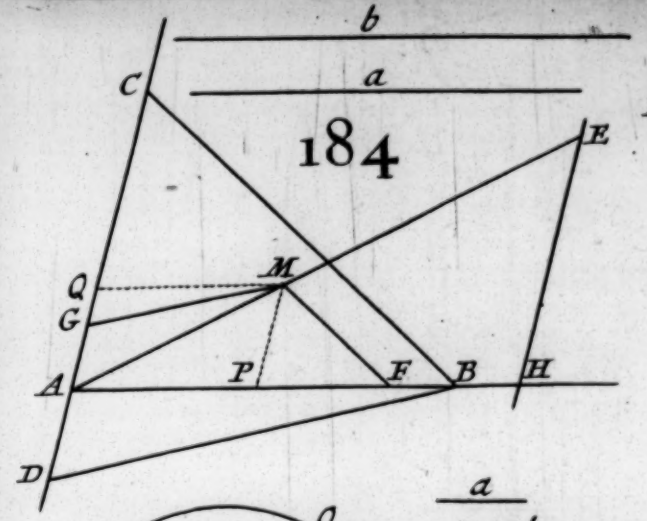
Here there are three different Caſes according as the given Angle $BMC$ is acute, obtuſe, or right.

*Art. 160. *Caſe* 1. When $BMC$ is an acute Angle, draw * $AD$ the Axis of the given Parabola $BAC$, meeting the Tangents $MB$, $MC$, in the Points $F$, $G$, alſo from the Points of Contact $B$, $C$, and the Point of Concurrence $M$ draw the Lines $BD$, $CE$, $MP$, perpendicular to the Axis. This being done, draw $MN$ making the Angle $FNM$ with the Axis $AD$ equal to $FMG$ the Complement of the given Angle $BMC$, to two right Angles, and call the unknown Quantities $AP$, $x$; $PM$, $y$; $AF$, $s$; $AG$, $t$; and the Parameter of the Axis $AD$, *viz.* $AV$, $a$; which is given, becauſe the Parabola is given. Now becauſe $FPM$ is a Right-angl'd Triangle, therefore the Square $\overline{FM}^2 = ss - 2sx + xx + yy$, which being divivded by $FG$ $(s-t)$ will give $\frac{ss-2sx+xx+yy}{s-t} = FN$, becauſe the Triangles $FGM$, $FMN$, are ſimilar; and therefore $PN$ or $FP - FN = \frac{sx+tx-st-xx-yy}{s-t}$. Now by means of the given Parabola $BAC$, ſeek the Values of $s+t$, $st$, and $s-t$ with reſpect to $x$ and $y$, that ſo they being ſubſtituted in the Value of $PN$, this Line may be expreſſed only by $x$ and $y$. Which muſt be done thus.

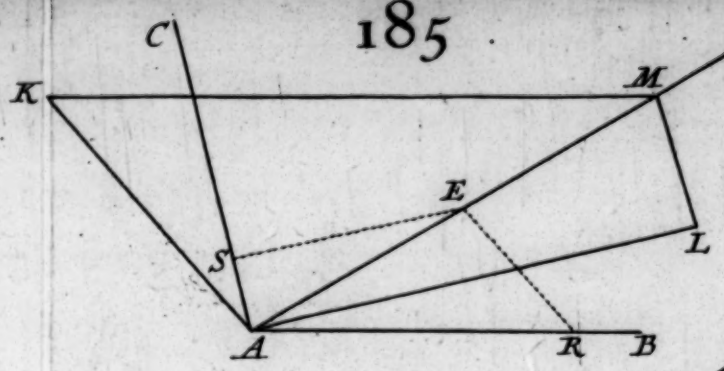
The ſimilar Triangles $FPM$, $FDB$; and $GPM$, $GEC$, do give
* *Art.* 22. $FP\,(s-x) : PM\,(y) :: FD^*\,(2s) : BD^*\,(\sqrt{as})$. And $GP\,(x-t) :$
* *Art.* 7. $PM\,(y) :: GE\,(2t) : CE\,(\sqrt{at})$. From whence may be form'd the following Equations, *viz.* $ss - 2xs - \frac{4yy}{a}s + xx = o$, and $tt - 2xt - \frac{4yy}{a}t + xx = o$; that is, (making $p = 2x + \frac{4yy}{a}t$ for brevities ſake) $ss - ps + xx = o$, and $tt - pt + xx = o$; then ſubſtract the ſecond Equation from the firſt, and you will get $ss - tt - ps + pt = o$, which being divided by $s-t$, and then $s+t$ is $= p$, and ſo $s = p - t$, and $ss = ps - ts = ps - xx$, by the firſt Equation, therefore $st = xx$. Laſtly, if $4xx$ the Value of $4st$ be taken from $pp$, the Value of

$ss +$

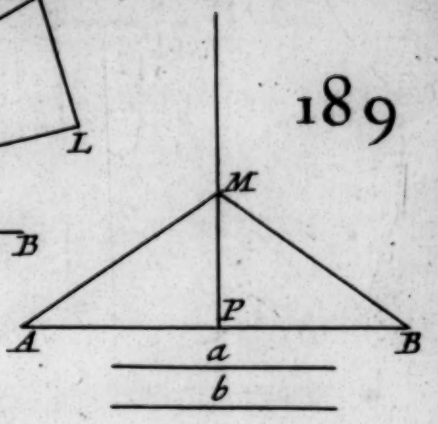
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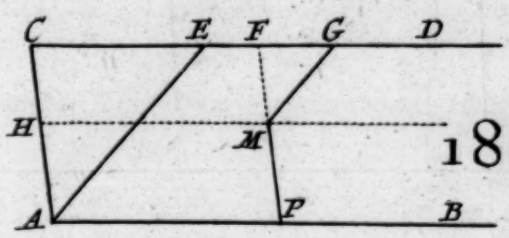

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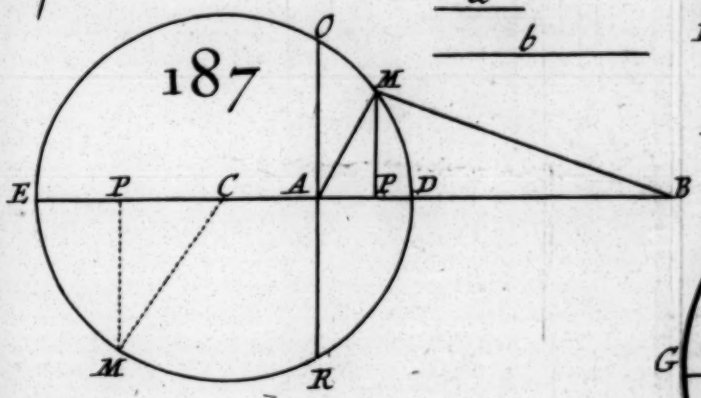

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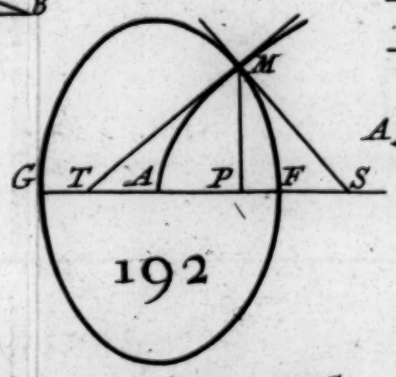

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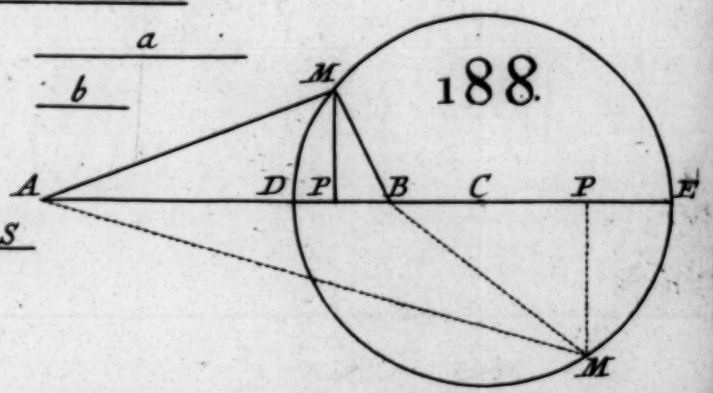

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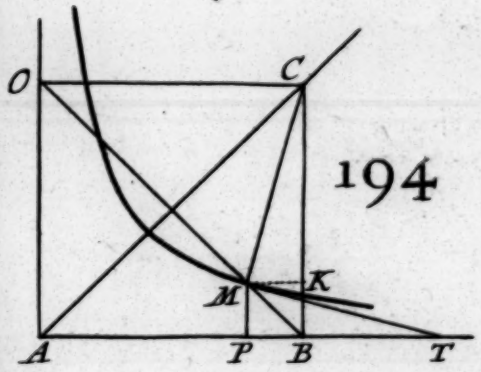

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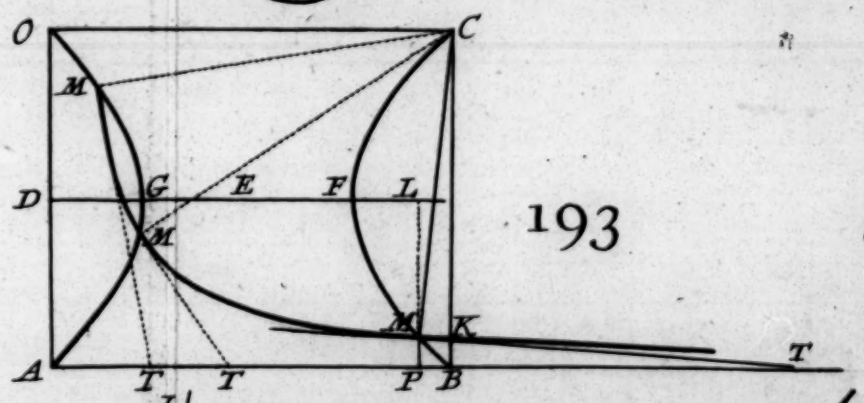

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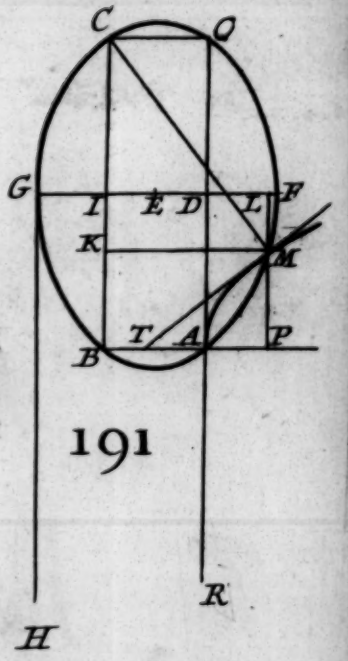

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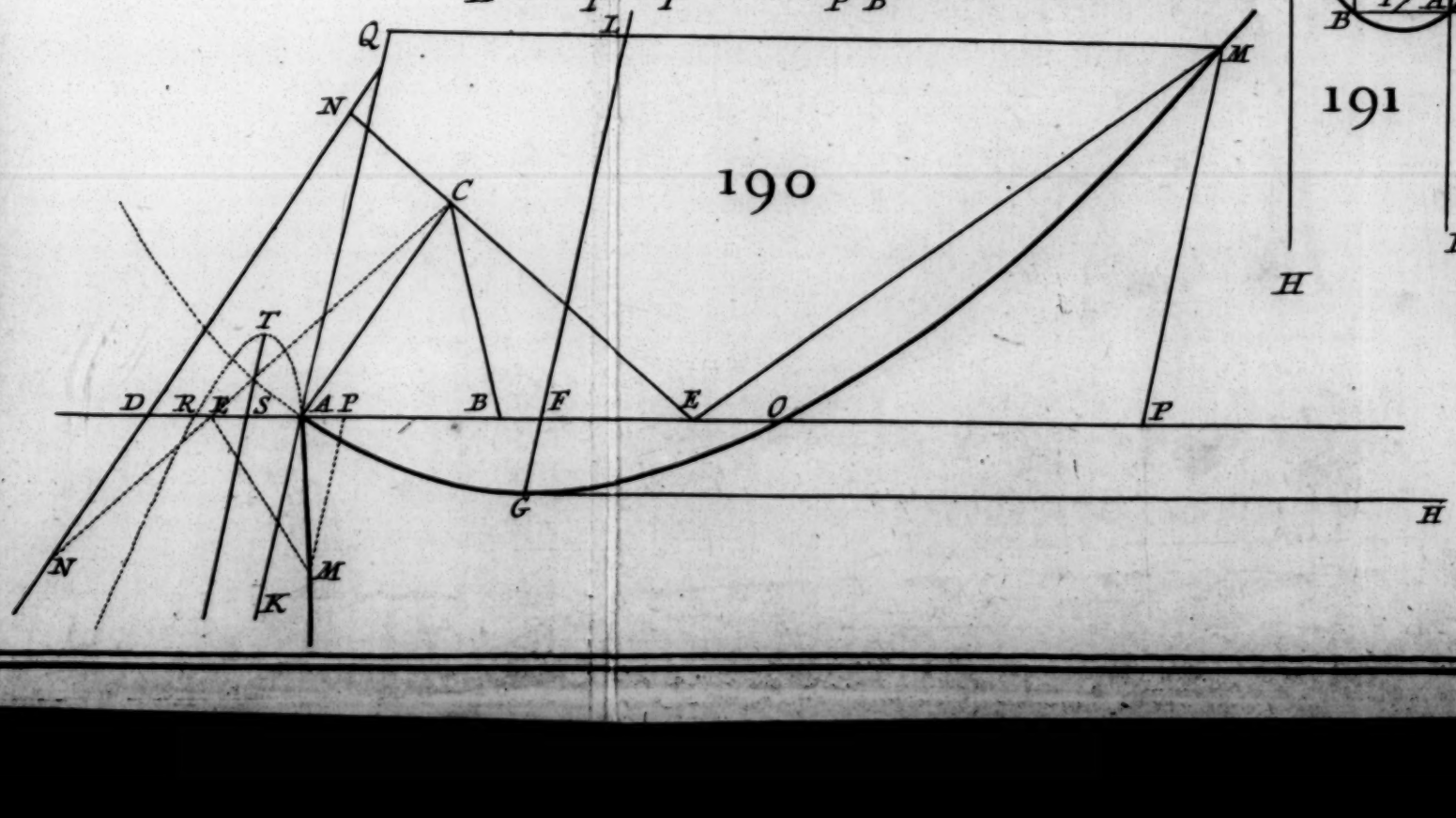

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$s\,s + 2\,t\,s + t\,t$, there will arise this Equation, *viz.* $s\,s - 2st + tt = pp - 4xx$, and extracting the Square Root of both Sides, we have $s - t = \sqrt{pp - 4xx} = \frac{4y\sqrt{ax + yy}}{a}$, substituting $2x + \frac{4yy}{a}$ for $p$.

Now if for $s + t$, $s\,t$, and $s - t$, you substitute their Values $2x + \frac{4yy}{a}$, $x\,x$, and $\frac{4y\sqrt{ax+yy}}{a}$, in $\frac{sx+tx-st-xx-yy}{s-t}$ and then $PN$ will be $= \frac{4xy-ay}{4\sqrt{ax+yy}}$. Now if $N\mathcal{Q}$ be taken in the Axis equal to its Parameter $AV$ ($a$), and $\mathcal{Q}T$ be drawn parallel to $PM$, meeting the right Line $MN$ (produced according to Necessity) in the Point $T$; then it is plain that the Line $\mathcal{Q}T$ will be given, because in the Right-angl'd Triangle $N\mathcal{Q}T$, the Angle $\mathcal{Q}NT$, which is equal to the given Angle $BMC$ is given; and moreover the Side $N\mathcal{Q}$ being equal to the Parameter $AV$ to the Axis. Now let the given Line $\mathcal{Q}T$ be $= b$, then because the Triangles $NPM$, $N\mathcal{Q}T$, are similar, therefore $NP\left(\frac{4xy-ay}{4\sqrt{yy+ax}}\right) : PM\,(y) :: a : b$, and so $4a\sqrt{yy + ax} = 4bx - ab$, and clearing the Equation from Surds, and we have $y\,y - \frac{bb}{aa}\,x\,x + a\,x + \frac{bb}{2a}\,x - \frac{1}{16}\,b\,b = o$, whose Locus (being that sought) may be * thus constructed. * *Art.* 330, *and* 332.

In $AD$, the Axis of the Parabola, assume $AH = \frac{1}{4}a + \frac{n^3}{2bb}$ from $A$ towards $PM$, and on both Sides the Point $H$ take $HI$, $HK$, each equal to $\frac{aa\sqrt{aa+bb}}{2bb}$; and describe an Hyperbola $KM$ with $IK$ as a first Axis, having the same Proportion to the Parameter $KL$ as $a\,a$ to $b\,b$; then I say, that Hyperbola will be the Locus of the Equation found as above.

For $HP = x - \frac{1}{4}a - \frac{a^3}{2bb}$, and by the Property of the Hyperbola $\overline{HP}^2 - \overline{HK}^2\,(x\,x - \frac{1}{2}a\,x - \frac{a^3}{bb}\,x + \frac{1}{16}a\,a) : \overline{PM}^2\,(yy) :: IK : KL :: aa : bb$. From whence may be gotten the aforesaid Equation, by multiplying the Means and Extremes.

Here it is necessary to observe, that $FN$ shall be always less than $FP$; because the Angle $FNM$, which was taken equal to the Complement of the given Angle, is obtuse. Therefore $\frac{4xy-ay}{4\sqrt{yy+ax}}$ the Value of $FP - FN$ must be positive; and consequently $x$ must always ex-

ceed $\frac{1}{4}a$. From whence it appears, that although there be a Portion of the Hyperbola oppoſite to $KM$, which is contain'd in the Angle $PAV$ made by the Line $AP$, and the right Line $AV$ drawn parallel to $PM$ towards the ſame Parts, yet that Portion cannot be the Locus of the Points $M$; becauſe $AI$ being leſs than $\frac{1}{4}a$, the variable Quantity $AP$, which then ſhall be leſs than $AI$, will be much more leſs than $\frac{1}{4}a$.

*Caſe* 2. When the given Angle is obtuſe. Suppoſing that the Points $M$ do fall in the Angle $PAV$, and reaſoning after the ſame manner as in the firſt Caſe, you will get the ſame Equation as there, and conſequently the Conſtruction of the Locus ſhall be the ſame alſo. But here it may be obſerv'd, that $FN$ ſhall be greater than $FP$, and ſo the Value $\frac{4xy-ay}{4\sqrt{yy+ax}}$ of $FP-FN$ will become negative; from whence it follows, that $x$ ſhall be always leſs than $\frac{1}{4}a$, and therefore the Locus ſought will be the Portion of the Hyperbola tending the ſame way as the Parabola, being contain'd in the Angle $PAV$. And becauſe the ſame Equation (as in *Caſe* 1.) ariſes, ſuppoſing the Points $M$ to fall in the Angle $DAV$, therefore that whole Hyperbola ſhall be the Locus of all the Points $(M)$ ſought.

From hence it is manifeſt, that if an Hyperbola, as $KM$, be the Locus of all the Points $M$, when the given Angle $BCM$ is acute; then the oppoſite Hyperbola ſhall be the Locus of all thoſe Points, when the given Angle ſhall be equal to the Complement of the Angle $BMC$ to two right ones, becauſe then the given Lines $a$ and $b$, which determine the Conſtruction of the Hyperbola's, will remain the ſame.

Fig. 196, 197. *Caſe* 3. When the given Angle is a right Angle. Here it is manifeſt, that $FN$ is equal to $FP$, and ſo $\frac{4xy-ay}{4\sqrt{yy+ay}}$ the Value of $FP-FN$ ſhall be $=o$. Whence if $AP=\frac{1}{4}a$ be taken in the Axis $AD$ produced towards its Origin $A$, and the indefinite right Line $PM$ be drawn perpendicular to the ſame; then it is * manifeſt, that $PM$, which is the Directrix of the Parabola, (as appears from the Definitions in Book 1.) ſhall be the Locus ſought.

* *Art.* 306.

## COROLLARY.

Fig. 196, 197. 359. IF the Semi-ſecond Axis $HO$ be drawn, as alſo the Hypothenuſe $KO$; then the right-angled Triangles $KHO$, $NQT$, ſhall be ſimilar: For becauſe the ſecond Axis is a mean Proportional between the firſt Axis $IK$, and its Parameter $KL$, therefore $\overline{KH}^2 : \overline{HO}^2 :: IK : KL :: aa : bb$, and ſo $KH : HO :: NQ\,(a) : QT\,(b)$. Therefore the Angle $HKO$ (which by *Def.* 11. *Book* 3. is equal to the half of the Angle

Angle form'd by the Aſymptotes of the Hyperbola $KM$) ſhall be equal to the Angle $\mathcal{Q}NT$, that is, to the given Angle $BMC$; and we ſhall have $N\mathcal{Q}(a) : \mathcal{Q}T(b) :: KH\left(\frac{aa\sqrt{aa+bb}}{2bb}\right) : HO = \frac{a\sqrt{aa+bb}}{2b}$;

and $N\mathcal{Q}(a) : NT(\sqrt{aa+bb}) :: KH\left(\frac{aa\sqrt{aa+bb}}{2bb}\right) : KO = \frac{a^3+abb}{2bb}$.

Now if the Hypothenuſe ($KO$) of the right-angled Triangle ($KHO$) made by the Semi-axes $HK$, $HO$, be laid from the Centre $H$ in the firſt Axis $IK$ to $R$, and $S$; then it is manifeſt, * that $R$ and $S$, ſhall be the two Foci of the Hyperbola $KM$, and that oppoſite to it; and $RA = \frac{1}{4}a$, becauſe $HR = \frac{a^3+abb}{2bb}$, and $AH = \frac{1}{4}a + \frac{a^3}{2bb}$. Whence the Focus $R$ of the Hyperbola $KM$, is * alſo the Focus of the Parabola $BAC$; and $SR\left(\frac{a^3+abb}{bb}\right) : HO\left(\frac{a\sqrt{aa+bb}}{2b}\right) :: HO\left(\frac{a\sqrt{aa+bb}}{2b}\right) : AR\left(\frac{1}{4}a\right)$, becauſe if the Means and Extremes be multiply'd, the ſame Product will be found. From whence ariſes the following Theorem.

* *Art.* 74.

* *Def.* 3, 4, 5. I.

If $RA$ be taken from $R$ towards $S$ in the focal Diſtance $SR$ of an Hyperbola $KM$, equal to a third Proportional to that Diſtance $SR$ and $HO$ half of the ſecond Axis; and if a Parabola $BAC$ be deſcrib'd *, having the Point $R$ for the Focus, and the Line $AR$ whoſe Origin is $A$, for the Axis; and if from any Point $M$ of the Hyperbola $KM$, there be drawn two Tangents $MB$, $MC$, to that Parabola. I ſay, the Angle $BMC$ contain'd by theſe Tangents, ſhall be always equal to $\frac{1}{2}$ the Angle form'd by the Aſymptotes; and if the Point $M$ be aſſum'd on the oppoſite Hyperbola, the Angle contain'd by the Tangents, will be equal always to the Complement of $\frac{1}{2}$ the Angle form'd by the Aſymptotes to two right Angles.

FIG. 196.

* *Art.* 4.

## EXAMPLE VIII.

360. AN indefinite right Line $BAP$ being given in Poſition on a Plane, together with two fixed Points $A$, $D$, the one in that Line, and the other without the ſame: It is required to find the Locus of all the Points $M$, whoſe Property may be ſuch, that two right Lines $MA$, $MD$, being drawn from any one of them, to the fixed Points $A$, $D$: The Line $AM$ may be always equal to $ME$, that part of the other Line $DM$, taken between the Point $M$, and the Point $E$ wherein it meets the Line $BAP$.

FIG. 198.

Draw the right Lines $BD$, $MP$, from the given Point $D$, and the Point $M$ (ſuppoſed to be one of the Points ſought) perpendicular to $AP$, and call the given Lines $AB$, $2a$; $BD$, $2b$; and the

unknown and indeterminate ones $AP$, $x$; $PM$, $y$; then we ſhall have $AP=PE$; becauſe (by the *Hyp.*) $AM=ME$. And becauſe the Triangles $EBD$, $EPM$, are ſimilar, therefore $EB$ or $AE-AB$ $(2x-2a)$ : $BD$ $(2b)$ :: $EP$ $(x)$ : $PM$ $(y)$. And multiplying the Extremes and Means, and we ſhall have this Equation $xy-ay=bx$, containing the Condition ſpecified in the Problem, and the Locus thereof, which is * an Equilateral Hyperbola between its Aſymptotes may be thus conſtructed. *Art. 337.

Draw the Line $AD$ which biſect in $C$, and through $C$ draw the right Lines $CF$, $CG$, the one parallel, and the other perpendicular to $AP$: Then between the Aſymptotes $CF$, $CG$, both ways indefinitely produced from the Point $C$ * deſcribe * two oppoſite Hyperbola's $DM$, $AM$, (which * are equilateral) through the Points $A$, $D$. I ſay, theſe Hyperbola's ſhall be the compleat Loci of all the ſought Points $M$. *Art. 130. 131. * Def. 16. III.

For the Aſymptotes $CF$, $CG$, do divide the right Lines $AB$, $BD$, into two equal Parts in the Points $L$, $K$, becauſe $AD$ is biſected in $C$; and therefore when the Points $P$ do fall on $AB$ infinitely produced towards $B$, as is ſuppoſed in the Calculus, the Line $PL$ or $CH$ is $=x-a$, and $HM=y-b$; and by * the Property of the Hyperbola $CH \times HM$ $(xy-ay-bx+ab)$ $=CK \times KD$ $(ab)$: That is, $xy-ay=bx$. *Art. 100.

Now if the Points $P$ be ſuppoſed to fall upon $BA$, produced infinitely towards $A$, or on the determinate Part $AB$, the ſame Equation $xy-ay=bx$ will always be had, from the Condition denoted in the Problem and the Nature of the Hyperbola $AM$, or $DM$, obſerving to make $AP=-x$, and $PM=-y$, when they fall on the other Side the Point $A$ and the Line $AP$. Therefore, &c.

## COROLLARY.

361. HENCE the Parts $MR$, $MS$, of the right Lines $AM$, $DM$, taken from the Point $M$ to the Aſymptotes, are equal to one another. For 1. when the Aſymptote, as $CF$, is parallel to the Line $AP$, then the Angle $RSM$ is equal to the Angle $AEM$, and the Angle $SRM$ equal to the Angle $MAE$. 2. When the Aſymptote, as $CG$, is perpendicular to $AP$, then the Angle $RSM$ ſhall be the Complement of the Angle $AEM$ to one right Angle, (ſince $SLE$ is a right Angle Triangle) and alſo the Angle $SRM$ or $ARL$ vertically oppoſite to it is the Complement of the Angle $EAM$ to a right Angle, becauſe $RAL$ is a Right-angl'd Triangle. Therefore becauſe the Angles $EAM$, $AEM$, are equal, the Triangle $RMS$ ſhall be an Iſoſceles Triangle, and ſo the Sides $MR$, $MS$, ſhall be equal to one another.

another. Now this Corollary furniſhes us with the following Theorem.

If from any Point *M* of an equilateral Hyperbola there be drawn two right Lines *M D*, *M A*, to the Extremities of (*AD*) one of the firſt Diameters thereof; meeting the Aſymptotes in the Points *R*, and *S*: I ſay the Parts *M R*, *M S*, of thoſe Lines ſhall be equal between themſelves.

## EXAMPLE IX.

362. TWO Circles *EG F*, *B N O*, whoſe Centres are *C* and *A*, being given, and if through any Point *G* of the Circle *EGF*, there be drawn the indefinite Tangent *GNO* cutting the Circle *BNO* in two Points *N*, *O*, and if the Tangents *NM*, *OM*, be drawn through *N* and *O*; it is requir'd to find the Locus of all the Points of Concurrence *M*. FIG. 199.

Draw *M P* perpendicular to *C A*, which paſſes through *C* and *A* the Centres of the given Circles; alſo draw the right Lines *C G*, *AM*, which ſhall be parallel, becauſe both of them is perpendicular to the right Line *G O* which meets them in the Points *G* and *Q*; and call the given Quantities *A B* or *A O*, *a*; *C E*, or *C F*, or *C G*, *b*; *C A*, *c*; and the unknown and variable Quantities *A P*, *x*; *P M*, *y*. Now becauſe the right-angl'd Triangles *A O M*, *A Q O*, are ſimilar, therefore $AM(\sqrt{xx+yy}) : AO(a) :: AO(a) : AQ = \frac{aa}{\sqrt{xx+yy}}$. And if *C H* be drawn parallel to *G O*, meeting *M A* (produced according to Neceſſity) in the Point *H*; then (by the Similarity of the Right-angl'd Triangles *M A P*, *C A H*) $PA(x) : AM(\sqrt{xx+yy}) :: AH$ or $CG - AQ\left(b - \frac{aa}{\sqrt{xx+yy}}\right) : AC(c)$; and ſo $b\sqrt{xx+yy} = aa + cx$, that is (freeing the Equation from Surds) $yy + \frac{bb-cc}{bb}xx - \frac{2aac}{bb}x - \frac{a^4}{bb} = o$, and the Locus of this is * a Parabola, Ellipſis or Hyperbola, according as *C E* (*b*) is equal, greater, or leſs, than *C A* (*c*). The Conſtruction of this laſt Caſe is as follows. * *Art.* 345.

In the Line *AP* aſſume $AR = \frac{aac}{cc-bb}$ from *A* the contrary way to *P*; and on both Sides the Point *R* take *R I*, *R K*, each equal to $\frac{aab}{cc-bb}$; and with *IK* as a firſt Axis, having $KL = \frac{2aa}{b}$ for its Parameter deſcribe an Hyperbola. I ſay, the indefinite Portion *D M* of this Hyperbola,

perbola, which is contain'd in the Angle $PAD$ made by the Line $AP$, and the right Line $AD$ drawn parallel to $PM$ on the ſame Side $AP$, with regard to $AP$, ſhall be the Locus of the aforeſaid Equation.

For by the Property of the Hyperbola, $\overline{RP}^2 - \overline{RI}^2 \left(\frac{a^4+2aacx}{cc-bb} + xx\right) : \overline{PM}^2 (yy) :: IK \left(\frac{2aab}{cc-bb}\right) : KL \left(\frac{2aa}{b}\right)$; which gives again the aforeſaid Equation.

Now if the Points $M$ be ſuppos'd to fall in the Angle $KAD$, adjoining to the Angle $PAD$, the aforeſaid Equation will yet be found, in making $AP = -x$; therefore $ID$ the determinate Portion of the Hyperbola $IM$, together with half of the whole Hyperbola oppoſite to the ſame, ſhall be the Locus of all the Points $M$, and ſo the two oppoſite Hyperbola's do make up the compleat Locus of all the ſought Points $M$: Where it muſt be obſerv'd, that the Portion $SIT$, contain'd in the Circle $BNO$, is uſeleſs, becauſe not one of the Points of Concurrence of the two Tangents $NO$, $OM$, to that Circle can fall within the ſame.

Here it is proper to obſerve, that $RA \left(\frac{aac}{cc-bb}\right)$ is $= \sqrt{\overline{KR}^2 + \frac{1}{4} IK \times KL}$, as will appear by putting for thoſe Lines their Analytick Values; and ſo ſince the Rectangle $IK \times KL$ is equal to the Square of $\frac{1}{2}$ the ſecond Axis, the Point $A$ ſhall be * one of the Foci of the Hyperbola $IM$. And becauſe $AI$ or $AR - RI$ is $= \frac{aac-aab}{cc-bb} = \frac{aa}{c+b}$, and $AK = AR + RK = \frac{aac+aab}{cc-bb} = \frac{aa}{c-b}$: Therefore the aforeſaid Conſtruction may be ſhortened after this manner.

* *Art.* 74.

In the Line $AC$ aſſume $AD$, $AK$ (from $A$ towards $C$) as third Proportionals to $AF(c+b)$, $AB(a)$, and to $AE(c-b)$, $AB(a)$; and with the firſt Axis $IK$, and the Focus $A$ deſcribe * two oppoſite Hyperbola's. Then it is evident that thoſe Hyperbola's ſhall be the Loci of all the ſought Points $M$.

* *Art.* 76.

When $CE(b)$ is greater than $CA(c)$, the Conſtruction of the Ellipſis, which is the Locus of all the ſought Points $M$, will be after the ſame manner as that for the Hyperbola, obſerving to aſſume $AK$ on the other Side the Point $A$ with regard to $C$. And finally when $CE(b) = CA(c)$, then you need only aſſume $AI$ in $AC$ from $A$ towards $C$, equal to a third Proportional to $AF$, and $AB$, and afterwards deſcribe a Parabola, with the Point $A$ as a Focus, and the Line $AI$, whoſe Origin is in $I$, as the Axis.

FIG. 200.

COROL-

### COROLLARY I.

*For the* ELLIPSIS *and oppoſite* SECTIONS.

363. HENCE if any Circle *B N O* be deſcrib'd about (*A*) one of the Foci of an Ellipſis or the oppoſite Sections, having *IK*, as a firſt Axis; and if *A E*, *A F*, be taken in that Axis, as third Proportionals to *A K*, *A B*, and to *A I*, *AB*, and the Circle *E G F* be deſcrib'd with *E F*, as a Diameter: Then it is evident, if two Tangents *M N*, *M O*, be drawn from any Point *M* of the Section to the Circle *B N O*, that the Line *O N* joining the Points of Contact (being produced according as is neceſſary) ſhall always touch the Circle *E G F*. FIG. 199.

### COROLLARY II.

*For the* PARABOLA.

364. HENC alſo if any Circle *B N O* be deſcribed from *A* the Focus of a Parabola *I M*, whoſe Axis is *L A*, and Origin *I*; and if *A F* be taken in that Axis from *A* towards its Origin, as a third Proportional to *A I*, *A B*, and a Circle *A G F* be deſcribed with the Diameter *A F*: And laſtly, if from any Point *M* of the Parabola, there be drawn two Tangents *M N*, *M O*, to the Circle *B N O*: Then the Line *NO* joining the Points of Contact, (being produced according to Neceſſity) ſhall touch the Circle *A G F* in the Point *G*. FIG. 200.

### EXAMPLE X.

365. AN indefinite right Line *A P* being given on a Plane, together with the fixed Point *F* without the ſame: It is requir'd to find the Locus of all the Points *M* being ſuch, that a right Line *M P* drawn from any one of them perpendicular to *A P*, and another right Line *M F* being drawn to the Point *F*; the Ratio of *M P* to *M F* may be always the ſame, ſuppoſe as the given Quantity *a* to *b*. FIG. 201, 202, 203.

Draw the right Line *F A* from the given Point *F*, perpendicular to *A P*, and the Line *M Q* from the Point *M* (ſuppoſed to one of thoſe ſought) parallel to *AP*; and call the given Quantity *AF*, *c*, and the unknown and variable ones *AP*, *x*; *PM*, *y*; (theſe are at right Angles to one another.) Now becauſe *M Q F* is a Right-angl'd Triangle, therefore $\overline{MF}^2 = \overline{FQ}^2\ (cc-2cy+yy) + \overline{MQ}^2\ (xx)$; and according to the State of the Problem $\overline{MP}^2\ (yy) : \overline{MF}^2$ (*cc*

$(cc - 2cy + yy + xx) :: aa : bb$; from whence (multiplying the Means and Extremes) there ariſes this Equation $aayy - bbyy - 2aacy + aaxx + aacc = o$, the Conſtruction of whoſe Locus, (wherein are three different Caſes, according as $a$ is greater, leſs, or equal to $b$,) is as follows.

*Caſe* 1. If the aforeſaid Equation be divided by $aa - bb$ there will ariſe $yy - \frac{2aac}{aa-bb}y + \frac{aa}{aa-bb}xx + \frac{aacc}{aa-bb} = o$, whoſe Locus is * an Ellipſis, and may be conſtructed thus. *Art. 324.

Fig. 201. In $AF$ aſſume $AC = \frac{aac}{aa-bb}$, from $A$ towards $F$; and drawing $KH$ through the Point $C$ parallel to $AP$, aſſume in the ſame the Parts $CH$, $CK$, on both Sides the Point $C$, each equal to $\sqrt{\frac{bbcc}{aa-bb}}$. This being done with the Axis $KH$, having the ſame Proportion to its Parameter $KL$ as $aa - bb$ to $aa$, deſcribe an Ellipſis. I ſay, the ſame will be the Locus of the aforeſaid Equation, and conſequently of all the ſought Points $M$.

For by the Property of the Ellipſis $KE \times EH$, or $\overline{CH}^2 - \overline{CE}^2$ $\left(\frac{bbcc}{aa-bb} - xx\right) : \overline{EM}^2 \left(\frac{a^4cc}{aa-bb^2} - \frac{2aacy}{aa-bb} + yy\right) :: KH : KL :: aa - bb : aa$; from whence, multiplying the Means and Extremes, and there will ariſe the ſame Equation as above.

Becauſe $\overline{CH}^2 : \overline{CB}^2 :: KH : KL :: aa - bb : aa$, therefore the Semi-axis $CB$ or $CD$ is $= \frac{abc}{aa-bb}$, and ſo $DF$ or $DC + CF$ is $= \frac{abc+bbc}{aa-bb} = \frac{bc}{a-b}$, and $FB$ or $CB - CF$ is $= \frac{abc-bbc}{aa-bb} = \frac{bc}{a+b}$. Therefore $DF \times FB$ is $= \frac{bbcc}{aa-bb} = \overline{CH}^2$; and conſequently the Point $F$ is * one of the Foci of the Ellipſis, and $BD$ is the great Axis. Now from hence ariſes a much eaſier Conſtruction than that foregoing, which is this. * *Art.* 35.

In $FA$ aſſume $FB = \frac{bc}{a+b}$ from $F$ towards $A$, and $FD = \frac{bc}{a-b}$ the contrary way, alſo aſſume $DG = BE$ from $D$ towards $F$; then deſcribe * an Ellipſis with the Points $F$, $G$, as Foci, and the Axis $BD$, and it is evident that the ſame ſhall be the Locus ſought. * *Art.* 36.

*Caſe* 2. Hence we have $yy + \frac{2aac}{bb-aa}y - \frac{aa}{bb-aa}xx - \frac{aacc}{bb-aa} = o$; becauſe $a$ is leſs than $b$. And the Locus of this Equation will be two oppoſite Sections, which may be deſcrib'd by the 332d Article. After having made

made the ſame Obſervations as in the precedent Caſe, the following Conſtruction may be gathered.

In $FA$ aſſume $FB = \frac{bc}{b+a}$, and $FD = \frac{bc}{b-a}$ from $F$ towards $A$: Alſo take $DG = BF$ from $D$ the contrary way to $F$; then with the Points $F$ and $G$, as Foci, and $BD$, as a firſt Axis, deſcribe * two oppoſite Sections $BM$, $DM$; and the ſame ſhall be the Locus of all the ſought Points $M$. FIG. 202. * *Art.* 76.

*Caſe* 3. Here the general Equation $aayy - bbyy - 2aacy + aaxx + aacc = o$ will become this $xx - 2cy + cc = o$, becauſe $a$ is $= b$, and the Locus thereof is a Parabola, which may be conſtructed eaſily by Article the 310th. But it appears at once, without any manner of Calculus, if a Parabola be deſcrib'd with the Line $AP$ as a Directrix, and the Point $F$ as a Focus, (according to the Directions of Def. 1.) that the ſame will be the Locus requir'd. FIG. 223.

## COROLLARY I.

366. IN the firſt Caſe, it is evident that $CF\left(\frac{bbc}{aa-bb}\right) : CB\left(\frac{abc}{aa-bb}\right) :: CB\left(\frac{abc}{aa-bb}\right) : CA\left(\frac{aac}{aa-bb}\right) :: a : b$; the ſame will be found alſo in the ſecond Caſe: From hence ariſes the following Theorem.

In an Ellipſis or the oppoſite Sections, whoſe Centre is the Point $C$, and two Foci the Points $F$ and $G$, and firſt Axis the Line $BD$, if $CA$ be taken equal to a third Proportional to $CF$, $CB$, from $C$ towards the Focus $F$; and if the indefinite Line $AP$ be drawn perpendicular to $BD$: Then if from any Point $M$ of the Section, be drawn the right Line $MP$ perpendicular to $AP$, and the right Line $MF$ to the Focus $F$: I ſay the Ratio of $MP$ to $MF$, ſhall be always the ſame as the Ratio of the firſt Axis $BD$ to $FG$, the Diſtance between the Foci. FIG. 201, 202.

In the following Corollaries the indefinite right Line $AP$ is called a *Directrix*, as well in regard to the Ellipſis and oppoſite Sections, as to the Parabola. From whence it appears to be eaſy to deſcribe a Conick Section, whoſe Focus is the given Point $F$, Directrix the right Line $AP$ given in Poſition, that ſhall paſs through a given Point $M$; for drawing the Line $MF$ to the Focus $F$, and $MP$ perpendicular to the Directrix $AP$, and calling the given Quantities $MP$, $a$; $MF$, $b$; then you need only deſcribe the Locus of the Points $M$ being ſuch, that $MP$ be always to $MF$ as $a$ to $b$.

### COROLLARY. II.

FIG. 204, 205. 367. IF any two Points *M* and *N*, of a Conick Section be joined by a right Line meeting the Directrix in *C*; and if the right Lines *FM*, *FN*, *FC*, be drawn from the Focus *F*: I ſay the Line *FC* does biſect *NFH* the Complement of the Angle *NFM* to two right Angles, when the Points *M* and *N* be taken on the Parabola, Ellipſis, or Hyperbola, and the Angle *NFM*, when they be taken on the oppoſite Sections.

For if the Perpendiculars *MP*, *NQ*, be drawn to the Directrix, and if the Line *ND* be drawn parallel to *MF*; then becauſe the Triangles *MPC*, *NQC*, and *MFC*, *NDC*, are ſimilar, therefore *MP*:*NQ* :: *MC*:*NC* :: *MF*:*ND*. And conſequently *MP*:*MF*:: *NQ*:*ND*. But by the Property of the Conick Section, having *PQ* for the Directrix, and *F* for a Focus, we have *MP*:*MF* :: *NQ*:*NF*. Therefore the Lines *ND*, *NF*, ſhall be equal between themſelves; that is, (in the firſt Caſe) the Angle *NDF* or *CFM*, ſhall be equal to the Angle *CFN*, and (in the ſecond) the Angle *FDN* or *CFH* ſhall be equal to the Angle *CFN*. *W. W. D.*

### COROLLARY III.

FIG. 204. 369. HENCE appears the manner of deſcribing a Parabola, Ellipſis, or Hyperbola, which ſhall paſs through three given Points *M*, *N*, *O*, and have the given Point *F* for a Focus.

Through the Focus *F* draw the right Lines *FC*, *FE*, which do biſect *NFH*, *NFK*, the Complements of the given Angles *MFN*, *OFN*; and through the Points *C*, *E*, wherein *FC*, *FE* meet the Lines *MN*, *ON*, (which join the given Points) draw the indefinite Line *CE*. Then if a Conick Section be deſcribed through the Point *M*, with the Line *CE* as the Directrix, and the Point *F*, as a Focus; it is manifeſt by the precedent Corollary, that that Conick Section will alſo paſs through the two other Points *N* and *O*.

### COROLLARY IV.

FIG. 205. 369. FROM the ſecond Corollary ariſes a way of deſcribing two oppoſite Hyperbola's, having the Point *F* as a Focus; ſo that one of them ſhall paſs through two given Points *M*, *O*, and the other through one given Point *N*.

Through the Point *F*, draw the Line *FE*, biſecting *HFO*, the Complement of the Angle *MFO* (formed by the right Lines *FM*, *FO*, drawn from the Point *F* to the Points *M*, *O*, being both in the ſame Hyperbola.) Alſo thro' the ſame Point *F*, draw the Line *FC*,

*FC*, biſecting the Angle *MFN*, (form'd by the right Lines *FM*, *FN*, drawn from the Point *F* to the Points *M* and *N*, falling on the oppoſite Hyperbola's.) Through the Points *E*, *C*, wherein the Lines *FE*, *FC*, meet the right Lines *MO*, *MN*, (that do join the given Points) draw the indefinite Line *EC*. Laſtly, deſcribe two oppoſite Hyperbola's with the Point *F*, as a Focus, and the right Line *EC*, for the Directrix, ſo that one of them may paſs through the Point *M*; and then it is evident, that theſe Hyperbola's will anſwer what is propos'd.

### COROLLARY V.

370. THE ſame Things being premis'd as in the ſecond Corollary; it is manifeſt, that the Angle *MFN*, the Difference between *CFH* or *CFN* the Complement thereof to two right Angles, diminiſhes more and more, according as the Point *N* accedes to *M*, ſo that the ſame will vaniſh quite, when the Point *N* coincides with *M*; therefore the Angle *CFM* ſhall then be equal to its Complement to two right Angles, and conſequently will be a right Angle. And becauſe the Line *MD* does then become the Tangent *MT*, ſince * the ſame paſſes through two Points of the Curve infinitely near to each other; therefore, from hence we have a general and new way of drawing a Line (*MT*) to touch a Conick Section in the given Point *M*, the Focus, together with the Axis paſſing through that Focus, being given. FIG. 204. **Art.* 188.

For finding the Directrix according to the Directions in *Coroll.* 2. from the given Point *M* draw the right Line *MF* to the Focus *F*, and draw the right Line *FT* perpendicular to *MF*, meeting the Directrix in *T*; then if *MT* be drawn through the Point *T* and the given Point *M*, the ſame will touch the Section in *M*.

### EXAMPLE XI.

371. TWO Angles *KAM*, *KBM*, moveable about the fixed Points *A*, *B*, being given upon a Plane, together with an indefinite right Line *FK*, not paſſing through thoſe Points; let the Point of Concurrence (*K*) of the two Sides *AK*, *BK*, move along the right Line *FK*: Now it is requir'd to find the Nature of the Curve deſcrib'd by the Interſection (*M*) of the other two Sides *AM*, *BM*, produced, when neceſſary, on the other Side the Points *A* and *B*. FIG. 206.

Upon *AB*, as a Chord, deſcribe the Segment of a Circle, on the other Side the Point *M*, containing an Angle *BDA* equal to four right Angles, minus the two given Angles *KAM*, *KBM*; and compleating the whole Circle whereof *BDA* is the Segment, it may happen that the indefinite right Line *FK* does fall quite without that

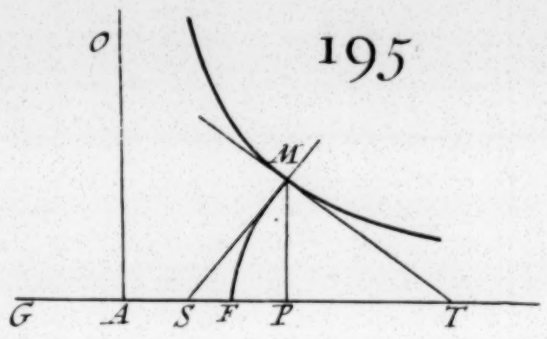
Circle, within the ſame, or finally touches it: And ſo there are three Caſes, which I ſhall explain in particular.

*Caſe* 1. From *C* the Centre of the Circle *B D A E*, draw *C F* perpendicular to *F K* meeting the Circle in the Points *D*, *E*; and through the Point *D* (being nigher to the Line *F K* than the Point *E*) let the two Sides *D A*, *D B*, of the two Angles *D A P*, *D B Q*, equal to the two Angles *KAM*, *KBM*, paſs, which Sides being produced towards *D*, let meet the Line *F K* in the Points *G* and *H*. And by Conſtruction, the Angle *B D A* plus the two Angles *D A P*, *D B Q*, are equal to four right Angles; and ſince the ſame Angle *B D A* plus the two Angles *D A B*, *D B A*, are equal to two right Angles; therefore the Angles *B A P*, *A B Q*, are equal to two right ones; and ſo the Lines *A P*, *B Q*, are parallel between themſelves. This being laid down.

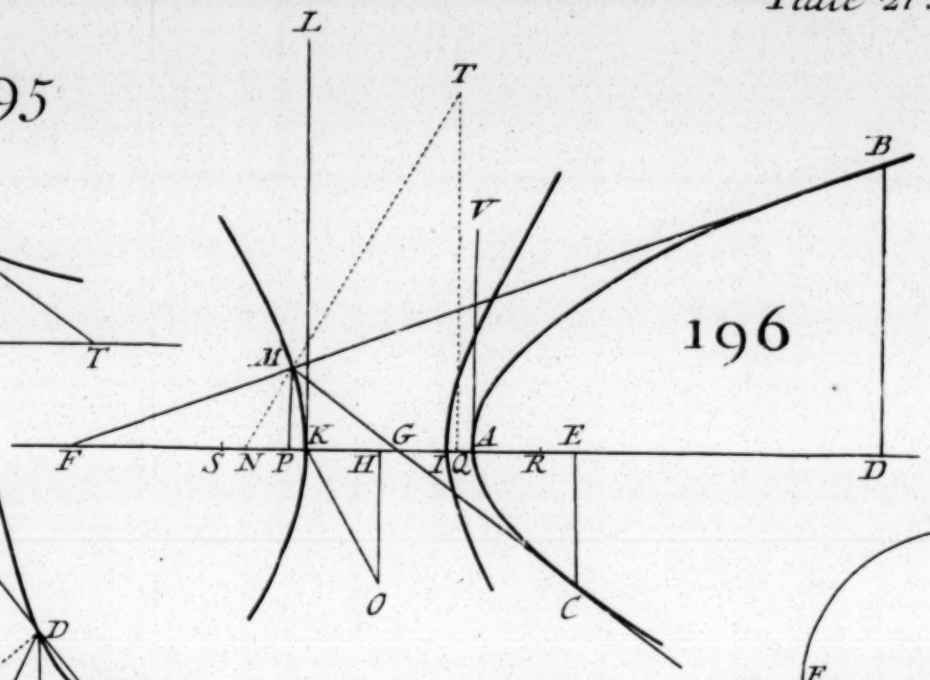
From the Point *K* draw *K R*, *K S*, perpendicular to the Sides *A D*, *B D*; and from the Points *A*, *M*, the Lines *A I*, *M P*, perpendicular to the two other Sides *B Q*, *A P*, which meet *B Q*, in the Points *I* and *Q*. Now let the given Quantities *F E* be $= a$, *F D* $= b$, *BI* $= c$, *A I* $= d$, *F G* $= g$, *F H* $= h$, *D G* $= m$, *D H* $= n$: and the unknown Quantities *F K* $= z$, *A P* $= x$, *P M* $= y$; then becauſe *G DF*, *G K R*, are Right-angl'd ſimilar Triangle: Therefore *G D* $(m)$ : *G F* $(g) :: G K\ (z-g) : G R = \frac{gz-gg}{m}$. And *G D* $(m)$ *DF* : : $(b)$ : : *GK* $(z-g)$ *KR* $= \frac{bz-bg}{m}$. But becauſe *G D F*, *E D A* are alſo Right-angl'd ſimilar Triangles, therefore *G D* $(m)$ : *D F* $(b)$ : : *E D* $(a-b)$ : *A D* $= \frac{ab}{m} - \frac{bb}{m}$; and conſequently *A D* + *D G* or *A G* is $= \frac{ab-bb+mm}{m}$, and *A G* + *G R* or *A R* $= \frac{ab-bb+mm+gz-gg}{m} = \frac{ab+gz}{m}$; ſince *D F G* being a right-angl'd Triangle, *m m* is $= b\,b + g\,g$. Again the right-angl'd Triangles *A R K*, *A P M*, are ſimilar: Becauſe taking the ſame Angle *K A P* from the equal Angles *K A M*, *D AP*, the remaining Angles *K A R*, *P A M*, ſhall be equal; and conſequently *A R* $\left(\frac{ab+gz}{m}\right)$ : *R K* $\left(\frac{bz-bg}{m}\right)$ : : *A P* $(x)$ : *P M* $(y)$, from whence we get $z = \frac{aby+bgx}{bx-gy}$.

But the Right-angl'd ſimilar Triangles *H D F*, *H K S*, give *H S* $= \frac{bz+bh}{n}$, and *K S* $= \frac{bz+bh}{n}$; and the Right-angl'd ſimilar Triangles *H F D*, *E B D*, give *D H* $(n)$ : *D F* $(b)$ :: *D E* $(a-b)$ : *D B* $= \frac{ab-bb}{n}$.

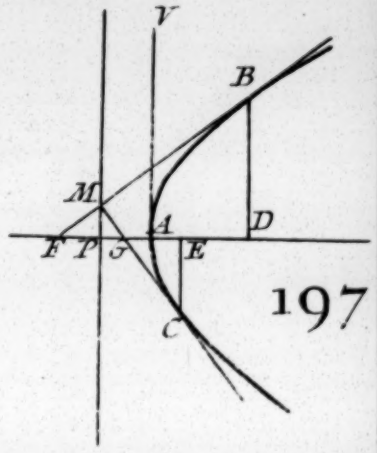
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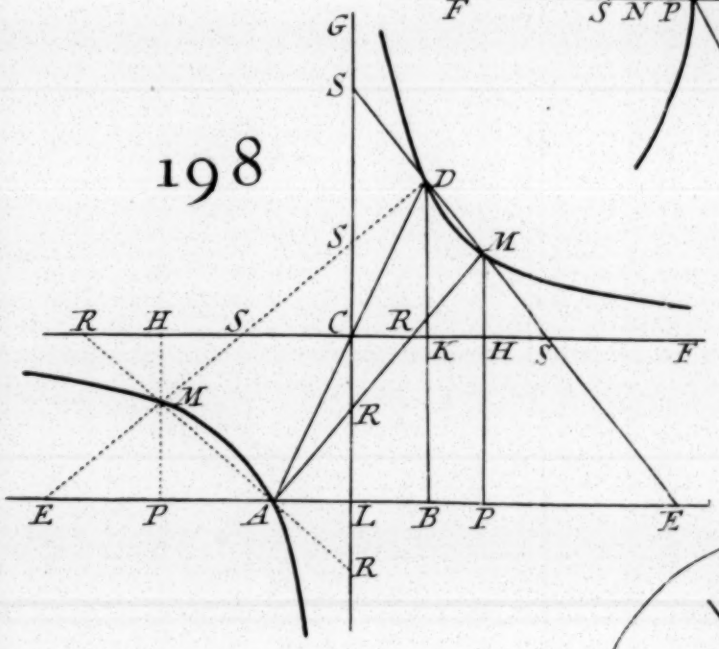

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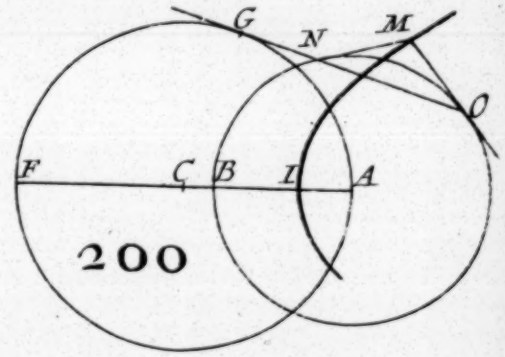

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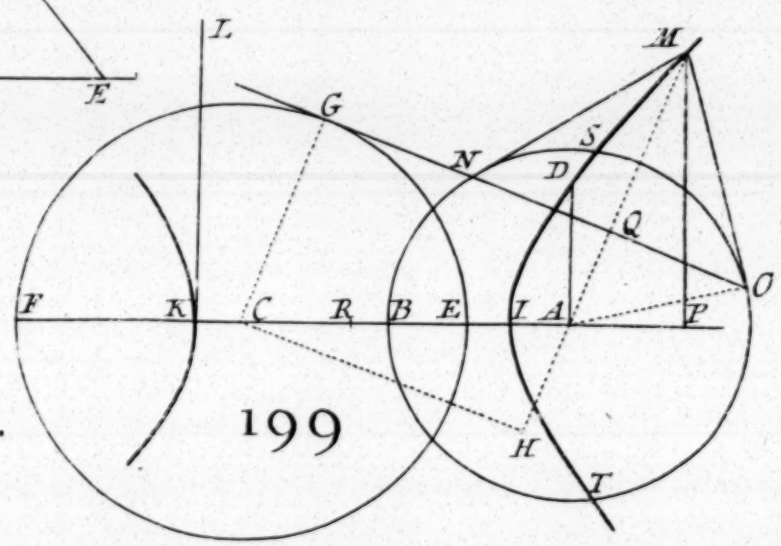

197



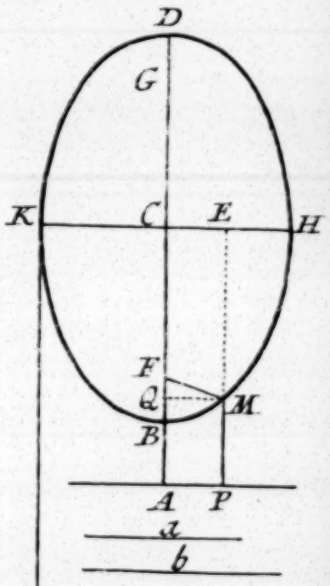

198



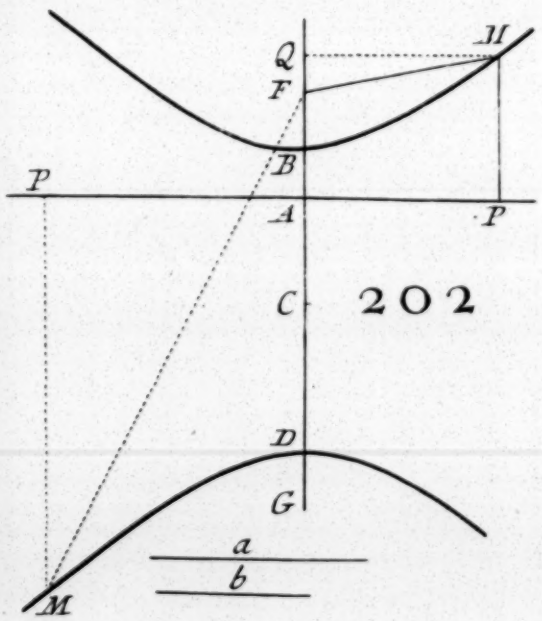

200



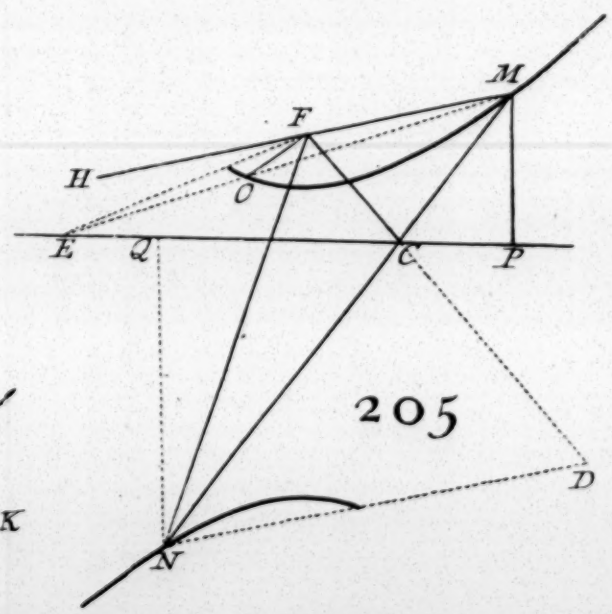

199



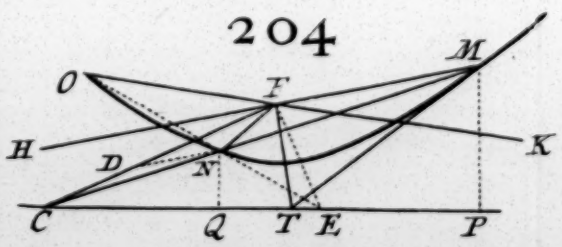

201



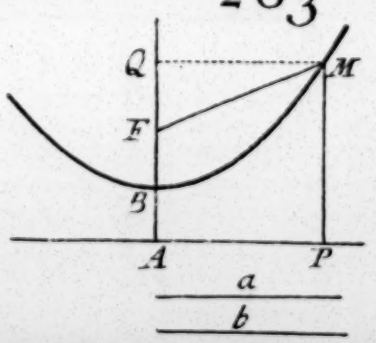

202




205




204




203

And therefore $BD + DH$ or $BH$ is $= \frac{ab-bb+nn}{n}$, and $BH - HS$ or $BS = \frac{ab-bb+nn-hz-hb}{n} = \frac{ab-hz}{n}$, ſince $DFH$ being a Right-angl'd Triangle $nn$ is $= bb + hh$. And the right-angl'd Triangles $BSK$, $BQM$, are ſimilar; becauſe the Angle $DBM$ being taken away from the equal Angles $DBQ$, $KBM$, the remaining Angles $KBS$, $MBQ$, ſhall be equal between themſelves; and conſequently $BS\left(\frac{ab-hz}{n}\right) : SK\left(\frac{bz+bh}{n}\right) :: BQ(x-c) : QM(y+d)$. From whence we get $z = \frac{aby-bhx+bch+abd}{bx+hy-bc+hd}$.

Now comparing this laſt Value of $z$ with the precedent one, multiplying croſs-wiſe, making $GH(g+b) = f$. and laſt of all dividing by $abf$, and we ſhall have the following Equation.

$$yy + dy + \frac{b}{a}xx - \frac{bc}{a}x = o$$
$$-\frac{bc}{f} \qquad -\frac{bd}{f}$$
$$+\frac{cgh}{af} \qquad +\frac{dgh}{af}$$

whoſe Locus, which may be conſtructed by Article the 324th, ſhall be an Ellipſis, becauſe the Term $\frac{b}{a}xx$ (in the firſt Caſe) ſhall always have the Sign $+$ prefixed to it, let the Situation of the Points $A, B, K$, be any how at pleaſure.

*Caſe* 2. The Lines being call'd by the ſame Letters here, as in the firſt Caſe, by reaſoning after the ſame Manner as there, you will get this Equation. FIG. 20?

$$yy + dy - \frac{b}{a}xx + \frac{bc}{a}x = o$$
$$-\frac{bc}{f} \qquad -\frac{bd}{f}$$
$$-\frac{cgh}{af} \qquad -\frac{dgh}{af}$$

differing from the former Equation only in Signs, and the Locus of the ſame, which may be conſtructed by Article 332, ſhall be always two oppoſite Hyperbola's; becauſe in this Caſe, the Term $\frac{b}{a}xx$ has the Sign $-$ always prefixed to it. Becauſe the Plane $xy$ is not found in either of the two precedent Equations, and the Angle $APM$ is a right Angle; therefore one of the Axes of the Ellipſis in the firſt

Caſe, and of the oppoſite Sections in the ſecond, muſt be parallel to the Lines $AP$, $B\mathcal{Q}$; and the Ratio thereof to its Parameter, ſhall be the ſame as of $EF$ $(a)$ to $FD$ $(b)$; becauſe the Fraction $\frac{b}{a}$ multiplying the Square $x\,x$ expreſſes that Ratio.

When the Point $K$ by its Motion along the indefinite right Line $KF$, comes to the Point $O$, wherein that Line meets the Circumference; then it is manifeſt, 1. That the Sides $AM$, $BM$, whoſe Point of Concurrence $M$ does deſcribe the Hyperbola $BAM$, do become parallel, 2. That they cut one another towards the oppoſite Side, during the Motion of the Point $K$ along $OL$, that Part of the Line $KF$, falling within the Circle; and then they will again become parallel, when the Point $K$ falls in $L$, after which they will begin again to cut one another towards the ſame Side. From whence it appears, that the Point $M$ does deſcribe the Hyperbola $BAM$ during the Motion of the Point $K$ along the two indefinite Parts of the right Line $KF$, that fall without the Circle; and the oppoſite Hyperbola, during the Motion of the Point $K$ through $OL$, that Part of $KF$ falling within the Circle.

FIG. 208. *Caſe* 3. Becauſe here the indefinite right Line $FK$ touches the Circumference of the Circle $BDAE$ in ſome Point $F$, it is manifeſt, that the Point $D$ (in the two other Caſes) does here coincide with the Point $F$; and ſo the Triangles $DFG$, $DFH$, do vaniſh; therefore we may uſe the Triangles $DAE$, $BBE$, for them, after the following manner.

Let the given Quantities $AE$ be $= a$, $EB = b$, $EF = m$, $AF = g$, $BF = h$, $BI = c$, $AI = d$; and the unknown Quantities $FK = z$, $AP = x$, $PM = y$. Now the right-angled Triangles $FKR$, $EFA$, are ſimilar; becauſe the Angle $KFR$ or $TFA$ (vertically oppoſite thereto) made by the Tangent $FT$, and the Chord $FA$, is meaſur'd by half the Arc $AF$, as well as the Angle $FEA$; and therefore $FE\ (m) : EA\ (a) :: KF\ (z) : FR = \frac{az}{m}$. And $EF\ (m) : FA\ (g) :: FK\ (z) : KR = \frac{gz}{m}$. But the right-angled ſimilar Triangles $ARK$, $APM$, do give $AR$ or $AF + FR \left(\frac{az+gm}{m}\right) : RK \left(\frac{gz}{m}\right) :: AP\ (x) : PM\ (y)$; from hence we get $z = \frac{gmy}{gx-ay}$: after the ſame manner, becauſe $EFB$, $FKS$, are right-angled ſimilar Triangles; therefore $FS$ is $= \frac{bz}{m}$, and $KS = \frac{bz}{m}$; and becauſe the right-angled Triangles $BSK$,

$B\mathcal{Q}M$,

$B\,\mathcal{Q}\,M$, are ſimilar; therefore $BS$ or $BF - FS\left(\frac{hm-bz}{m}\right) : SK\left(\frac{hz}{m}\right)$ $:: B\,\mathcal{Q}\,(x-c) : \mathcal{Q}M\,(y+d)$; from whence ariſes $z = \frac{hmy+hmd}{hx-ch+bd+by}$.

Now comparing the two Values of $z$, multiplying croſs-wiſe, and putting the Terms in Order, and we ſhall have this Equation, $yy + dy - \frac{cgh}{ah+bg}y - \frac{dgh}{ah+bg}x = o$, the Locus of which ſhall always be a Parabola, whoſe Axis is parallel to the right Lines $AP$, $B\,\mathcal{Q}$; and the ſame may be conſtructed by Article the 310th.

Hence it is evident, 1. That the Locus of all the ſought Points $M$ ſhall be always a Conick Section, whoſe Axis, or one of the Axes, ſhall be parallel to the Lines $AP$, $B\,\mathcal{Q}$; and particularly the ſame ſhall be an Ellipſis in the firſt Caſe, two oppoſite Hyperbola's in the ſecond, and a Parabola in the third; and in the firſt and ſecond Caſes, the Axis which is parallel to $AP$ ſhall have the ſame Ratio to its Parameter as $EF$ to $FD$. 2. That in the firſt and third Caſes, the two fix'd Points $A$ and $B$, about which the moveable Angles $KAM$, $KBM$, revolve, do always fall on the ſame Side the Line $FK$: But in the ſecond Caſe, thoſe Points may not only fall on the ſame Side of that Line, but on both Sides thereof; becauſe the Circumference of the Circle $ADBE$ upon which they are ſituate, is then cut into two Portions by the Line $FK$.

## SCHOLIUM I.

372. 1. ANY right Line, as $AM$, paſſing through one of the fixed Points $A$ or $B$, being given, the Point $M$ wherein the ſame meets the Locus ſought, may always be found after the following manner. Draw the right Line $AK$ forming the Angle $MAK$ with $AM$, equal to the given Angle revolving about the fixed Point $A$, and from the Point $K$ wherein $AK$ meets $FK$, through the fixed Point $B$, draw the Angle $KBM$ equal to the other given Angle revolving about the other fixed Point $B$; then the Point $M$, wherein the Side $BM$ of that Angle meets the Line $AM$, ſhall be the Locus ſought. 2. When the Point $K$, in moving along the Line $FK$, is ſo ſituate, that the Side $(AM)$ of the Angle $KAM$ does coincide with the Line $AB$: Then it is plain that $M$ the Point of Concurrence of the two Sides $AM$, $BM$, does fall on the Point $B$; and ſo the Locus of the Points $M$ does paſs through the fixed Point $B$; after the ſame Manner we prove that the ſame paſſes through the Point $A$. FIG. 206, 207, 208.

Hence if it be requir'd to deſcribe the Conick Section being the Locus of all the ſought Points $M$, without the Help of the foregoing FIG. 206.

Equa-

Equations, you need but draw (as in the Example) the right Lines *A P*, *A I*, and finding the Points wherein they meet the Section, and compleating the Rectangle, having thoſe two Lines for the Sides thereof, then deſcribe * an Ellipſis or two oppoſite Hyperbola's about the Rectangle, (according as *F K* falls without or within the Circle) whoſe Axis being parallel to *A P* may be to its Conjugate, as the Square of *E F* to the Square of *D F*. And if the Section be a Parabola, (which happens when the Line *K F* touches the Circle *B D A*;) then find a Point on the Line *A I* wherein the ſame meets the Section, and deſcribe a Parabola (by Article 170) paſſing through that Point and the two given Points *A*, *B*; ſo that the Diameters thereof be parallel to the Lines *A P*, *B Q*.

*Art.* 176, *and* 178.

Fig. 208.

## SCHOLIUM. II.

Fig. 209. 373 WHEN the Point *K* in its motion along the Line *F K*, is ſo ſituate, that *A M* the Side of the Angle *K A M*, does coincide with *A B*; then it is manifeſt that the Point *M* falls on *B*; and alſo that *B M* the Side of the Angle *K B M* does touch * the Locus of the Point *M* in *B*, becauſe in this Caſe *M* may be taken as being infinitely near to *B*. Whence if it be requir'd to draw a Line to touch that Locus in *B*, you need only draw a right Line *A C* through the Point *A* making an Angle (*B A C*) with *B A* equal to the given Angle *K A M*; and then a Line *B D*, making with *B C* the Angle *CBD*, equal to the other given Angle *K B M*. For the Side (*B M*) of that Angle, which does become *B D*, ſhall touch the Section in *B*. Underſtood the ſame with regard to the other fixed Point *A*.

*Art.* 188.

Fig. 209. From hence ariſes a very eaſy way of deſcribing the Conick Section being the Locus of all the Points *M*, without uſing the foreſaid Equations, or even any manner of Calculus.

Through the fixed Point *B* draw the Tangent *B D*, and through the other fixed Point *A* draw *A E* parallel to that Tangent, and on *A E* find * the Point *E* wherein the ſame meets the Section, and biſecting it in *H*, draw *B H*, upon which find * alſo the Point *G* wherein the ſame meets the Section: Now if a Conick Section be deſcrib'd * with the Diameter *B G*, and Ordinate *H A* or *H E*, the ſame ſhall be the Locus ſought. For it is plain that the Line *B G*, which does biſect the Line *A E* terminated by the Section, and being parallel to the Tangent in *B*, ſhall be a Diameter thereof, and the Line *A H*, an Ordinate to that Diameter. Where it muſt be obſerved, that when the Point *H* falls between the Points *B*, *G*, the Section is an Ellipſis: when the ſame falls on either Side of thoſe two Points, the Section is two oppoſite Hyperbola's; and finally, when the Line *B G* is infinite, the Section is a Parabola.

*Art.* 372.
*Art.* 372.
*Art.* 161, 162.

COROL-

## COROLLARY I.

374. BY means of the laſt Example we may deſcribe a Conick Section of a given Species, through four given Points *A*, *B*, *H*, *M*. FIG. 210.

For, 1. Let the Section be an Ellipſis, whoſe great Axis to its Parameter let be as the given Quantity *a* to *b*. Form the Triangle *ABH*, in joining three of the given Points by ſtraight Lines; and at the Points *A*, and *B*, let the Angles *M A K*, *M B K*, be made equal to the Angles *G A H*, *R B A*, the Complements of the Angles *H A B*, *H B A*, to two right Angles, ſo that their Sides *B M*, *A M*, do interſect one another in the fourth given Point *M*. This being done, upon *A B* as a Chord deſcribe the Segment *B D A* of a Circle (on the other Side the Point *M*) containing an Angle equal to four right Angles minus the two Angles *K A M*, *K B M*; and about *C* the Centre of that Circle, deſcribe another Circle whoſe Radius *C F* may be to the Radius *C D* of the former Circle, as $a + b$ to $a - b$; then from *K*, the Point of Concurrence of *A K*, *B K*, the Sides of the Angles *M A K*, *M B K*, draw the Tangent *K F* to this latter Circle. Now if the Point *K* be moved along the indefinite right Line *F K*, the Point of Concurrence (*M*) of the two other Sides *A M*, *B M*, produced on the other Side of the Points *A*, *B*, ſhall deſcribe the Ellipſis required. For it is evident from what has been ſaid in the firſt Caſe of the Example, that the Locus of the Points (*M*) ſhall be an Ellipſis, whoſe great Axis ſhall be to the Parameter as *E F* (*a*) to *D F* (*b*); and moreover that the ſame ſhall paſs through the Points *A*, *M*, *B*, *H*, becauſe when the Point *K* is in *G*, the Side *A M* will fall in *A H*, and the Side *B M*, in *B R*.

2. When it is an Hyperbola or two oppoſite ones that is required to be deſcribed through four given Points *A*, *B*, *H*, *M*, the great Axis being to the Parameter in the given Ratio of *a* to *b*; then the Conſtruction will be the ſame as above, only *C F* the Radius of the Circle Concentrick to the Circle *B D A E*, muſt here be to the Radius *C D*, as $a - b$ to $a + b$.

3. If a Parabola be to be deſcribed through four given Points *A*, *B*, *H*, *M*. Deſcribe the Circle *B D A E* as in the firſt Caſe, and from *K* the Point of Concurrence, draw a Tangent to that Circle, and the ſame ſhall be the indefinite right Line, along which, if the Point *K* be moved, the other Point of Concurrence *M* will deſcribe the Parabola required.

Becauſe there can be drawn two Tangents to a Circle from one Point; therefore we may deſcribe two different Conick Sections, both anſwering the Problem when the ſame is poſſible; for when the

Point *K* falls within the Circle, having *C F* for its Radius, it is plain then that the Problem is impoſſible.

The Conick Section ſought may be deſcribed by means of its Axes according to Article 372, or elſe by help of one of its Diameters, and an Ordinate to the ſame, by Article 373.

### COROLLARY II.

FIG. 211. 375. FROM the foregoing Example we have alſo a new way of deſcribing a Conick Section paſſing through five given Points *A*, *B*, *H*, *M*, *N*. For joining any three of thoſe Points, as *A*, *B*, *H*, by right Lines, and through the other two Points, *M*, *N*, and the fixed Points *A*, *B*, let the Angles *MAK*, *NAS*, paſs, each equal to the Angle *HAG*, the Complement of the Angle *HAB* to two right Angles, and the Angles *MBK*, *NBS*, each equal to the Angle *ABR*, the Complement of the Angle *ABH* to two right Angles; then if through the Points of Concurrence *K*, *S*, the indefinite Line *SK* be drawn, and the Point *K* be moved along the ſame, it is manifeſt that the Point of Concurrence *M* by that Motion, ſhall deſcribe the Conick Section requir'd; becauſe it paſſes through the five given Points *A*, *B*, *H*, *M*, *N*.

*The End of the Eighth Book.*

BOOK

# BOOK IX.

## *Of the Conſtruction of* EQUATIONS.

### PROPOSITION I.

### Problem.

376. *TO conſtruct any given Equation, wherein the unknown Quantity is but of one Dimenſion.*

Firſt, Let the unknown Quantity $x$ be equal to one or more ſimple Fractions, as $\frac{ab}{c}$, or $\frac{abe}{cf}$, or $\frac{abeh}{cfg}$, &c. If we make $c : b :: a : l$, then it is plain, that this four Proportional $l$ ſhall be $= \frac{ab}{c}$; and if we make $f : l :: e : m$, then $m$ will be $= \frac{el}{f} = \frac{abe}{cf}$; and finally, making $g : m :: b : n$, and we ſhall have $n = \frac{mb}{g} = \frac{abeh}{cfg}$, by ſubſtituting $\frac{abe}{cf}$ for its Value $m$. Therefore the unknown Quantity $x$ will be equal to $l$, or $m$, or $n$, &c. according as it was equal to $\frac{ab}{c}$, or $\frac{abe}{cf}$, or $\frac{abeh}{cfg}$, &c. Now by augmenting the Number of Proportions, (according to Neceſſity) and we ſhall always find a right Line equal to a given ſimple Fraction, be the Number of Dimenſions of its Numerator what it will. From whence it appears, that a Line $x$ may be always found equal to a Quantity compounded of ſeveral ſimple Fractions; for right Lines being found equal to each of thoſe Fractions, we need only add them together, or ſubſtract them from each other, according to the Signs + and —. For Example: Suppoſe it be requir'd to find a Line $x = a + \frac{ab}{c} + \frac{aab}{cf} - \frac{aacc}{b^3}$. Add the two Lines $b = \frac{ab}{c}$, and $l = \frac{aab}{cf}$ to the Line $a$, that ſo all three of them may make but one Line only, and from the Sum take the Line $m = \frac{aacc}{b^3}$, and what remains ſhall

be the Value ſought of the unknown Quantity $x$, that is, $x$ ſhall be $= a + b + l - m$.

2. Let the unknown Quantity $x$ be equal to one or more compound Fractions, that is, Fractions whoſe Denominators conſiſt of ſeveral Terms. Firſt, according to the Directions above, find a Line equal to the Denominator divided by a Line taken at pleaſure, when the Terms of the Denominator do not ariſe to above two Dimenſions; by a Plane, when they ariſe not to above three; and by a Solid, when they do not ariſe to above four, *&c.* and by this means all the Terms of the Denominator may be brought into one only, which being ſubſtituted inſtead of them, ſhall change the compound Fraction into one or more ſimple Fractions, according as the Numerator conſiſts of one or more Terms; and then if a Line be found (as above) equal to them, the ſame ſhall be that ſought. This will be manifeſt by the following Examples.

It is requir'd to find a Line $x = \frac{age - bce}{bb + af}$. Firſt, find a Line $m = f + \frac{bb}{a}$, that is, equal to the Denominator $af + bb$ divided by $a$; from whence $bb + af = am$, and then finding a Line $n = \frac{age - bce}{am} = \frac{ge}{m} - \frac{bce}{am}$; and we ſhall have the ſought Line $x = n$. After the ſame manner, if it be requir'd to find a Line $x = \frac{a^3b + aacc - abcf}{aaf + ccf + bff}$, we muſt firſt find a Line $m = a + \frac{cc}{a} + \frac{bf}{a}$, that is, equal to the Denominator $aaf + ccf + bff$ divided by the Plane $af$; from whence we get $afm = aaf + ccf + bff$, and afterwards another Line $= \frac{a^3b + aacc - abcf}{afm} = \frac{aab}{fm} + \frac{acc}{fm} - \frac{bc}{m} = x$. The ſame muſt be underſtood of any other Example, which any one may form at pleaſure.

### COROLLARY I.

377. BY means of this Propoſition it is eaſy, 1. To find a ſimple Fraction $\frac{x}{a}$ or $\frac{a}{x}$, whoſe Denominator or Numerator $a$ is given, equal to one or more ſimple or compound Fractions; for we need only find a Line $x$ equal to the Line $a$ multiply'd or divided by thoſe Fractions. For Example; If it be requir'd to find a Fraction $\frac{x}{a} = \frac{cc + ff}{af + cf} + \frac{aa}{gg}$, then it is plain, that we need only find a Line $x$

$=$

$= \frac{acc+aff}{af+cf} + \frac{a^3}{gg}$. 2. To find a Plane $ax$, one of whoſe Sides $a$ is given, equal to one or more Planes be they never ſo much compounded; for here we need only find a Line $x$ equal to all thoſe Planes divided by $a$. 3. To find a Solid $aax$ or $abx$, whoſe Sides $a$, $a$, or $a$, $b$, are given, equal to ſeveral Solids; becauſe here you need but find a Line $x$ equal to all the Solids divided by the Square $aa$, or Plane $ab$. 4. To find a ſurd Solid $a^3 x$, or $abcx$, whoſe three Sides $a, a, a$, or $a, b, c$, are given, equal to ſeveral ſurd Solids; becauſe you need only find a Line $x$ equal to all the ſurd Solids divided by the Cube $a^3$, or the Solid $abc$. And the ſame muſt be underſtood of ſeveral Products of five Dimenſions, of ſix, *&c.* which may be always reduced to one only, whoſe Sides are all given except one.

### COROLLARY. II.

378. HENCE, if it be requir'd to find a Square equal to ſeveral given Planes, the Planes muſt be brought into one, and then a mean Proportional found between the Sides of it; and then that mean Proportional ſhall be the Side of the Square requir'd. For Example: If it be requir'd to find a Square $xx = ss - \frac{ccee-eehh}{bb+af}$ (the Lines $a, b, c, e, f, b, s$, being given) firſt find a Line $m = \frac{ss}{e} - \frac{cce-ehh}{bb+af}$, that ſo we may have a Plane $em = ss - \frac{ccee-eehh}{bb+af}$, and then finding a mean Proportional $x$ between $e$ and $m$, the Sides of the Plane $em$; and it is plain, that $xx$ is $= em = ss - \frac{ccee-eehh}{bb+af}$.

If it be requir'd to find a Line $x$, whereof the Square $x^4$ ſhall be equal to ſeveral given ſurd Solids; find (as above) a Square $zz$ equal to all the given ſurd Solids divided by the Square $aa$ either given or taken at pleaſure. When this is done, find a mean Proportional as $x$, between the two Lines $a$ and $z$, and that mean Proportional ſhall be the Line ſought; for $xx = az$, and (ſquaring both Sides) $x^4 = aazz$, that is, $x^4$ is equal to all the given ſurd Solids.

### SCHOLIUM.

379. ALtho' the Method above laid down is general for all poſſible Caſes, yet it is not always the moſt ſimple; therefore I ſhall here lay down ſome particular Examples, and reſolve them more eaſy after a manner ſomething different from the general Rule, which may ſerve as Rules for all the like Caſes.

1. Let

1. Let $x$ be $= \frac{abcc-aabb}{abc+c^3}$. First find a Line $m = \frac{ab}{c}$, and ſubſtituting $cm$ for its Value $ab$, we ſhall have $x = \frac{c^3m-ccmm}{ccm+c^3} = \frac{cm-mm}{m+c}$; from whence making $c+m : c-m :: m : n$, I find that this fourth Proportional $n$ is $= x$. Therefore it is manifeſt, that we here find the Value of $x$ by only two Proportions, whereas by the general Method there muſt have been three at leaſt.

2. Let $x$ be $= \sqrt{aa+bb}$. Make a right-angl'd Triangle, one of whoſe Sides let be $= a$, and the other $= b$; then the Hypothenuſe thereof ſhall be the Value of $x$. If it be requir'd to find a Line $x = \sqrt{aa-bb}$, then find a mean Proportional as $x$, between the two Lines $a+b$ and $a-b$; for the Square thereof muſt be equal to the Product $aa-bb$ of the Extremes. Or elſe make a right-angled Triangle, whoſe Hypothenuſe let be $= a$, and one of the Sides $= b$; then the other Side ſhall be the Value of $x$.

3. Let $xx$ be $= ss + 4ee - \frac{4ccee}{aa}$. Aſſume $m$ equal to the Hypothenuſe of a right-angled Triangle having one Side $= s$, and the other $= 2e$, and finding another Line $n = \frac{2ce}{a}$, we have $xx = mm - nn$, and $x = \sqrt{mm-nn}$, which may be reſolved in the ſame manner as in the laſt Example where $x$ was $= \sqrt{aa+bb}$.

Laſtly, Let $xx$ be $= ss - \frac{ccee-eehh}{bb+af}$. Aſſume a mean Proportional between $a, f$, the Sides of the Plane $af$, that ſo we may have $ll = af$, then find a Square $mm = bb + ll$, and another Square $nn = cc + hh$, by means of two right-angled Triangles, as in the ſecond Example, and (by Subſtitution) we have $xx = ss - \frac{eenn}{mm}$; and finally, finding a Line $g = \frac{en}{m}$, and there ariſes $x = \sqrt{ss-gg}$, which may be reſolved as above.

## PROPOSITION II.

### Problem.

380. *TO find the Roots of all kinds of Equations of the ſecond Degree, [or of two Dimenſions.]*

All Equations of the Second Degree may be reduced to one of the following Forms $xx \mp ax - bb = o$, or $xx \mp ax + bb = o$, by finding a * Line $a$ equal to all the known Quantities that do

*Art. 376. multi-

multiply the unknown one $x$, and a * Square $bb$ equal to all the known Rectangles. This being premiſed. * *Art.* 378.

1. Let $xx + ax - bb$ be $= o$. Make a right-angl'd Triangle $CAB$, one of whoſe Sides $CA$ let be $= \frac{1}{2}a$, and the other Side $AB = b$, and drawing the Hypothenuſe $BC$, and producing the ſame beyond $C$, about $C$ as a Centre with the Radius $CA$, deſcribe a Circle cutting $BC$ in the Points $E$, $D$. I ſay the right Lines $BD$, $BE$, are the two Roots of the propoſed Equation $xx \mp ax - bb$; $BE$ being the affirmative Root, and $BD$ the negative Root of the Equation $xx + ax - bb = o$, and contrariwiſe $BD$ the affirmative, and $BE$ the negative Root of the Equation $xx - ax - bb = o$. FIG. 212.

For making $BE = x$, and we ſhall have $BD$ or $BE + ED = a + x$; and if $BD$ be made $= -x$, then $BE$ or $BD - ED$ ſhall be $= -x - a$. And ſo in both Caſes $DB \times BE$ is $= xx + ax = \overline{AB}^2 (bb)$ by the Nature of the Circle, that is, $xx + ax - bb$ is $= o$. And contrariwiſe, if $BD$ be made $= x$ or $BE = -x$, we ſhall find $DB \times BE = xx - ax = bb$ or $xx - ax - bb = o$.

2. Let $xx \mp ax + bb$ be $= o$. Make a right-angl'd Triangle (as in Caſe 1.) $CAB$, having one Side $CA = \frac{1}{2}a$, and the other $AB = b$; and the indefinite right Line $BD$ being drawn parallel to $AC$, about the Centre $C$ with the Radius $CA$ deſcribe a Circle cutting the Line $BD$ in the Points $E$, $D$. I ſay the right Lines $BE$, $BD$, are the Roots of the propoſed Equation $xx \mp ax + bb = o$; *viz.* the two affirmative Roots of $xx - ax + bb = o$, and the two negative ones of $xx + ax + bb = o$. FIG. 213.

For compleating the Semi-circumference $AEDH$, and drawing $EF$, $DG$, parallel to $AB$; then if $BE$ or $AF$ be made $= x$, we ſhall have $AF \times FH = ax - xx = \overline{FE}^2 (bb)$ by the Nature of the Circle. In like manner, if $BD$ or $AG$ be made $= x$, we ſhall have $AG \times GH = ax - xx = \overline{GD}^2 (bb)$: That is, $xx - ax + bb = o$ in both Caſes. If $BE$ or $AF$ be $= -x$, and $BD$ or $AG = -x$, then ſhall $AF \times FH$ and $AG \times GH$ be $= -xx - ax = \overline{FE}^2$ or $\overline{GD}^2$ $(bb)$; that is, $xx + ax + bb = o$.

If the Circle, whoſe Centre is $C$, and Radius $CA$, does not cut or touch the parallel $BD$ (which happens always when $AB$ exceeds $CA$); then both the Roots of the Equation will be imaginary; but if the Circle touches the ſame in one Point, the two Roots $BE$, $BD$, do each become equal to the Radius $CA$.

## SCHOLIUM.

381. WHEN the unknown Quantity in an Equation hath only four and two Dimenſions; then that Equation may be always brought to another wherein the unknown Quantity ariſes no

higher than the ſecond Degree; and ſo Equations of this kind may be taken for thoſe of the ſecond Degree.

FIG. 214. For Example, let $z^4 - aazz - aabb$ be $= o$. Suppoſe an unknown Quantity $x$ to be ſuch that the Rectangle under the ſame and the known Quantity $a$ be equal to the Square $zz$; that is, let $ax$ be $= zz$. Then ſubſtituting $ax$ for $zz$, and $aaxx$ for $z^4$, and the given Equation $z^4 - aazz - aabb = o$, will be brought to this $xx - ax - bb = o$, wherein the unknown Quantity $x$ ariſes no higher than the ſecond Degree. And if the Roots ($x$) thereof be found as above, and mean Proportionals be found between the known Quantity $a$, and the Values of thoſe Roots; then it is evident that thoſe mean Proportionals ſhall expreſs the ſought Values of the unknown Quantity $z$: Becauſe $zz$ is $= ax$.

## PROPOSITION III.

### Problem.

382. *TO find the Roots of Equations of the ſecond Degree another way without neceſſarily charging the laſt Term into a Square.*

FIG. 215. 1. Let $xx \mp ax - bc$ be $= o$, wherein $b$ exceeds $c$. Deſcribe any Circle $ABD$ having its Diameter not leſs than the given Quantities $a$ and $b - c$, and within this Circle, inſcribe two Chords $AB = a$, $AD = b - c$, both from any Point $A$ thereof: And producing $AD$ to $F$, ſo that $DF = c$, about the Centre $C$ with the Radius $CF$, deſcribe another Concentrick Circle, cutting the Chords $AD$, $AB$, (produced) in the Points $F$, $E$, $G$, $H$. I ſay $AG$ is the affirmative, and $AH$ the negative Root of $xx + ax - bc = o$; and contrariwiſe $AG$ the negative, and $AH$ the affirmative Root of the Equation $xx - ax - bc = o$.

For $AF$ or $AD + DF = b$, and $DF$ or $AE = c$, and making $AG$ or $BH = x$, we ſhall have $AH = a + x$. And by the Property of the Circle $EGFH$, the Rectangle $EA \times AF\,(bc) = GA \times AH\,(xx + ax)$. Now if $AH$ be made $= -x$, we ſhall have $AG$ or $BH$ or $AH - AB = -x - a$, and conſequently $GA \times AH = xx + ax$ as before. Therefore whether $AG$ be $= x$, or $AH = -x$, we ſhall always have $xx + ax - bc = o$. After the ſame manner we prove that $AG$ is the negative, and $AH$ the affirmative Root of $xx - ax - bc = o$.

FIG. 216. 2. Let $xx \mp ax + bc$ be $= o$. Deſcribe any Circle $ABD$, whoſe Diameter is not leſs than the given Quantities $a$ and $b + c$, and within the ſame inſcribe two Chords $AB = a$, $AD = b + c$, both from any Point $A$ thereof: Then in $AD$ aſſume $DF = c$, and about the Centre $C$ with the Radius $CF$ deſcribe another Concentrick Circle, cutting the Chords $AD$, $AB$, in the Points $F$, $E$, $G$, $H$. I ſay

ſay $AG$ and $AH$, are the two affirmative Roots of $xx - ax + bc = o$, and the two negative Roots of $xx + ax + bc = o$. This is demonſtrated after the ſame manner as in the firſt Caſe.

If the Circle, whoſe Radius is $CF$, does neither cut nor touch the Line $AB$; then the two Roots of the Equation ſhall be imaginary.

### ADVERTISEMENT.

All the Contrivance that I make uſe of, in the Conſtruction or Inveſtigation of the Roots of an Equation, conſiſting of but one unknown Quantity, lies in bringing a new unknown Quantity into that Equation; that thereby ſeveral Equations may be had, each containing the two unknown Quantities; and moreover may be ſuch, that any two of them do contain together all the known Quantities of the propos'd Equation; becauſe otherwiſe, when the new unknown Quantity is ſtruck out, the propos'd Equation will not again ariſe. Then among thoſe Equations I pick out two of the moſt ſimple, and conſtruct their Loci ſeparately, and the Interſection of thoſe Loci will give the Roots ſought. Now that unknown Quantity muſt be ſo taken, that the Loci of the Equations ariſing from the propos'd Equation, be the moſt ſimple poſſible. For Example: If the Equation be one of the fourth Degree, the Loci of the two Equations muſt not exceed the ſecond Degree: Among which Loci there muſt be always a Circle, as being moſt ſimple, and alſo a Parabola, Equilateral Hyperbola, *&c.* All this will fully appear in the following Lemmata and Propoſitions.

### *A* FUNDAMENTAL LEMMA *for the Conſtruction of Equations of the third and fourth Degree, by means of a Circle and a given Parabola.*

383. LET there be a propos'd Equation $x^4 + 2bx^3 + acxx - aadx - a^3f = o$, wherein $x$ is unknown, and $a, b, c, d, f$, are known; and ſuppoſe another unknown Quantity $y$ to be ſuch, that the Rectangle under the ſame, and the known one $a$, be equal to the Rectangle under $x + b$ and $x$; and from hence we have the following Equations.

1. $ay = xx + bx$, both Sides of which being ſquared, and there ariſes $x^4 + 2bx^3 + bbxx = aayy$, and ſubſtituting $aayy - bbxx$ in the Equation propos'd for its Value $x^4 + 2bx^3$, and the ſame ſhall be chang'd into this Equation.

2. $yy - \frac{bb}{aa}xx + \frac{c}{a}xx - dx - af = o$, wherein ſubſtituting for $xx$ its Value $ay - bx$ found by means of the firſt Equation, 1. in $-\frac{bb}{aa}xx$. 2. in $\frac{c}{a}xx$. 3. in $-\frac{bb}{aa}xx + \frac{c}{a}xx$, and the following three Equations ſhall be had.

3. $yy - \frac{bb}{a}y + \frac{b^3x}{aa} + \frac{c}{a}xx - dx - af = o.$

4. $yy - \frac{bb}{aa}xx + cy - \frac{bc}{a}x - dx - af = o.$

5. $yy + cy - \frac{bb}{a}y - \frac{bc}{a}x + \frac{b^3}{aa}x - dx - af = o.$ If the firſt Equation $xx + bx - ay = o$, be taken from this fifth Equation, and afterwards added to it, then we ſhall have theſe two others, *viz.*

6. $yy + cy - \frac{bb}{a}y + ay - xx - bx - \frac{bc}{a}x + \frac{b^3}{aa}x - dx - af = o.$

7. $yy + cy - \frac{bb}{a}y - ay + xx + bx - \frac{bc}{a}x + \frac{b^3}{aa}x - dx - af = o.$

Now if for the unknown Quantities $x$ and $y$, there be taken two right Lines $AP$, $PM$, making any Angle $APM$ with each other; then it is evident, * that the Locus of the firſt Equation is a Parabola; that of the ſecond may be a Parabola, Ellipſis, or Hyperbola, according as $bb$ is equal, leſs, or greater than $ac$; that of the third, an Ellipſis, which does become * a Circle when $c$ is $= a$, and the Angle $APM$ is a right Angle; that of the fourth, an Hyperbola, which does become equilateral when * $b$ is $= a$; that of the ſifth is alſo a Parabola; that of the ſixth an Equilateral Hyperbola; and laſtly, the Locus of the ſeventh is a Circle, when the Angle $APM$ is a right Angle.

*Art. 310.

*Art. 328, and 329.

*Art. 335, and 336.

## SCHOLIUM I.

384. IF $2bx^3$ had been negative in the propos'd Equation, then the Signs of all the Terms wherein $b$ is found of odd Dimenſions in all the Equations muſt have been changed; and if the ſecond Term was wanting, then all the Terms affected with $b$ muſt have been ſtruck out. The ſame muſt be underſtood with regard to the other Terms of the propoſed Equation in reſpect of the Letters $c$, $d$, $f$, contain'd in them. But it muſt be obſerved here, that in all the different Alterations that can happen, the Locus of the firſt Equation ſhall be a Parabola, that of the ſixth an Equilateral Hyperbola; and laſtly, that of the ſeventh a Circle, when the Angle $APM$ is a right Angle.

## SCHOLIUM II.

385. THE Reaſon why we have choſen the firſt Equation $xx + bx = ay$, rather than $xx - bx = ay$, or ſimply $xx = ay$, is, becauſe when both Sides thereof are ſquared, the two firſt Terms of one Side are the ſame as the two firſt Terms of the propoſed Equation

$x^4 + 2bx^3$, *&c.* and ſo they may be made to deſtroy one another. And by that means we ſhall get a new Equation, whoſe Locus is no higher than the ſecond Degree, which being combin'd different ways with the firſt, gives other Equations (as appears above) whoſe Loci, not being higher than the ſecond Degree, may be eaſily conſtructed, becauſe the Plane $xy$ is not contain'd in thoſe Equations; among which the Locus of the laſt is always a Circle, ſuppoſing the unknown Quantities $x$ and $y$ to make a right Angle with one another.

## PROPOSITION IV.

### Problem.

386. *TO find the Roots of the propoſed Equation* $x^4 + 2bx^3 + acxx - aadx - a^3f = 0$, *by means of a Parabola and Circle.* FIG. 217.

Aſſume two right Lines $AP$, $PM$, making a right Angle $APM$ with one another, for the unknown and indeterminate Quantities $x$ and $y$; and then conſtruct * the Parabola which is the Locus of the firſt Equation of the Lemma, and afterwards the Circle which is the Locus of the ſeventh; and by means of the Interſection of theſe two Loci, the different Values of the unknown Quantity $x$ which ſhall be the Roots of the propoſed Equation, may be found. This may be done after the following Manner. * *Art.* 310.

In the Line $AP$ produced on the other Side of $A$, aſſume $AD = \frac{1}{2}b$, and through the Point $D$ draw a Parallel to $PM$, in which Parallel take $DC = \frac{bb}{4a}$ on the contrary Side of $AP$ with regard to $PM$, and with $CD$ as an Axis, (the Point $C$ its Origin,) and a Line equal to $a$ for its Parameter, deſcribe a Parabola $MCM$. This being done, through the fixed Point $A$ draw $AQ$ parallel to $PM$, and in the ſame aſſume $AB = \frac{1}{2}a + \frac{bb}{2a} - \frac{1}{2}c = \overline{+}g$ for brevities ſake, and parallel to $AP$ draw the right Line $BE = \frac{1}{2}d \pm \frac{bg}{a}$, *viz.* $-\frac{bg}{a}$ when $AB$ is $= +g$, that is, when the Value of $AB$ is affirmative, and $+\frac{bg}{a}$ when $AB = -g$; obſerving to take or draw both the Lines $AB$, $BE$, on the ſame Side $AP$ as $PM$ is, when their Values are poſitive, and on the contrary Side when the ſame are negative. Laſtly, calling $EA$, $m$; about the Centre $E$, with the Radius $EM = \sqrt{mm + af}$ deſcribe a Circle: Then if Perpendiculars $MP$ be drawn from the Points $M$ wherein the Circle cuts the Parabola, to the Line $AP$, the parts ($AP$) of that Line ſhall denote the Roots of the Equation, the affirmative Roots being on the ſame Side $A$ as $PM$ was ſuppoſed to be in the Conſtructing the Parabola, and the negative ones, on the contrary Side.

For producing $MQ$ (parallel to $AP$) until it meets the Axis $CG$ in the Point $L$, we have $ML$ or $AP + AD = x + \frac{1}{2}b$, $CL$ or $MP + DC = y + \frac{bb}{4a}$; and by the Property of the Parabola $\overline{ML}^2 = CL \times a$, that is, $xx + bx + \frac{1}{4}bb = \frac{1}{4}bb + ay$, or $xx + bx = ay$, which is the firſt Equation of the Lemma. Now if $EB$ be produced until it meets $PM$ in $R$, and the Radius $EM$ be drawn, then becauſe $ERM$ is a right-angled Triangle, the Square $\overline{EM}^2$ ſhall be $= \overline{ER}^2 + \overline{RM}^2 = \overline{EB}^2 + 2EB \times BR + \overline{BR}^2 + \overline{PM}^2 - 2AB \times PM + \overline{AB}^2 = \overline{EB}^2 + \overline{BA}^2 + af$ by Conſtruction, and ſtriking out the Squares $\overline{EB}^2$, $\overline{BA}^2$ from both Sides, ſubſtituting $a + \frac{bb}{a} - c$, for $2AB$, $\frac{2bg}{a} - d$, or $b + \frac{b^3}{aa} - \frac{bc}{a} - d$, for $2BE$, and $x, y$, for $BR$, or $AP$ and $PM$, and then we ſhall get the ſeventh Equation $yy + cy - ay - \frac{bb}{a}y + xx + bx + \frac{b^3}{aa}x - \frac{bc}{a}x - dx = af$; wherein if $\frac{xx+bx}{a}$ be ſubſtituted for its Value $y$, and the Square thereof for $yy$, then we ſhall get again the propoſed Equation $x^4 + 2bx^3 + acxx - aadx - a^3f = 0$. From whence it appears, that the Line $AP$ expreſſes an affirmative Root of this Equation.

If $-x$ and $-y$ be taken for $AP$ and $PM$, when theſe Lines do fall the contrary way to that they are ſuppos'd to fall in the Conſtruction; then the firſt Equation will be always found by the Property of the Parabola, and the ſeventh by the Property of the Circle. Therefore, *&c.*

## COROLLARY I.

387. HENCE the foregoing Conſtruction may be made general for all Equations of the third and fourth Degree; and a Parabola having the Parameter of its Axis equal to the given Line $a$, ſhall be always uſed in that Conſtruction. If you, (1.) multiply an Equation
*Art. 376. of the third Degree by the Root $x$, and take a Line * $2b$ equal to all
*Art. 377. the Lines that do multiply $x^3$, a Plane * $ac$ equal to all thoſe that multiply $xx$, a Solid $aad$ equal to the Solids multiplying $x$; and laſtly, a ſurd Solid $a^3f$ equal to the known Terms of the Equation. (2.) And change the Signs wherein $b$ is found of odd Dimenſions, in the Values of the Lines $AD$, $DC$, $AB$, $BE$, $EM$, which determine the Conſtruction of the Parabola and Circle, if $2bx^3$ be negative in the propoſed Equation, becauſe the ſame is poſitive in the Problem; and more-

moreover, deſtroy all the Terms affected with $b$, if the Term $2bx^3$ be wanting, becauſe then $b$ is $=0$; and do the ſame with regard to the Terms wherein $c$, $d$, $f$, do happen. (3.) And take or draw the Lines when affirmative towards, or on the ſame Side as $PM$; and when negative, the contrary way. Then we ſhall have $AD = \pm\frac{1}{2}b$, *viz.* $-\frac{1}{2}b$ when $2bx^3$ is affirmative, and $+\frac{1}{2}b$ when the ſame is negative; $AB = \frac{1}{2}a + \frac{bb}{2a} \pm \frac{1}{2}c = \pm g$, *viz.* $-\frac{1}{2}c$ when $acxx$ is affirmative, and $+\frac{1}{2}c$ when the ſame is negative; $BE = \pm\frac{bg}{a} \mp \frac{1}{2}d$, *viz.* $-\frac{bg}{a}$ when $AB$ is $= +g$, and when $2b^3$ is affirmative, or elſe when $AB$ is $= -g$, and $2bx^3$ is negative; and contrariwiſe $+\frac{bg}{a}$ when $AB = +g$, and $2bx^3$ is negative, or elſe when $AB = -g$, and $2bx^3$ is affirmative (that is, $-\frac{bg}{a}$ when the Values of $AB$, and $AD$, are one poſitive, and the other negative, and $+\frac{bg}{a}$ when their Values are either both poſitive, or both negative) as likewiſe $+\frac{1}{2}d$ when $aadx$ is negative, and $-\frac{1}{2}d$ when the ſame is poſitive; and laſtly, $EM = \sqrt{mm \mp af}$, *viz.* $+af$ when $a^3f$ is negative, and $-af$ when the ſame is poſitive. From hence ariſes the following Conſtruction, which is general for all Caſes.

A Parabola $MCM$ whoſe Axis is the Line $CG$ having a Line equal to $a$ for its Parameter, being given, and the propoſed Equation being reduced to this Form $x^4 \mp 2bx^3 \mp acxx \mp aadx \mp a^3f = 0$, draw a Line $AB$ parallel to the Axis $CG$ diſtant therefrom by the Quantity $\frac{1}{2}b$, on the right Side of the Axis when $2bx^3$ is affirmative in the propoſed Equation, and on the left Side of the ſame when $2bx^3$ is negative. Through the Point $A$ wherein the Line $AB$ meets the Parabola, draw $AD$ perpendicular to the Axis $CG$, and in the Axis aſſume $DF = \frac{1}{2}a$, $FG = 2CD$ (always from $D$ the contrary way to the Origin $C$) and $GK$ towards $C$, when $acxx$ is poſitive; but the contrary way, when the ſame is negative. Then through the determinate Points $A$, $F$, draw an indefinite right Line $AF$, and thro' the Point $K$ a perpendicular to the Axis meeting $AF$ in $H$; and in this Perpendicular take $HE = \frac{1}{2}d$ on the right Side thereof when $aadx$ is negative, and on the left Side when the ſame is affirmative. This being done about the Centre $E$, and with the Radius $EM = AE$, when the Term $a^3f$ is wanting in the given Equation; that is, when the ſame is but of the third Degree; but when it is of the fourth, call $AE$, $m$, and take the Radius $EM = \sqrt{mm \mp af}$, *viz.* $+af$ when $a^3f$ is negative, FIG. 217.

tive, and $-af$ when the ſame is affirmative. Laſtly, Drawing Perpendiculars $MQ$ from the Points $M$, wherein the Circle meets the given Parabola, to the Line $AB$; and theſe Perpendiculars ſhall be the Roots of the given Equation; thoſe that fall on the right Side of the Line $AB$, being the affirmative Roots; and thoſe falling on the left, the negative Roots.

For producing $HK$ till it meets the Line $AB$ in the Point $B$, then by Conſtruction $AD$ is $= \pm \frac{1}{2}b$, *viz.* $-\frac{1}{2}b$, when $2bx^3$ is affirmative, and $+\frac{1}{2}b$ when the ſame is negative; but by the Property of the Parabola $CD = \frac{bb}{4a}$. Therefore $DG$ or $DF + FG = \frac{1}{2}a + \frac{bb}{2a}$, and $DK$ or $AB = \frac{1}{2}a + \frac{bb}{2a} \pm \frac{1}{2}c = \mp g$, *viz.* $-\frac{1}{2}c$ when $+acxx$ is affirmative, and $+\frac{1}{2}c$ when the ſame is negative; and it muſt be here obſerved, that the Point $B$ falls on the ſame Side as $PM$, when $AB = +g$, that is, when the Value thereof is poſitive, and on the contrary Side when the Value is negative. Now becauſe the Triangles $ADF$, $ABH$, are ſimilar, therefore $DF(\frac{1}{2}a) : DA\ (\pm\frac{1}{2}b) :: AB(\mp g) : BH = \pm\frac{bg}{a}$, *viz.* $+\frac{bg}{a}$, when the Values of $AD$ and $AB$ are both poſitive or both negative, and $-\frac{bg}{a}$ when the Value of one is poſitive, and of the other negative. And therefore $BE = \pm\frac{bg}{a} \pm \frac{1}{2}d$, *viz.* $-\frac{1}{2}d$ when $aadx$ is negative, and $+\frac{1}{2}d$ when $aadx$ is affirmative; but you muſt obſerve that the Point $E$ will fall on the ſame Side as $PM$, when the Value of $BE$ is poſitive, and on the oppoſite Side when the ſame is negative. Hence it appears that the Centre $E$ of the Circle ſhall be always determined, as requſite in all the poſſible Caſes, by this Conſtruction.

If the ſecond Term $2bx^3$ be wanting in the propoſed Equation, then it is manifeſt that the Lines $AB$, $AF$, will fall in the Axis $CG$, in ſuch manner that the Points $A$, $D$, ſhall coincide with $C$ the Origin thereof; ſince $b = o$. And conſequently the Point $G$ will coincide with the Point $F$, and the Points $H$ and $B$ with the Point $K$: And ſo
FIG. 218. the general Conſtruction in this Caſe will be much ſimpler than that above. For here you need but take $CF = \frac{1}{2}a$ in the Axis of the Parabola, always within the ſame, and $FK = \frac{1}{2}c$ from $F$ towards the Origin $C$ when $acxx$ is affirmative, and the contrary way when the ſame is negative; and then draw $KE = \frac{1}{2}d$ perpendicular to the Axis on the left Side thereof when $aadx$ is affirmative, and on the right when the ſame is negative; and proceed afterwards as in the general Conſtruction, obſerving that $EC$ is here $= m$.

In

In like manner if the Term $a\,c\,x\,x$ be wanting, then the Point $K$ ſhall fall on the Point $G$; and if the Term $a\,a\,d\,x$ be wanting, the Centre $E$ of the Circle ſhall fall in $H$. FIG. 217.

## COROLLARY II.

388. THERE may be had yet a more ſimple Conſtruction for Equations of the third Degree that have their ſecond Terms, in multiplying them by the unknown Quantity plus or minus, the known Quantity of the ſecond Term, *viz.* plus that Quantity when the ſecond Term is affected with the Sign $-$; and minus that Quantity when the ſame is affected with the Sign $+$; for in doing thus, we get an Equation of the fourth Degree wanting the ſecond Term. For Example, to find the Roots of this Equation of the third Degree, $x^3 - b\,x\,x + a\,p\,x + a\,a\,q = o$: Multiply it by $x + b$, and then you will have the following Equation of the fourth Degree $x^4 + a\,p\,x\,x + a\,a\,q\,x + a\,a\,b\,q = o$, wanting $-b\,b\,x\,x + a\,b\,p\,x$ the ſecond Term; now uſing the Conſtruction already laid down for theſe Equations that want the ſecond Term, and then we ſhall have $C\,K\,(\frac{1}{2}\,a \pm \frac{1}{2}\,c) = \frac{1}{2}\,a + \frac{bb}{2a} - \frac{1}{2}\,p$, $K\,E\,(\frac{1}{2}\,d) = \frac{1}{2}\,q + \frac{b\,p}{2a}$, and the Radius of the Circle $E\,M = \sqrt{m\,m - b\,q}$: From whence ariſes the following Conſtruction.

Draw a Parallel to the Axis $C\,D$ diſtant therefrom to the left by a Line equal to $b$, and meeting the Parabola in the Point $A$, alſo draw the Line $C\,A$ through $C$ the Origin of the Axis, and upon $O$ the middle of $C\,A$ raiſe the indefinite Perpendicular $O\,G$ meeting the Axis in the Point $G$. This being done in the Axis aſſume $G\,K = \frac{1}{2}\,p$ from $G$ towards $C$, and through the Point $K$ draw a Perpendicular to the Axis meeting the Line $C\,G$ in the Point $H$, in which perpendicular produced towards $H$, take $H\,E = \frac{1}{2}\,q$, and about the Centre $E$, with the Radius $E\,A$ deſcribe a Circle. I ſay this Circle ſhall cut the Parabola in Points ($M$,) from whence letting fall Perpendiculars $M\,Q$ to the Axis; and thoſe Perpendiculars on the right of the Axis ſhall be the affirmative, and on the left the negative Roots of the propoſed Equation $x^3 - b\,x\,x + a\,p\,x + a\,a\,q = o$. FIG. 218.

For if the right Lines $A\,D$, $O\,L$, be drawn perpendicular to the Axis; then by Conſtruction we have $A\,D = b$, and by the Property of the Parabola $C\,D = \frac{bb}{a}$. Therefore ſince $C\,A$ is biſected in $O$, the ſimilar Triangles $C\,A\,D$, $C\,O\,L$, ſhall give $O\,L = \frac{1}{2}\,b$, $C\,L = \frac{bb}{2a}$; and becauſe the right angl'd Triangles $C\,L\,O$, $O\,L\,G$, are ſimilar, therefore

therefore $CL\left(\frac{bb}{2a}\right) : LO\left(\frac{1}{2}b\right) :: LO\left(\frac{1}{2}b\right) : LG = \frac{1}{2}a$, and consequently $CK$ or $CL + LG - GK = \frac{1}{2}a + \frac{bb}{2a} - \frac{1}{2}p$. Moreover, because the Triangles $GLO$, $GKH$, are ſimilar, therefore $KH$ ſhall be $= \frac{bp}{2a}$, and $KH + HE$ or $KE = \frac{1}{2}q + \frac{bp}{2a}$, which tends to the left of the Axis, as is preſcribed in the Conſtruction, when $aadx$ is affirmative. Therefore the Point $E$ is the Centre of the Circle, whoſe Interſections with the given Parabola ſhall determine all the Roots of the Equation of the fourth Degree $x^4 + apxx$, &c. And becauſe the Roots of this Equation are the Roots of the propoſed Equation $x^3 - bxx + apx + aaq = o$, together with a negative Root $AD\,(b)$; whence the Circle muſt paſs through the Point $A$. Therefore, &c.

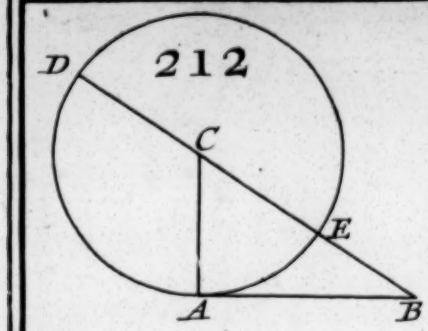
We can prove likewiſe by Calculation, that $EA$ is the Radius of the Circle ſought. For if $EB$ be drawn parallel to the Axis, then becauſe the Triangles $EAB$, $EKC$, are right-angled, the Squares of the Hypothenuſes $\overline{EA}^2 = \overline{EB}^2 + \overline{BA}^2$, and $\overline{EC}^2 = \overline{CK}^2 + \overline{KE}^2$, and conſequently it muſt be proved, that $\overline{EB}^2 + \overline{BA}^2 = \overline{EK}^2 + \overline{KC}^2 - bq$, becauſe we muſt take $EM = \sqrt{mm - bq}$. But ſubſtituting on both Sides, inſtead of thoſe Lines, their Analytick Values, and theſe ſame Quantities will ariſe, as they muſt if the Radius ſought $EM$ be $= EA$.

Now to render the aforeſaid Conſtruction general, you muſt, (1) Draw the Parallel to the Axis, being diſtant therefrom by a Line equal to $b$, on the left Side the ſame, when $bxx$ is negative in the propoſed Equation; and on the right Side, when the ſame is affirmative. (2.) And you muſt take $GK = \frac{1}{2}p$ in the Axis from $G$ towards its Origin $C$, when $apx$ is affirmative; and the contrary way, when the ſame is negative. (3.) And $HE = \frac{1}{2}q$ muſt be taken to the left, when $aa$ is affirmative; and to the right, when the ſame is negative.

### SCHOLIUM.

389. HENCE it is neceſſary to obſerve, 1. If the Circle does cut the given Parabola in two Points only, then the propoſed Equation ſhall have but two real Roots, when it is one of the fourth Degree; and but one real Root, when it is an Equation of the third Degree, and the two others imaginary; as in *Fig.* 219. where the Circle cuts the Parabola in two Points $A$, $M$, only; the Equation $x^4 + apxx - bbxx$, &c. has but two real Roots $AD$, $MQ$, which are both negative ones, becauſe they fall on the left Side the Axis. 2. If the Circle does neither cut nor touch the Parabola (which cannot happen

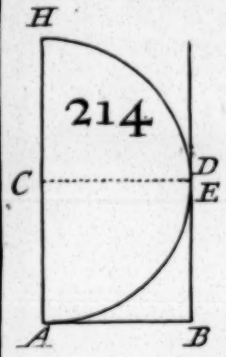
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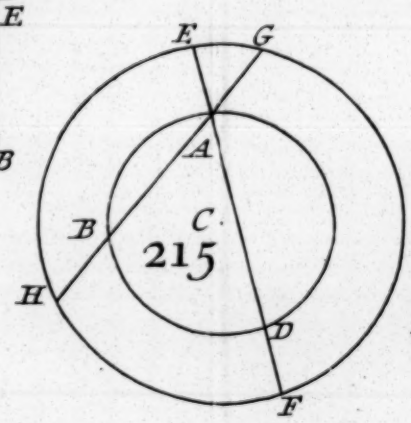

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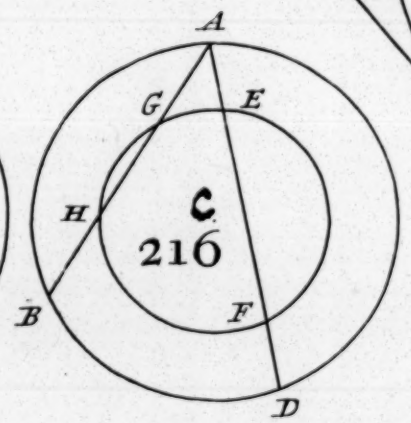

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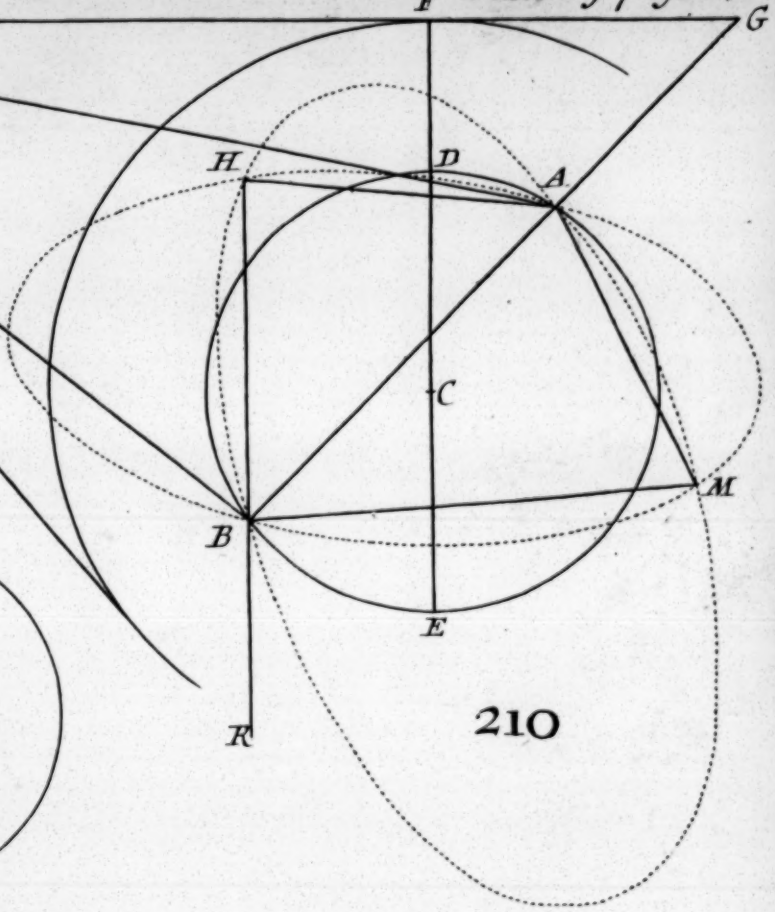

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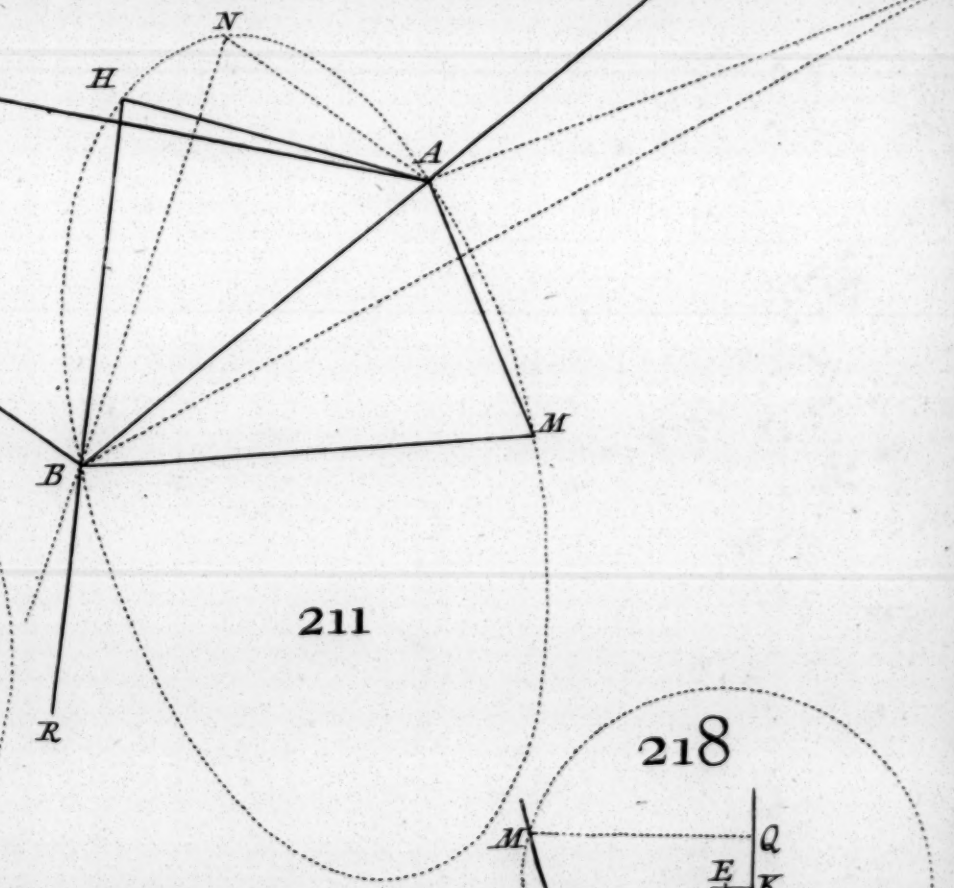

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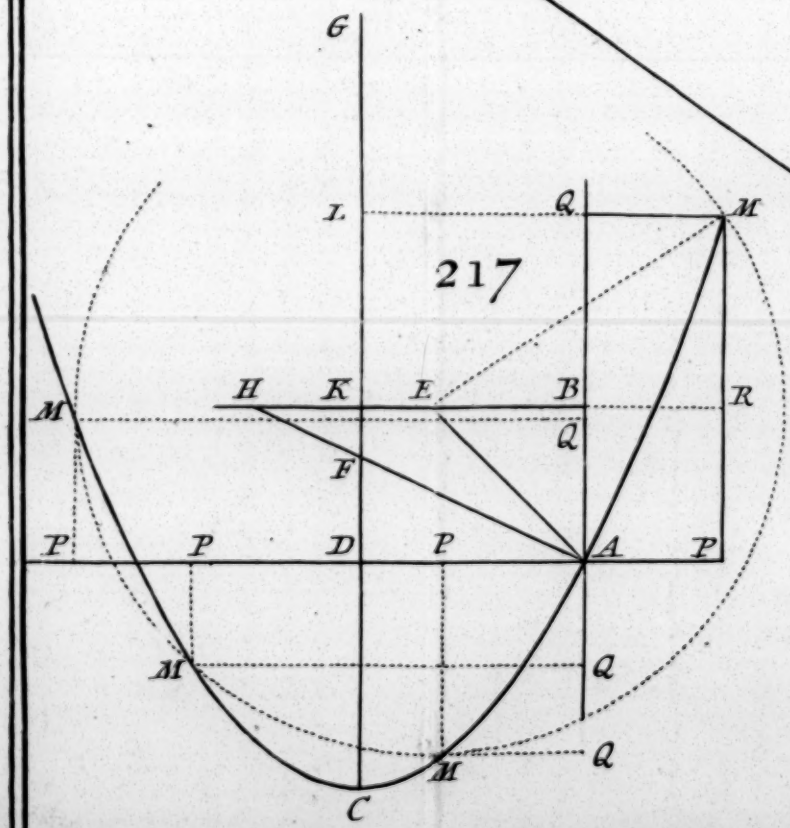

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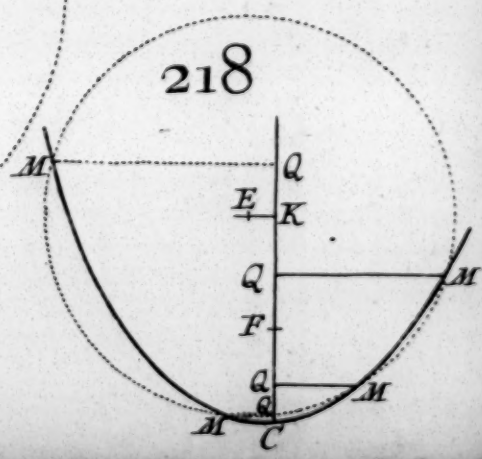

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218



happen when the Equation is one of the third Degree, as appears by the foregoing Conſtructions) then the four Roots ſhall be imaginary.

3. If the Circle touches the Parabola, then the propoſed Equation will have two Roots, each being equal to the Perpendicular drawn from the Point of Contact; as appears from hence, *viz.* that a Circle touching a Parabola, may be conſider'd as cutting the ſame in two Points infinitely near to each other, which do coincide in the Point of Contact: But then the propoſed Equation may be brought down to one of the ſecond Degree by common Algebra, and ſo there will be no Neceſſity of conſtructing a Parabola for finding the Roots.

## SCHOLIUM II.

390. IF Regard be had to what is demonſtrated in Algebra, *viz.* That in any Equation wanting the ſecond Term, and having all real Roots, the Sum of the affirmative Roots is equal to the Sum of the negative ones; we may get the following Theorem.

If a Circle cuts a Parabola in four Points $M$, and Perpendiculars, as $MQ$, be drawn from them to the Axis $CF$; I ſay, the Sum of the Perpendiculars that fall on the right Side of the Axis, ſhall be equal to the Sum of the Perpendiculars that fall on the left Side. FIG. 218.

For if $CF$ be taken in the Axis of the Parabola from $C$, its Origin within the Parabola, equal to the Parameter, which call $a$; and if $EK$ be drawn from $E$ the Centre of the Circle perpendicular to the Axis, and you make $FK = \frac{1}{2}c$, $KE = \frac{1}{2}d$, $\overline{EC}^2 - \overline{EM}^2 = af$; then it is plain * from the Conſtruction at the End of *Cor.* 1. that the Perpendiculars $MQ$ ſhall be the Roots of the Equation $x^4 - acxx + aadx + a^3 f = o$ wanting the ſecond Term, *viz.* thoſe falling on the right Side the Axis being the affirmative Roots, and thoſe on the left, the negative ones; therefore, *&c.* *Art. 387.

If the Circle paſſes through $C$ the Origin of the Axis, then it is plain, that one of the Perpendiculars $MQ$ will become $= o$, and ſo the Perpendicular on one Side the Axis ſhall be equal to the Sum of the Perpendiculars on the other.

If the Circle touches the Parabola in one Point, and cuts the ſame in two others, then you muſt aſſume twice the Perpendicular drawn from the Point of Contact, becauſe that Circle may be look'd upon * as cutting the Parabola in two Points infinitely near to one another, which do coincide in the Point of Contact. *Art. 389.

## SCHOLIUM III.

391. BEcauſe in Geometry there cannot be ſuppos'd Products of more than three Dimenſions, ſince a Solid, which is the moſt compounded, has but three; therefore you may divide all the Terms

of a propoſed Equation exceeding the third Degree by any given Line (taken at pleaſure) rais'd to a Power, leſs by Unity than the Power of the higheſt Term thereof; and by this Means the Equation will not be at all embaraſs'd, and each Term thereof ſhall expreſs right Lines only. For Example: Let there be an Equation of the fourth Degree, *viz.* $x^4 + 2bx^3 + acxx - aadx - a^3f = o$; divide the ſame by $a^3$, and there will ariſe $\frac{x^4}{a^3} + \frac{2bx^3}{a^3} + \frac{cxx}{aa} - \frac{dx}{a} - f = o$, each of whoſe Terms hath but one Dimenſion, and conſequently do expreſs right Lines only. We commonly chuſe that Line which is found repeated moſt times in all the Terms of the propoſed Equation, as the Line $a$ is here; and even ſometimes it is underſtood in eſteeming the ſame as Unity, which will produce no Alteration in the Quantities it multiplies or divides; ſo making $a = 1$, and then we may write $x^4 + 2bx^3 + cxx - dx - f = o$, inſtead of $x^4 + 2bx^3 + acxx - aadx - a^3f = o$, or of $\frac{x^4}{a^3} + \frac{2bx^3}{a^3} + \frac{cxx}{aa} - \frac{dx}{aa} - f = o$. The ſame muſt be underſtood of Equations of the fifth Degree, ſixth Degree, &c.

## SCHOLIUM IV.

392. AFter the Circle, which is the Locus of the laſt Equation of the Lemma, is conſtructed, if a Conick Section, which is the Locus of any other of the Equations of that Lemma, be conſtructed; then the Interſections of thoſe two Loci ſhall determine the Roots of the propoſed Equation; becauſe making the unknown Quantity $y$ vaniſh by means of their Equations, the propoſed Equation will be again gotten.

From whence it is evident, that that Equation may be conſtructed, 1. By means of a Circle and Equilateral Hyperbola, being the Loci of the ſeventh and ſixth Equations of the Lemma. 2. By means of a Circle and an Ellipſis, whoſe Axis (being parallel to $AP$) is to the Parameter as $a$ to $c$, uſing the ſeventh and third Equations. 3. By means of a Circle and Hyperbola, whoſe Axis (being parallel to $AP$) is to the Parameter as $aa$ to $bb$, uſing the ſeventh and fourth Equations. And becauſe the Line $a$, by means of which all the Quantities multiplying $xx$ are reduced to the Expreſſion $ac$, all the Quantities multiplying $x$ to the Expreſſion $aad$, and all the Quantities multiplying the known Terms to the Expreſſion $a^3f$, is a Quantity at pleaſure; therefore, if for $a$ be taken for an infinite Number of different Lines, the propoſed Equation may be conſtructed by means of an infinite Number of Circles, Ellipſes, or Equilateral or not Equilateral Hyperbola's, all different between themſelves.

It appears, (in Article 387.) that if the arbitrary Quantity *a* being the Parameter of the Axis of a given Parabola be taken for Unity, then uſing the firſt and ſeventh Equation, the propoſed Equation may be conſtructed by means of a Circle and a given Parabola: And here I ſhall ſhew how to take the Quantity *a* ſuch, that the Equation may be conſtructed by means of a Circle, and an Ellipſis or an Hyperbola ſimilar to a given Ellipſis or Hyperbola. For the Ratio of the Axes being given, by ſuppoſition, the Ratio of the Axis (parallel to *A P*) to its Parameter ſhall be given alſo. Therefore if that given Ratio be called $\frac{n}{m}$; we ſhall have $\frac{c}{a}=\frac{n}{m}$, (when the Section is an Ellipſis) and ſo $aa=\frac{acm}{n}$; from whence if for the arbitrary Quantity *a* (being Unity) there be taken the Root of a Square *a a* equal * to the known Quantity *a c* which multiplies *x x* in the given Equation, and is multiplied by $\frac{m}{n}$; the Equation may be conſtructed from the ſeventh and third Equations, by means of a Circle and an Ellipſis, whoſe Axis being parallel to *A P*, is to its Parameter as *m* to *n*, becauſe $\frac{n}{m}=\frac{c}{a}$: But when the Section is to be an Hyperbola, we have $\frac{n}{m}=\frac{bb}{aa}$, and therefore $a=b\sqrt{\frac{m}{n}}$; whence if that Value be taken for unity (*a*) and the *E*quation be conſtructed from the ſeventh and fourth Equations, then the Axis of the Hyperbola (being parallel to *A P*) which is the Locus of the fourth Equation, ſhall be to the Parameter thereof, as *m* is to *n*, becauſe $\frac{n}{m}=\frac{bb}{aa}$. And this is what was propos'd.

* *Art.* 378.

### SCHOLIUM V.

393. HENCE the arbitrary Line *a*, doing the Office of Unity, is ſufficient for conſtructing the propoſed Equation, by means of a Circle and a given Parabola, or elſe by means of a Circle and an Ellipſis, or Hyperbola ſimilar to a given one. But when it is requir'd to Conſtruct the ſame by means of a Circle, and an Ellipſis, or Hyperbola given, then one arbitrary Line only is not enough; there muſt be others brought into the Equation propoſed, to the End that they may be determin'd afterwards ſo as the given Section ſerves. This we ſhall ſhow how to do in the following Lemma.

*A* FUNDAMENTAL LEMMA *for the Conſtruction of Equations of the Third and Fourth Degree, with a Circle and an Ellipſis, or a given Hyperbola.*

394. LET there be an Equation of the fourth Degree, *viz.* $z^4 + abzz - aacz + a^3d = o$, wherein the Letters, $a$, $b$, $c$, $d$, do denote given Lines, and the Letter $z$, expreſſes the unknown Roots of the Equation. Aſſume another unknown Quantity $x = \frac{fz}{a}$ (the Letter $f$ denoting a Line taken at pleaſure); and in the Room of $z$, $zz$, and $z^4$, ſubſtituting their Values $\frac{ax}{f}$, $\frac{aaxx}{ff}$, and $\frac{a^4x^4}{f^4}$ in the foreſaid Equation, and the ſame will be changed into this, *viz.* $x^4 + \frac{bff}{a} x x - \frac{cf^3}{a} x + \frac{df^4}{a} = o$; alſo aſſume a third unknown Quantity $y$ ſuch, that being multiply'd by $f$, the Product $fy$ may be equal to $xx$ the Square of the ſecond unknown Quantity; and then we ſhall have the following Equations.

1. $xx - fy = o$, and ſubſtituting $fy$, and $ffyy$ for $xx$ and $x^4$, in the Equation $x^4 + \frac{bff}{a} xx$, *&c.* and we ſhall have a ſecond Equation.

2. $yy + \frac{bf}{a} y - \frac{cf}{a} x + \frac{dff}{a} = o$, which being added to the firſt Equation, and then,

3. $yy + \frac{bf}{a} y - fy + xx - \frac{cf}{a} x + \frac{dff}{a} = o$, and the Locus of this Equation is * a Circle, when the unknown and indeterminate Quantities $x$ and $y$ make right Angles with one another. Again, multiply the firſt Equation by the Fraction $\frac{g}{a}$ ($g$ expreſſing any Line at pleaſure) and then we have $\frac{g}{a} xx - \frac{fg}{a} y = o$; and adding this Equation to the ſecond, and then ſubſtracting the ſame from it, and the two following Equations will be had.

*Art. 324, and 329.

4. $yy + \frac{bf}{a} y - \frac{gf}{a} y + \frac{g}{a} xx - \frac{cf}{a} x + \frac{dff}{a} = o$, whoſe Locus is * an Ellipſis.

*Art. 324.

5. $yy + \frac{bf}{a} y + \frac{gf}{a} y - \frac{g}{a} xx - \frac{cf}{a} x + \frac{dff}{a} = o$, whoſe Locus is * an Hyperbola or the oppoſite Sections.

*Art. 332.

SCHO-

SCHOLIUM.

395. IF the Signs of ſome Terms of the propoſed Equation ſhould be different from what they are here; or if ſome Terms be wanting, notwithſtanding this, the Loci of the five Equations ſhall always be Conick Sections of the ſame Species; that is, the Loci of the firſt and ſecond Equations ſhall be always Parabola's, the Locus of the third, a Circle, *&c.*

## PROPOSITION V.

### Problem.

396. *TO conſtruct the Equation* $z^4 + a\,b\,z\,z - a\,a\,c\,z + a^3\,d = 0$ *of the four Degrees, by means of a Circle given, and an Hyperbola ſimilar to a given Hyperbola; or elſe by means of a given Hyperbola and a Circle.*

Conſtruct * the Loci of the third and fifth Equations, taking the ſame Lines $AP$, $PM$, (making a right Angle $APM$ with each other) for the unknown and variable Quantities $x$ and $y$; and then the Values of the unknown Quantity $z$ may be determin'd by means of the Interſections of theſe Loci, after the following manner. * *Art.* 324, *and* 332. FIG. 220, *and* 221.

Through the Point $A$ the Origin of the $x^s$, draw the Line $AD = \frac{af - bf}{2a}$ parallel to $PM$, and on the ſame Side as $PM$, when $a$ exceeds $b$, and on the oppoſite Side when the ſame is leſs. Alſo draw the indefinite right Line $DG$ parallel to $AP$, and in $DG$ aſſume $DC = \frac{cf}{2a}$ from $D$ towards $PM$, then about the Centre $C$ with the Radius $CF$ or $CG = \frac{f}{2a}\sqrt{cc + aa - 2ab + bb - 4ad}$ deſcribe a Circle. This being done, draw $AH = \frac{bf + gf}{2a}$ parallel to $PM$ but towards contrary Parts, and draw the indefinite Right Line $HK$ parallel to $AP$, in which take $HI = \frac{cf}{2g}$ from $H$ the contrary way to $PM$, and on both Sides the Point $I$ aſſume $IK$, $IL$, each equal to $\frac{f}{2g}\sqrt{cc - bg + 4dg}$ or $\frac{f}{2g}\sqrt{bg - 4dg - cc}$ ($b$ being taken $= \frac{\overline{b+g}^2}{a}$ for brevities ſake). Laſtly, with the Axis $LK$ (which muſt be a firſt Axis when $cc + 4dg$ is greater than $bg$, and a ſecond one when it is leſs) having the ſame proportion to its Parameter $KO$, as $a$ is to $g$, deſcribe an Hyperbola or the oppoſite Sections meeting the Circle in the Points $M$, $M$, from which Points let $MP$, $MP$,

$M P$, be drawn perpendicular to the Line $A P$. I ſay the Parts $A P$, $A P$, of this Line ſhall be the Roots of the Equation $x^4 + \frac{bff}{a} x x - \frac{cf^3}{a} x + \frac{df^4}{a} = o$, the affirmative Roots falling from $A$ towards the Line $P M$ which was drawn in the Conſtruction, and the negative ones the contrary way.

For by the Properties of the Circle and Hyperbola, we ſhall have the third and fifth Equations; and ſubſtracting the third Equation from the fifth, and then $\frac{gf}{a} y + f y - \frac{g}{a} x x - x x = o$, and ſo $y = \frac{x x}{f}$; and ſubſtituting $\frac{x x}{f}$ for $y$, and $\frac{x^4}{ff}$ for $y y$, in either of thoſe two Equations, and then there will ariſe the Equation $x^4$, *&c.* But the Values of $x$ being had, the Values of $z$ will be ſo likewiſe; becauſe $z = \frac{a x}{f}$.

Now to ſatisfy the firſt thing requir'd in the Problem, call the Radius of the given Circle $C F$, $r$; and then $r$ ſhall be $= \frac{f}{2a} \sqrt{cc + aa - 2ab + bb - 4ad}$; whence if you take $f = \frac{2ar}{\sqrt{cc + aa - 2ab + bb - 4ad}}$ the Radius $CF$ or $CG$ of the Circle being the Locus of the third Equation, ſhall be equal to the given Quantity $r$. And that the Hyperbola be ſimilar to a given one, that is, that its firſt or ſecond Axis $L K$, be to the Parameter $K O$ in the given Ratio of $m$ to $n$; you need only aſſume $g = \frac{a n}{m}$, becauſe $L K : K O :: a : g :: m : n$.

Laſtly, To order it ſo that the Hyperbola be given, or, which is the ſame thing, that the firſt or ſecond Axis $L K$, and $K O$ the Parameter of that Axis, be equal to given Lines; call the firſt Axis $L K$, $2t$; and its Parameter $K O$, $p$; then $K O\ (p) = \frac{2gt}{a}$, and $L K\ (2t) = \frac{f}{g} \sqrt{cc + 4gd - bg}$ (remembring that $b = \frac{\overline{b+g}^2}{a}$); whence $g = \frac{ap}{2t}$, and $f = \frac{2gt}{\sqrt{cc + 4dg - bg}}$: And ſo it appears, if $cc + 4dg$ does exceed $bg$, and thoſe Values be taken for $g$ and $f$, the given Lines $2t$ and $p$ ſhall be found for the firſt Axis $L K$, and its Parameter $K O$, in the Conſtruction of the fifth Equation. But if $cc + 4dg$ be leſs than $bg$, then you muſt call the ſecond Axis $L K$, $2t$; and its Parameter $KO$, $p$; from

from whence there ariſes (as above) $g = \frac{ap}{2t}$, and $f = \frac{2gt}{\sqrt{bg - cc - 4dg}}$.

And if $bg$, in this laſt Suppoſition, wherein $2t$ does repreſent the ſecond Axis, exceeds $cc + 4dg$; then if thoſe Values be taken for $g$ and $f$, in the Conſtruction of the fifth Equation, the given Lines $2t$, and $p$, ſhall be found for the ſecond Axis $LK$, and its Parameter $KO$.

Here it muſt be obſerv'd, that the Value of $f$ may be imaginary in both theſe Suppoſitions; and ſo it appears, that the Conſtruction in this Caſe is impoſſible, at leaſt by this Method. And ſince all thoſe who have uſed the ſame after *Sluſius*, who invented it, have affirm'd, that it is general; I ſhall here orderly examine all the Caſes that can happen, and ſhew, that even in this Example there may be an infinite Number of Caſes wherein that Method will not ſucceed.

If the given Hyperbola's be two Conjugate ones, the Conſtruction will be always poſſible; for if you call the firſt Axis of one of the Hyperbola's $LK$, $2t$; and its Parameter $KO$, $p$; the Value of $f = \frac{2gt}{\sqrt{cc + 4dc - bg}}$ will be imaginary, that is, $bg$ does exceed $cc + 4dg$; then you need but uſe the Hyperbola, that is a Conjugate to this Hyperbola, and its ſecond Axis inſtead of it, and the firſt Axis thereof; becauſe the ſecond Axis of the latter Hyperbola being the ſame as the firſt Axis of the other, the Value of $f$ will not then include a Contradiction. *Note*, if $cc + 4dg$ be $= bg$, then the Equation of the fourth Degree may be brought down to one of the ſecond.

## SCHOLIUM.

397. 1. IF the given Hyperbola be an Equilateral one, we have $g = a$, and then the firſt Axis of that Hyperbola muſt be uſed in the Conſtruction of the Problem, when $cc + 4dg$ does exceed $bg$, that is, ſubſtituting $\frac{\overline{b+g}^2}{a}$ for its Value $b$, and $a$ for $g$, when $cc + 4ad$ does exceed $\overline{b+a}^2$; and the ſecond Axis, when the ſame is leſs. And the Conſtruction ſhall be always poſſible.

2. If the firſt Axis of the given Hyperbola exceeds its Parameter: The firſt Axis thereof muſt be uſed in the Conſtruction of the Problem, when $cc + 4ad$ is greater than $\overline{b+a}^2$; for it follows from thence, that $cc + 4dg$ does exceed $bg$, that is, (multiplying by $\frac{a}{g}$, and ſubſtituting $\frac{\overline{b+g}^2}{a}$ for its Value $b$) $\frac{acc}{g} + 4ad$ is greater than $\overline{b+g}^2$, ſince

$g\left(\frac{ap}{2t}\right)$ being leſs than $a$ in this Suppoſition, the Quantity $\frac{acc}{g}+4ad$ ſhall be greater than $cc+4ad$, and $\overline{b+g}^2$ ſhall be leſs than $\overline{b+a}^2$ And contrariwiſe, when $cc+4ad$ is leſs than $\overline{b+a}^2$; then the ſecond Axis muſt be uſed; for it follows from thence, that $cc+4dg$ is leſs than $bg$, or $\frac{acc}{g}+4ad$ leſs than $\overline{b+g}^2$, ſince $2t$ denoting here the ſecond Axis, which is leſs than the Parameter $p$, the Quantity $\frac{ap}{2t}$ is greater than $a$. Whence it appears, that the Conſtruction is always poſſible, not only when the Hyperbola given is an Equilateral one, but alſo when the firſt Axis is greater than its Parameter.

3. If the firſt Axis be leſs than its Parameter. Then there is a Neceſſity of uſing the firſt Axis, when $cc+4ad$ does exceed $\overline{b+a}^2$; for if the ſecond Axis be uſed, then $cc+4dg$ muſt be leſs than $bg$, or $\frac{acc}{g}+4ad$ leſs than $\overline{b+g}^2$; which cannot be, becauſe $2t$, which then would expreſs the ſecond Axis being greater than $p$, the Quantity $g\left(\frac{ap}{2t}\right)$ would be leſs than $a$. But in uſing the firſt Axis, it may happen that $\frac{acc}{g}+4ad$ be leſs than $\overline{b+g}^2$, becauſe $g\left(\frac{ap}{2t}\right)$ is greater than $a$; and then it is evident, that the Conſtruction of the Problem will become impoſſible, becauſe there is a Contradiction imply'd in the Value of $f\left(\frac{2gt}{\sqrt{cc+4dg-bg}}\right)$. In like manner, when $cc+4ad$ is leſs than $\overline{b+a}^2$, there is a Neceſſity of uſing the ſecond Axis; and becauſe then the Value of $g\left(\frac{ap}{2t}\right)$ is leſs than $a$, it may ſo fall out, that $\frac{acc}{g}+4ad$ be greater than $\overline{b+g}^2$, and ſo the Value of $f=\frac{2gt}{\sqrt{bg-cc-4dg}}$ may be imaginary.

Therefore it is evident, that there may a Multitude of Caſes happen, wherein the Conſtruction of the Equation in the Problem is impoſſible; and that is, when the firſt Axis of the given Hyperbola is leſs than its Parameter, for otherwiſe the ſame ſhall always ſucceed.

### COROLLARY I.

399. IF the fourth Equation in the laſt Problem ſhould have been taken inſtead of the fifth, and the Locus of that Equation, which is an Ellipſis, been conſtructed, inſtead of the Hyperbola being the Locus of the fifth Equation; then it is plain, that the propoſed Equation $z^4$, *&c.* might have been conſtructed by means of a given Circle, and an Ellipſis ſimilar to one given; or elſe by means of a given Ellipſis and a Circle.

### COROLLARY II.

399. THE foregoing Conſtruction may be made general for all Equations of the third and fourth Degree, after the following manner: 1. Get the ſecond Term out of the given Equation, when the ſame has one; afterwards multiply the ſame by its Root $z$, if it be but of the third Degree; take a Plane $ab$, equal to all the Planes multiplying $zz$, a Solid $aac$ equal to all the Solids which multiply $z$; and finally, a ſurd Solid $a^3 d$ equal to all the given ſurd Solids. 2. Strike out the Terms wherein $b$ is found in the Values of $AD$, $DC$, $CF$, $AH$, $IH$, $LK$, when $zz$ is not in the given Equation, as alſo the Terms wherein $c$ or $d$ happen, when the fourth or fifth Term is wanting: And change the Signs of all the Terms wherein $b$ is found of an odd Dimenſion, if the third Term of the Equation given has a Sign different from the third Term of the aforeſaid Equation; alſo change the Signs wherein $c$ or $d$ are found of an odd Dimenſion, when the fourth or fifth Terms have Signs different from the fourth or fifth Terms of the precedent Equation. 3. Take thoſe Lines towards $PM$ when their Values are poſitive, and the contrary way when negative.

### SCHOLIUM.

400. THE Conſtruction aforegoing may always be render'd more ſimple in particular Equations propoſed to be conſtructed, if it be ſo ordered that $a$ be equal to $b$; for then the given Equation need only be reduced to this Form, *viz.* $z^4 \mp aazz \mp aacz \mp a^3 d = o$, inſtead of $z^4 \mp abzz \mp aacz \mp a^3 d = o$.

## PROPOSITION VI.

### Problem.

401. *TO find the Roots of the following Equation,* $z^4 - bz^3 - aczz + aadz + aahh = o$ *by means of a given Hyperbola between its Aſymptotes, and a Circle.*

FIG. 222. Make $z = \frac{ax}{f}$, and then the given Equation may be transform'd into this $x^4 - \frac{bf}{a} x^3 - \frac{cff}{a} xx + \frac{df^3}{a} x + \frac{bbf^4}{aa} = o$.

From any Point $M$ of the given Hyperbola, whoſe Centre is the Point $A$, draw $MP$ parallel to $A\mathcal{Q}$ one of the Aſymptotes, meeting the other Aſymptote in the Point $P$; then call $AP$, $x$; $PM$, $y$; and ſuppoſing $mm$ to be the Power of the Hyperbola, we ſhall have $xy = mm$.

Now if you take $f = \sqrt{\frac{a}{b}}$, then $bff = amm$, and $\frac{bbf^4}{aa} = m^4 = xxyy$; and ſubſtituting $xxyy$ for $\frac{bbf^4}{aa}$, which is the laſt Term of the given Equation, and dividing by $xx$, then there will ariſe $xx - \frac{bf}{a}x - \frac{cff}{a} + \frac{df^3}{ax} + yy = o$, which will be changed (by ſubſtituting in the Term $\frac{df^3}{ax}$ inſtead of $x$ its Value $\frac{bff}{ay}$ found by means of the Equation $xy = mm = \frac{bff}{a}$) into this, *viz.* $xx - \frac{bf}{a}x - \frac{cff}{a} + \frac{df}{b}y + yy = o$,

*Art. 328, 329. the Locus of which is * a Circle, when the Angle $APM$ is a right Angle.

But when the Angle $APM$ is not a right Angle, (or, which is the ſame thing) when the given Hyperbola is not an Equilateral one; then it is evident, that the Locus of the laſt Equation is not a Circle, but an Ellipſis. But to find an Equation, whoſe Locus is a Circle, in the Aſymptote $AP$ aſſume $AB = 2a$; draw $BE$ parallel to the other Aſymptote $A\mathcal{Q}$, and draw $AE$ from the Centre $A$ perpendicular to $BE$: Then calling the given Quantities $BE$, $g$; $AE$, $e$; multiply the Equation $xy - mm = o$, whoſe Locus is the given Hyperbola, by $\frac{g}{a}$, and then there will ariſe $\frac{gxy}{a} - \frac{gmm}{a} = o$. This being done, add this laſt Equation to the preceding one, when the Angle form'd by the Aſymptotes is acute, and ſubſtract it from the ſame when that Angle is obtuſe, as it is ſuppos'd to be in the Figure; then we ſhall have $yy - \frac{g}{a}xy + \frac{df}{b}y + xx - \frac{bf}{a}x - \frac{cff+gmm}{a} = o$, and the Locus of this

*Art. 327, 329. is * a Circle, which may be thus conſtructed.

FIG. 222. In the Aſymptote $A\mathcal{Q}$ aſſume $AD = \frac{df}{2b}$ from $A$ towards $PM$: Draw the Line $DC = \frac{bf}{e} - \frac{dgf}{2eb}$ parallel to $AE$, on the ſame Side

$AP$

$AP$ with regard to $PM$, when the Value of $DC$ is poſitive; and on the contrary Side, when negative: Then about $C$ as a Centre, with a Radius $CM = \sqrt{\overline{AC}^2 + \frac{cff - gmm}{a}}$ deſcribe a Circle. I ſay, this Circle ſhall cut the given Hyperbola, and that oppoſite to it, in Points *(M)* from which Parallels $MP$ being drawn to the Aſymptote $AQ$; the Parts $AQ$ of the other Aſymptote ſhall expreſs the Roots of the Equation $x^4 - \frac{bf}{a}x^3 - \frac{cff}{a}xx + \frac{df^3}{a}x + \frac{bbf^4}{aa} = o$; the affirmative Roots falling from $A$ towards $PM$, and the negative ones the contrary way.

For from the Nature of the Circle, will be had the following Equation $yy - \frac{g}{a}xy + \frac{df}{b}y + xx - \frac{bf}{a}x - \frac{cff+gmm}{a} = o$, which will become (putting $mm$ for $xy$) $xx - \frac{bf}{a}x - \frac{cff}{a} + \frac{af}{b}y + yy = o$, and ſubſtituting $\frac{mm}{x}$, or $\frac{bff}{ax}$ for $y$, and its Square for $yy$, in this laſt Equation, and then there will be had again the propoſed Equation $x^4 - \frac{bf}{a}x^3$, *&c.*

If the Angle form'd by the Aſymptotes be acute, then in the Values of $AD$ and $CM$, the Signs of the Terms where $g$ happens muſt be changed, becauſe $BE$ *(g)* will then be negative. But when the Hyperbola is an Equilateral one, the Terms wherein $g$ is found muſt be ſtruck out, and $2a$ muſt be put for its Value $e$; becauſe $AE$ then falls in $AB$; from whence the Conſtruction is render'd much more ſimple.

Now the Values of $x$ being found, the Values of $z$ will be alſo had, ſince $z = \frac{ax}{f}$. And this is the thing propos'd.

## COROLLARY I.

402. IF the laſt Term of a propoſed Equation of the fourth Degree has the Sign —, it is manifeſt by working as above, that an Equation will be gotten, wherein is the Term $yy$ with the Sign —, and conſequently the Locus of the ſame will not be a Circle, but an * Hyperbola. Whence it appears, that the foregoing Method will not do for Equations of the fourth Degree having the Sign + prefix'd to their laſt Term. * *Art.* 332.

### COROLLARY II.

403. BY the Method aforeſaid may be conſtructed any given Equation of the third Degree, as $x^3 \mp nxx \mp apx \mp aaq = o$, with a given Hyperbola between its Aſymptotes and a Circle. For multiplying the ſame by $x + r$, when $aaq$ is affirmative, and by $x - r$ when $aaq$ is negative, it may be always changed into this Equation of the fourth Degree.

$$x^4 \mp nx^3 \mp apxx \mp aaqx + aaqr = o,$$
$$\mp r \quad \mp nr \quad \mp apr$$

whoſe laſt Term ſhall have always the Sign +, and ſo it will be one of thoſe Equations that may be conſtructed in manner aforeſaid.

But this Conſtruction may be very much ſhorten'd, if you obſerve, (1.) To take, for the arbitrary Quantity a repreſenting Unity, the Line $m$, being the Root of the Power of the given Hyperbola, which is the Locus of $xy = mm = a'a$, becauſe $m$ is $= a$. (2.) To make an Advantage of the indeterminate Quantity $r$ for comparing the laſt Term $aaqr$ with $a^4 = xxyy$; by which we get $r = \frac{aa}{q}$. (3.) That the Circle, whoſe Interſections do determine the Roots of the Equation, ſhall neceſſarily cut the Hyperbola when $aaq$ is negative, and that oppoſite to it when $aaq$ is poſitive, in the Point $K$; from whence drawing $KH$ parallel to the Aſymptote $AQ$, the Part $AH$ of the other Aſymptote muſt be equal to $r$; ſince $x = \mp r$ is one Root of the Equation of the fourth Degree. From hence ariſes the following Conſtruction, which may be eaſily made general for all Equations of the third Degree.

FIG. 223.

Suppoſe the Angle form'd by the Aſymptotes to be an acute Angle, and having taken for the arbitrary Quantity $a$ equal to Unity, the Root of the Power of the given Hyperbola, ſuppoſe the given Equation of the third Degree to be thus expreſs'd, $x^3 - nxx - apx - aaq = o$. In the Aſymptote $AP$ aſſume $AB = 2a$, and draw $BE$ parallel to the other Aſymptote $AQ$; alſo from the Centre $A$ draw $AE$ perpendicular to $BE$; then in $AQ$ take $AL = q$ from $A$ towards $PM$, becauſe $aaq$ is negative, in the given Equation; alſo draw $LK$ parallel to $AP$, meeting the Hyperbola in the Point $K$. This being done, call the given Quantities $BE$, $g$; $AE$, $e$; $LK$, $r$; and in the Aſymptote $AQ$ aſſume $AD = \frac{pr}{2a} - \frac{1}{2}q = \mp d$ for Brevity's Sake, and draw $DC = \frac{an \mp ar \mp dg}{e}$ parallel to $AE$, in obſerving to take or draw thoſe Lines towards the ſame Parts as $PM$, when their

their Values are poſitive, and towards contrary Parts when negative. Laſtly, About the Centre *C*, with the Radius *CK*, deſcribe a Circle, which ſhall cut the oppoſite Hyperbola's in Points (*M*) from which the right Lines *MP* being drawn parallel to the Aſymptote *AQ*; the Parts *AP* of the other Aſymptote ſhall be the Roots of the propoſed Equation $x^3 - nxx - apx - aaq = o$.

For if the right Lines *MP*, *KH*, be produced, till they meet the Line *DC* produced likewiſe (if neceſſary) we ſhall have (by means of the right-angled Triangles *CFK*, *CGM*) theſe two Equations $\overline{GM}^2 + \overline{CG}^2 = \overline{CM}^2$, and $\overline{FK}^2 + \overline{CF}^2 = \overline{CK}^2$; and conſequently $\overline{GM}^2 + \overline{CG}^2 = \overline{FK}^2 + \overline{CF}^2$, becauſe the Lines *CM*, *CK*, are Radii of the ſame Circle. But by Conſtruction (ſuppoſing here for avoiding the Confuſion ariſing from the Signs + and —, that $\frac{pr}{2a} - \frac{1}{2}q = +d$, that is, that this Value is poſitive) $GM$ or $PM + PG = y + \frac{g}{2a}x + d$, $CG$ or $DG - DC = \frac{ex}{2a} - \frac{an - ar - dg}{e}$, $FK$ or $KH + HF = q + \frac{g}{2a}r + d$, $CF$ or $CD - DF = \frac{an + ar + dg}{e} - \frac{er}{2a}$. Now, if the Analytick Values of thoſe Lines be ſubſtituted in the aforeſaid Equation $\overline{GM}^2 + \overline{CG}^2 = \overline{FK}^2 + \overline{CF}^2$, we ſhall have firſt $yy + \frac{g}{a}xy + 2dy + \frac{gg+ee}{4aa}xx - nx - rx = qq + \frac{g}{a}rq + 2dg + \frac{gg+ee}{4aa}rr - nr - rr$, becauſe there is no Neceſſity of writing down the Squares of *d* and of $\frac{an + ar + dg}{e}$, which mutually do deſtroy each other. Now ſince $xy = rq$, from the Nature of the Hyperbola, and $4aa = gg + ee$, becauſe *AEB* is a right-angled Triangle, the laſt Equation may be brought to this, *viz.* $yy + 2dy + xx - nx - rx = qq + 2dq - nr$, wherein ſubſtituting $\frac{pr}{a} - q$ for $2d$, and $\frac{aa}{x}$ and $\frac{a^4}{xx}$ for *y* and *yy*, and there will come out

$$x^4 - nx^3 - apxx - aaqx + a^4 = o.$$
$$- r + nr + apr$$

which being divided by $x - r$, and we ſhall get $x^3 - nxx - apx - aaq = o$, which is the Equation propoſed in the Problem.

The foregoing Conſtruction may be made general, if you, 1. Aſſume *AL* in the Aſymptote *AQ* on the contrary Parts as *PM*, when

*a a q* is affirmative in the given Equation; and change the Signs of the Terms when *q* and *r* happen in the Values of *A D*, *D C*. And 2. Change the Signs of the Term wherein *p* happens in the Value of *AD*, when *a p x* is affirmative in the given Equation, and ſtrike out the ſame when that Term is wanting: And do the ſame with regard to the Term wherein *m* happens in the Value of *D C*, when *nxx* is affirmative. 3. You muſt change the Term wherein *g* happens in the Value of *D C*, when the Angle form'd by the Aſymptotes is obtuſe, and ſtrike it out when this is a right Angle, always obſerving that *e* is = 2*a*.

### SCHOLIUM.

404. ALgebra furniſhes us with eaſy ways of transforming any Equation of the fourth Degree into another of the ſame Degree, wherein the Signs of the Terms may be alternate. And becauſe then the laſt Term will have always the Sign + prefixed to it, therefore it is plain uſing this Preparation, when the laſt Term of an Equation to be conſtructed hath the Sign — prefixed to it, that the Method laid down in the Problem ſhall be general for all Equations of the fourth Degree. But becauſe all the real Roots of an Equation are poſitive when the Signs of the Terms are alternate; therefore there will then be only need of uſing the given Hyperbola, becauſe the oppoſite one by means of which the negative Roots are determin'd, does become uſeleſs.

## PROPOSITION VII.

### Problem.

405. *TO conſtruct the following Equation of the ſixth Degree* viz. $x^6 - bx^5 + acx^4 + aadx^3 + a^3exx - a^4fx + a^5g = o$, or $x^6 - bx^5 + cx^4 + dx^3 + exx - fx + g = o$, (*wherein the Line* a, *which renders the Number of Dimenſions in each Term equal, and is eſteem'd as Unity, is underſtood*) *by means of a Circle and a Locus of the third Degree.*

Take $x^3 - mxx - nx + q = -pxy$ for the Locus of the third Degree, wherein the Quantities *m*, *n*, *p*, *q*, which are look'd upon as being given, muſt be determin'd in ſuch a manner as to ſatisfy the Problem; and this may be done thus: If each Side of this Equation be ſquared, then

$$x^6 - 2mx^5 + mmx^4 + 2mnx^3 + nnxx - 2nqx + qq = pyxxyy$$
$$- 2n \quad + 2q \quad - 2mq,$$

and if the Terms $-2mx^5$, $-2nqx$, $+qq$, be compar'd with the correſpondent Terms $-bx^5$, $-fx$, $+g$, in the propoſed Equation, we ſhall have $m = \frac{1}{2}b$, $q = \sqrt{g}$, $n = \frac{f}{2\sqrt{g}}$; and conſequently $x^6 - 2$

$-2mx^5-2nqx+qq=x^6-bx^5-fx+g$. Now if $ppxxyy-mmx^4$, &c. be ſubſtituted for its Value $x^6-2mx^4-2nqx+qq$, found by means of the preceding Equation, and $-cx^4-dx^3$, &c. for its Value $x^6-bx^5-fx+g$ found by means of the given Equation; and if you divide by $xx$, and bring all the Quantities over to one Side; then the following Equation will be had, *viz.*

$$\begin{array}{llll} ppyy- & mmxx- & 2mnx- & nn=0, \\ +2n & -2q & +2mq & \\ +c & +d & +e & \end{array}$$

whoſe Locus ſhall be a Circle * if the Quantity $c+2n-mm$, which does multiply $xx$, be poſitive, and $pp$ be $=c+\frac{f}{\sqrt{g}}-\frac{1}{4}bb$; for dividing by $pp$, and making $2r=\frac{2mn+2q-d}{pp}$ and $ss=\frac{2mq+e-nn}{pp}$ or $\frac{nn-2mq-e}{pp}$, for brevity's ſake, then we ſhall have $yy+xx-2rx\pm ss=0$: The Square $ss$ being affirmative, when $2mq+e$ exceeds $nn$, and negative when the ſame is leſs,

* *Art.* 328, *and* 329.

Now the Curve which is ths Locus of the Equation $x^3-xx-nx+q=-pxy$ may be drawn after the following Manner. Suppoſe $AP(x)$, $PM(y)$ to be two unknown and indeterminate right Lines forming a right Angle $APM$ with one another; and through the Point $A$, the Origin of the $x^s$, draw the indefinite right Line $AQ$ parallel to $PM$, in which aſſume $AG=\frac{n}{p}$, from $A$ tending the ſame way as $PM$, and $GB=\frac{q}{mp}$, the contrary way, and draw the right Line $BC=m$ perpendicular to $AQ$. This being done, upon ſome ſeparate Plane deſcribe a Parabola $MEM$, with the Line $p=\sqrt{c+2n-mm}$ for the Parameter of the Axis, and place the Plane of this Parabola on the Plane wherein $AP$, and $PM$ are drawn, ſo that the Axis of the Parabola coincides with the Line $AQ$, and the Parabola itſelf tends the contrary way to that which $PM$ does, and in the Axis aſſume $EF=BG=\frac{q}{mp}$, (from $E$ the Origin) within the Parabola. Then take a long Rule $CF$, and place it in the Point $C$, ſo as to be moveable about the ſame, and always paſs through the Point $F$: This being done, move that Rule about the Point $C$, ſo as to ſlide the part $EF$ of the Axis of the Parabola along the Line $AQ$. I ſay the two continual Interſections $M$, $M$, of that Rule, and the Parabola $MEM$ will by this Motion deſcribe two Curves that ſhall be the Locus ſought.

FIG. 224.

For

For by Conſtruction $AB$ or $AG-GB=\frac{n}{p}-\frac{q}{mp}$, and by the Property of the Parabola $EQ=\frac{xx}{p}$ becauſe $AP$ or $MQ=x$. But becauſe the Triangles $FQM$, $MDC$, are ſimilar, therefore $FQ$ or $EQ-EF\left(\frac{xx}{p}-\frac{q}{mp}\right):QM(x)::DM$ or $PM-AB\left(y+\frac{q}{mp}-\frac{n}{p}\right):CD\,(m-x)$. Whence multiplying the Means and Extremes, and we ſhall have $x^3-mxx-nx+q=-pxy$, and if the Points $M$ be taken ſucceſſively in the three Angles that follow this here Angle, the ſame Equation will be found always, obſerving to take $AP=-x$, and $PM=-y$, when the Points $P$ and $M$, do fall the contrary way to what they do here: So that the two Curves, which may be called *parabolick Conchoides*, ſhall be the compleat Locus of all the affirmative and negative Values of the unknown Quantity $y$, anſwering to all the affirmative and negative Values of the other unknown Quantity $x$, in the Equation $x^3-mxx-nx+q=-pxy$.

But now to conſtruct the Circle being the Locus of the Equation $yy+xx-2rx\mp ss=o$, in the indefinite right Line $AP$ aſſume $AH=r$, from $A$ towards $PM$, when the Value of $r$ is poſitive, and the contrary way, when the ſame is negative; then about the Centre $H$ with the Radius $HM=\sqrt{rr\mp ss}$, (*viz.* $-ss$ when $ss$ is poſitive in the Equation, and $+ss$, when the ſame is negative) deſcribe a Circle, which will be that ſought; for becauſe $HPM$ is a right-angl'd Triangle, therefore we have always $\overline{HM}^2=\overline{HP}^2+\overline{PM}^2$, that is, $yy+xx-2rx\mp ss=o$, by ſubſtituting the analytick Values, and bringing all the Terms over to one Side.

Now if from the Points $M$ there be drawn Perpendiculars ($MQ$) on the indefinite right $AQ$, I ſay theſe Lines ſhall be the Roots of the Equation propoſed; the affirmative Roots being on the right, and the negative ones on the left of $AQ$; for if $MP$ be drawn parallel to $AQ$, we ſhall have the following Equation from the Property of the Conchoides, *viz.* $x^3-mxx-rx+q=-pxy$, that is (ſquaring both Sides) $ppxxyy=-x^6-2mx^5$, &c. and by the Nature of the Circle, $yy+xx-2rx\mp ss=o$, which being multiplied by $ppxx$ and then $ppxxyy$ is $=-ppx^4+2ppprx^3\mp ppssxx$. And comparing theſe two Values of $ppxxyy$ together, there will be an Equation form'd, wherein if the Values of $2r$, $ss$, $pp$, $m$, $n$, $q$, be ſubſtituted, there will be found again the propoſed Equation $x^6-bx^5$, &c.

If $dx^3$ had been negative in the propoſed Equation, then you need but have taken $2r=\frac{2mn+2q+d}{pp}$, and the reſt of the Conſtruction would have been the ſame as above, becauſe $d$ does not happen in the Value

Value of *r*. And becauſe then all the Signs of the Terms of the propoſed Equation are alternate; it is a received Maxim in Algebra, that all the real Roots thereof are affirmative, that is, if the ſaid Equation has two real Roots, and four imaginary ones, the two real Roots are affirmative; if it has four real Roots, and two imaginary ones, the four real Roots are affirmative; and if all the ſix Roots be real ones, then they ſhall be all affirmative. From whence it appears, that in this Caſe we have Occaſion only for the Conchoid, which is deſcrib'd by that half of the Parabola next to the fixed Point *C*, becauſe the other Conchoid only determines the negative Roots.

If the Value of the Radius ſhould happen to be equal to nothing, or imaginary only; or if it ſhould be ſo ſmall, as not to touch or cut the two Conchoids; then we may be aſſured, that all the Roots of the Equation will be imaginary. If it cuts them in ſix Points, all the Roots will be affirmative. And laſtly, if it cuts them in but four or two Points, then there will be only four or two affirmative Roots, and the others are imaginary. Here it muſt be always obſerv'd, that if the Circle touches one of the Conchoides, the Point of Contact muſt be eſteem'd as two Points infinitely near to one another, ſo that then the propoſed Equation will have two Roots, each equal to the Perpendicular drawn from the Point of Contact to *BE*.

## SCHOLIUM I.

406. FROM the Deſcription of the two parabolick Conchoids, it follows, (1.) That the right Line *BE*, both ways indefinitely produced, is a common Aſymptote to both thoſe Curves. (2.) That one of the Conchoids does paſs through the fixed Point *C*, and the Rule *CF* touches it in the Point *C*; becauſe the Point *M* coinciding with the Point *C*, the Rule paſſes through two Points of the Curve infinitely near to one another. (3.) That when the Point *F* does fall on *B*, the Rule *CF*, whoſe Interſections *(M, M)* with the Parabola, do deſcribe the Conchoids, falls in *CB*; and ſo the Line *MFM* going through the Point *F*, does become a double Ordinate; that is, the Line *CB* meets the Conchoids in two Points *K*, *L*, being ſuch that *BK* and *BL* are each equal to the Ordinate to the Axis of the Parabola that paſſes thro' the Point *F*. From whence it is evident, if *BC* was equal to that Ordinate, that the Point *K* would then fall on *C*; and ſo the Line *BC*, which would paſs through two Points *K* and *C*, of the Conchoid infinitely near to one another, would touch the Curve in the Point *C*, wherein *K* and *C* do coincide.

Any Number of Points of the Conchoids may be found without uſing the Parabola *MEM*, after the following manner. In *BE* take *BO* equal to the Parameter of the Parabola, and with any right FIG. 225.

Line $OR$ greater than $OB$, as a Diameter, deſcribe a Circle cutting $BC$ in the Points $D, D$; in that Diameter take $RS = EF$, and through the fixed Point $C$, draw the two right Lines $CM, CM$, parallel to $DS, DS$, which ſhall meet $DM, DM$, (parallel to $EB$) in $M, M$, which ſhall be two Points of the Conchoids. For if $CM$ be produced till it meets the Aſymptote $BE$ in the Point $F$, and $MQ$ be drawn parallel to $BC$; then it is plain, that the right-angl'd Triangles $MQF$, $DBS$, ſhall be equal, and ſo $FQ$ is $= BS$. But if $RS$ be taken $= EF$, we ſhall have $EF + FQ$, or $EQ = RS + SB$ or $RB$; and the Parabola $EM$, whoſe Vertex is $E$, and having a Line equal to $BO$, for the Parameter, ſhall paſs through the Point $M$; becauſe by the Nature of the Circle, the Square of $BD$ or $MQ$ is equal to the Rectangle under $BD$ or $MQ$, and the Parameter $BO$; which likewiſe follows from the Property of the Parabola: Therefore the Point $M$, found by this Conſtruction, is the ſame as that gotten by the Interſection of the Rule $CF$, and the Semi-parabola $EM$.

If the Point $D$ was given, and the Point $R$ was requir'd, you muſt draw $DR$ perpendicular to $OD$; and the reſt of the Conſtruction will be the ſame as before.

It is proper to obſerve here by the way, (1.) If in $BC$ you take $BD$ (from $B$ towards $C$) equal to the affirmative Root of the Equation of the third Degree $x^3 - \frac{1}{2} m x x - \frac{1}{2} m n p = o$, (the given Quantities $BC$ being $= m$, $EF = n$, $BO = p$); and then the Point $M$ be found as above: This Point ſhall be further diſtant from the right Line $BC$, than any other Points of the Portion $KMC$, ſo that the Tangent paſſing through that Point ſhall be parallel to $BC$. (2.) If $BD$ be taken in $BC$, produced on the other Side of the Point $B$, equal to the affirmative Root of the Equation $x^3 - m n p = o$; then the Point $M$ of the Conchoid anſwering to the Point $D$, ſhall be the Point of Inflexion of the Curve; that is, the Point wherein the Curve, from being concave, begins to become convex. Becauſe this does depend upon the Principles that are laid down in my Book, *Des infiniment petits*, therefore I refer you to that Book, or ſome other of the like Nature; for the ſame may be here ſuppos'd as true, without enquiring into the Reaſon thereof; ſince this has nothing to do in the Conſtruction of Equations of the ſixth Degree, which is the Buſineſs here in hand.

## SCHOLIUM II.

407. THE Deſcription of the two parabolick Conchoids require, (1.) That the Line $BC$ ($\frac{1}{2} b$) be of ſome Magnitude, and ſo the 2d Term of the propoſed Equation muſt not be wanting. (2.) That the Term $q$ cannot be $= o$, in the Equation $x^3 - m x x - n x + q = - p x y$, ſince dividing by $x$, there will ariſe $x x - m x - n = - p y$, the

the Locus whereof is a Parabola; from whence it is plain, that the laſt Term $g$ in the propos'd Equation muſt have the Sign $+$ prefixed to it, for $q = \sqrt{g}$.

Further, if the Term $fx$ ſhould have the Sign $+$, you muſt give the Sign $-$ to the ſame, in likewiſe changing the Signs of the 2d and 4th Term; and doing this will not bring any Inconvenience to the Conſtruction, but only will change the negative Roots into affirmative ones, and the affirmative ones into negative ones. And in order for the Locus of the ſecond Equation to be a Circle, $\frac{f}{\sqrt{g}} \mp c$ muſt be greater than $\frac{1}{4} bb$, ($c$ being affirmative when $cx^4$ is ſo, and negative when that is) from whence it appears, that if the Term $fx$ be wanting, the Term $cx^4$ muſt be affirmative, and $c$ muſt be greater than $\frac{1}{4} bb$; and if the Term $cx^4$ be wanting, $\frac{f}{\sqrt{g}}$ muſt be greater than $\frac{1}{4} bb$.

Therefore it is evident, that the propoſed Equation of the ſixth Degree muſt neceſſarily have theſe Conditions, in order to conſtruct the ſame immediately by means of parabolick Conchoids, and a Circle, according to the Rules above preſcribed.

## SCHOLIUM III.

408. WHEN an Equation given is one of the fifth Power, by raiſing it to the ſixth, we can very often bring the ſame to ſuch Conditions, as to be immediately conſtructed; as will appear by the following Examples.

1. Let $x^5 - a^4 b = 0$, and ſuppoſe $a$ to be greater than $b$. Multiply this Equation by $x - b$, and then the following Equation of the ſixth Degree will ariſe, *viz.* $x^6 - bx^5 - a^4 bx + a^4 bb = 0$, which has all the requiſite Conditions ſpecified in the laſt Scholium.

2. Let $x^5 - 5aax^3 + 5a^4 x - a^4 b = 0$, and let $a$ be greater than $b$. Multiply this Equation by $x - b$, and then we ſhall have $x^6 - bx^5 - 5aax^4 + 5aabx^3 + 5a^4 xx - 6a^4 bx + a^4 bb = 0$, which hath all the neceſſary Conditions.

3. Let $x^5 - ax^4 - 4aax^3 + 3a^3 xx + 3a^4 x - a^5 = 0$. Multiply this Equation by $x - 4a$; and then there will ariſe $x^6 - 5ax^5 + 19a^3 x^3 - 9aaxx - 13a^5 x + 4a^6 = 0$, which is an Equation of the ſixth Degree, wherein all the Terms do happen to have the neceſſary Conditions.

Here it may be obſerved that by Means of the firſt Equation $a^5 - a^4 b = 0$, can be found four mean Proportionals between two Extremes $A$ and $B$; and by means of the ſecond $x^5 - 5aax^3$, &c. a given Angle may be divided into five equal Parts; and by means of the third $x^5 -$ $ax^4$

$a x^4$, &c. a regular Polygon of Eleven Sides may be inſcribed in a given Circle; as will appear in the following Book. I ſhall now proceed to the Conſtruction of the firſt of theſe Equations, that ſo it may be compar'd with the Conſtruction laid down by *Deſcartes* at the End of his third Book of Geometry.

FIG. 226. Deſcribe a Parabola $ME$ with a Line $p = \sqrt{aa - \frac{1}{4}bb}$ for the Parameter, and aſſume the Line $AG = \frac{aa}{2p}$, $GB$ or $EF = 4AG$, $BC = \frac{1}{2}b$, $AH = \frac{5aab}{4pp}$, and a Line $s = \frac{a}{2p}\sqrt{4bb - aa}$, or $\frac{a}{2p}\sqrt{aa - 4bb}$; then deſcribe a parabolick Conchoid (as is directed in Article 404.) by means of the Parabola $ME$, and long Rule $CF$, freely turning about the fixed Point $C$, and always paſſing through the Point $F$, while the Part $EF$ of the Axis of the Parabola ſlides along the Line $AQ$; after this, about the Centre $H$, with the Radius $HM = \sqrt{\overline{AH}^2 + ss}$ deſcribe a Circle, $ss$ being affirmative, when $4bb$ is greater than $aa$, and negative when it is leſs. Now if from the Points $O$, $M$, wherein this Circle cuts the Conchoid, there be drawn the Lines $OR$, $MP$, perpendicular to the Axis $AP$; I ſay the Parts $AR$, $AP$, ſhall be the Roots of the Equation $x^6 - bx^5 - a^4bx + a^4bb = o$. This is proved as in Art. 404.

The trouble of finding a Line $s = \frac{a}{2p}\sqrt{4bb - aa}$, or $\frac{a}{2p}\sqrt{aa - 4bb}$ may be ſpared, if you conſider that the Circle deſcrib'd with the Centre $H$, muſt cut the Conchoid $COM$ in the Point $O$ being ſuch, that $OR$ drawn perpendicular to $AP$, we have $AR = b$; becauſe one Root of the Equation is $x = b$. From whence in $AP$, aſſume $AR = b$, and draw $RO$ perpendicular to $AP$, meeting the Conchoid $COM$ in $O$; and then you may deſcribe the Circle about the Centre $H$, with the Radius $HO$. For the Circle cuts the Conchoid in another Point $M$, being ſuch that $MP$ drawn perpendicular to $AP$, the Line $AP$ ſhall be the greateſt of four mean Proportionals required. Becauſe the Circle deſcribed about the Centre $H$ does cut the Conchoid paſſing thro' the Point $C$, only in two Points $O$, and $M$, and does not meet the other Conchoid; therefore the propoſed Equation $x^6 - bx^5$, &c. hath only two affirmative Roots $AR$, $AP$, and four imaginary Roots.

## SCHOLIUM IV.

409. WHEN the given Equation of the ſixth Degree hath not the neceſſary Conditions for being conſtructed immediately by the Method above explain d, or elſe being an Equation of the fifth Degree, the laſt Scholium is found uſeleſs; then the Preparation aſſign'd by *Deſcartes* in the third Book of his Geometry may be uſed. Wherein

is

is ſhewn how to transform any Equation of the fifth or ſixth Degree into another of the ſixth, having the Signs of all the Terms alternate, and where the known Quantity in the third Term does exceed the Square of ½ the known Quantity in the ſecond: For by this means the Conſtruction of the Problem is made general for all Equations of the fifth and ſixth Degree. The Method of doing of this I ſhall not here explain, ſince it does depend upon pure Algebra, which is not my Deſign here to treat of; and becauſe in the following Propoſition I ſhall lay down a general Way of conſtructing all Equations of the fifth and ſixth Degree, without any other Preparation than only getting out the ſecond Term.

## PROPOSITION VIII.

### Problem.

410. *TO find the Roots of the following Equation* $x^6 - bx^4 - cx^3 + dxx - fx + g = 0$, *by means of a given Cubick Parabola, and a Conick Section.*

Let $aay = x^3$ be an Equation whoſe Locus is a Cubick Parabola *MAM* (*AP* being $= x$, $PM = y$, $AB = a$). Inſtead of $x^6$ in the propoſed Equation, ſubſtitute its Value $a^4yy$, in the Room of $x^4$ its Value $aaxy$, and inſtead of $x^3$ its Value $aay$; then the propoſed Equation will be changed into this Equation of the ſecond Degreee, $yy - \frac{b}{aa}xy - \frac{c}{aa}y + \frac{d}{a^4}xx - \frac{f}{a^4}x + \frac{g}{a^4} = 0$, whoſe Locus is * an Ellipſis, when *d* is greater than $\frac{1}{4}bb$, that is, when the known Quantity which does multiply *xx*, is greater than the Square of half the known Quantity multiplying $x^4$, as is here ſuppoſed. And if you would have the Line that repreſents Unity, and which is conceiv'd to be in the propoſed Equation, equal to the Parameter *a* of the given Cubick Parabola; then the Equation will be changed into this $yy - \frac{b}{a}xy - cy + \frac{d}{a}xx - fx + ag = 0$ whoſe Conſtruction is as follows. FIG. 227. *Art. 323.

In the indefinite Line *AP* take $AB = a$, and draw the right Lines $BE = \frac{1}{2}b$, $AD = \frac{1}{2}c$, parallel to *PM* and towards the ſame Parts: Alſo through the Point *A* draw the right Line *AE* (*e*), and through the Point *D*, the right Line *DG* parallel to *AE*, in which take $DC(s) = \frac{2afe + bce}{4ad - bb}$ from *D* towards *PM*, and on both Sides *C* aſſume *CK*, *CL*, each equal to $t = \sqrt{ss + \frac{ccee - 4agee}{4ad - bb}}$. This being done, with the

the Diameter $LK$ ($2t$) having a Line $KH = \frac{4adt - bbt}{2ee}$, and its Ordinates parallel to $PM$, deſcribe an Ellipſis, and the ſame ſhall be that ſought.

Now if from the Points of Concurrence of this Ellipſis, and the given Cubick Parabola there be drawn Lines as $PM$, making with $AP$, the Angle $APM$, given or taken at pleaſure, then the Parts ($AP$) of the indefinite right Line whereon the indeterminate Quantity $x$ tends, ſhall be the Roots ſought; the affirmative Roots falling on that Side the Point $A$ as $PM$ is ſuppoſed to fall in the Conſtruction, and the negative Roots on the contrary Side, For by the Property of the Conick Section $yy - \frac{b}{a}xy - cy + \frac{d}{a}xx - fx + ag$ is $= o$, and by the Property of the Cubick Parabola, $y$ is $= \frac{x^3}{aa}$; and ſubſtituting that Value inſtead of $y$, and the Square thereof for $yy$, in the precedent Equation, and then we ſhall get the given Equation $x^6 - abx^4 - aacx^3$, &c. $= o$.

## SCHOLIUM I.

411. ANY Equation of the fifth or ſixth Degree being given, if you get out the ſecond Term, and afterwards multiply it by the unknown Quantity $x$, if the ſame be an Equation of the fifth Degree, and if by means of ($a$) the Parameter of the given Cubick Parabola, you reduce the known Quantities that multiply $x^4$ to the Expreſſion $ab$; thoſe that multiply $x^3$, &c. to the Expreſſion $aac$, then if the Subſtitution as above be uſed, the given Equation ſhall be always transform'd to a Locus of the ſecond Degree. Whence if a Cubick Parabola be once well deſcrib'd, with any Line $a$ for a Parameter, and any Angle ($APM$) made by the Ordinates ($PM$) to the Diameter $AP$, then it is manifeſt that the Roots of any Equation of the fifth and ſixth Degree may be found by means of that Curve, and a proper Conick Section.

## SCHOLLIUM II.

412. IF after the ſecond Term of a given Equation of the fifth or ſixth Degree be gotten out, and the Equation is multiply'd by $x$, if it be one of the fifth Degree; the known Quantity that multiplies the Square $xx$ be poſitive, and does exceed the Square of half the known Quantity that multiplies $x^4$: Then from the Subſtitution by means of $aay = x^3$, we ſhall have always an Equation of the ſecond Degree, whoſe Locus is an Ellipſis, as appears in the Problem. And it may be ſo ordered always, that that Ellipſis may become a Circle, but then the Cubick Parabola will not be given. For Example.

Find

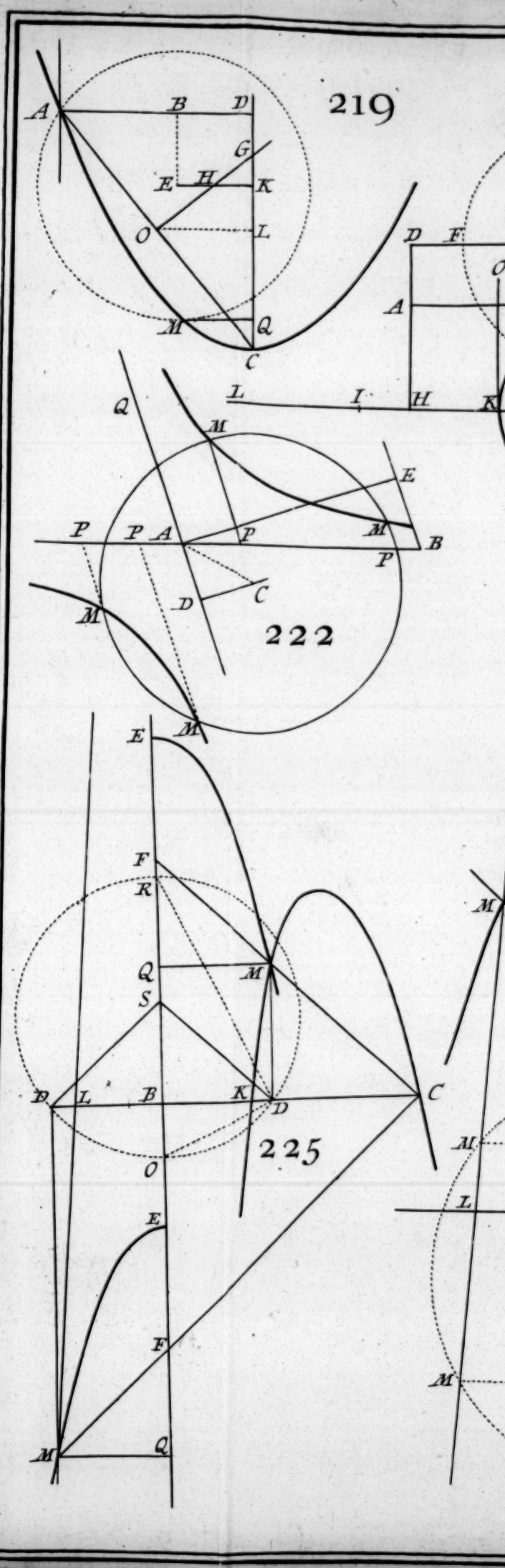
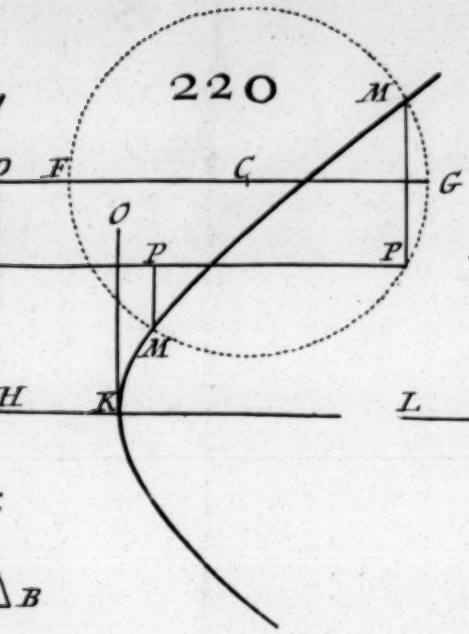
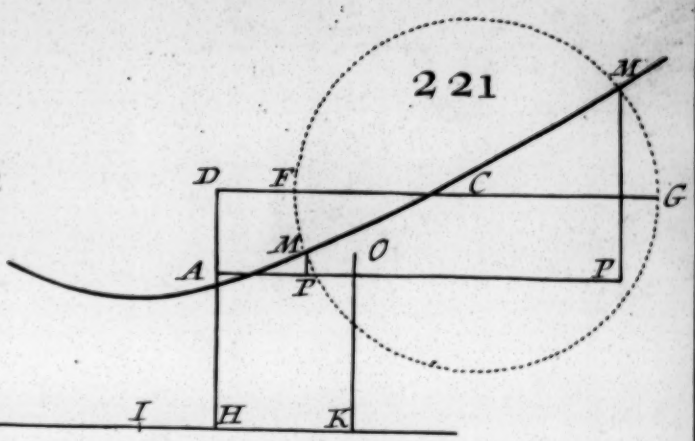

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Plate 24. page 262.

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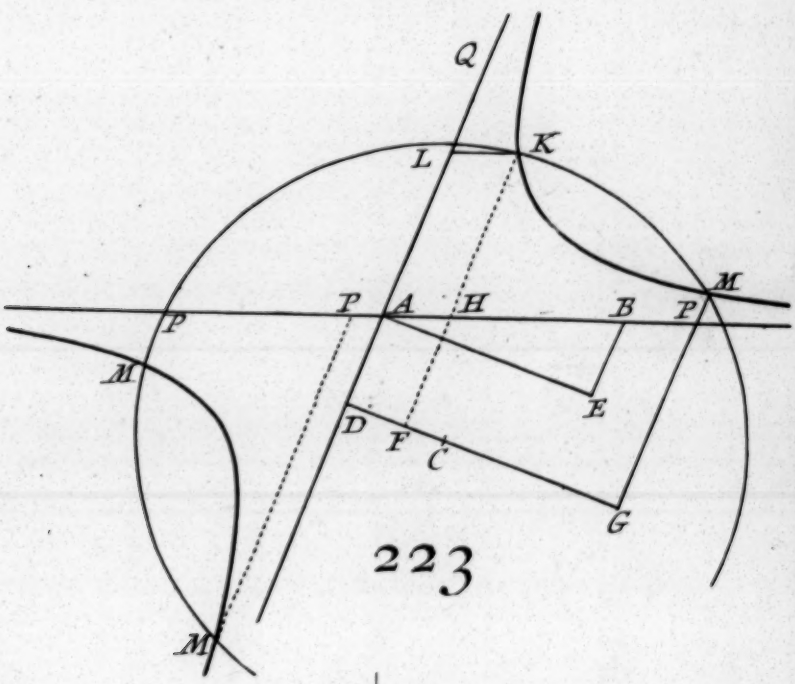



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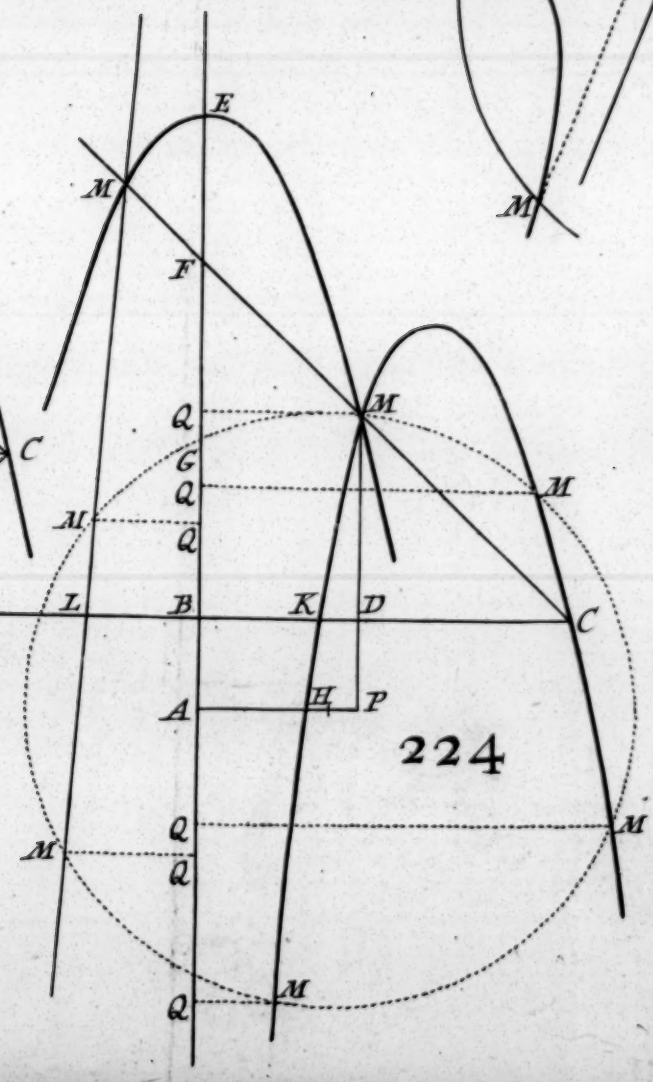




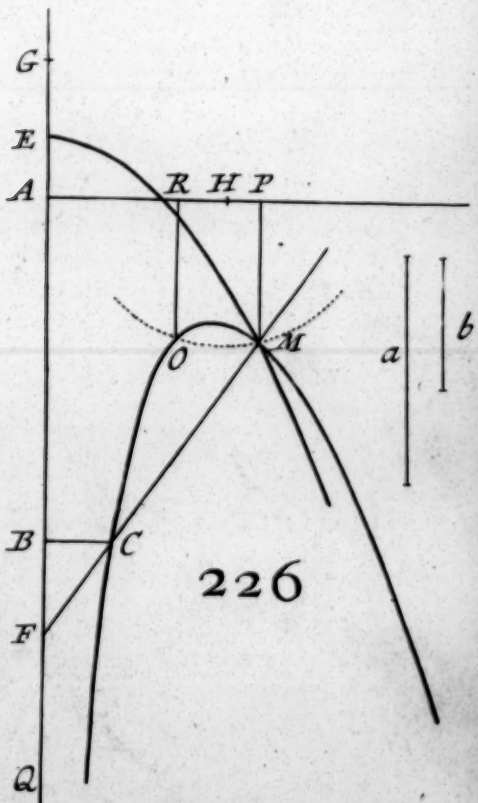





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Find * a Line $a$ whereof the ſquared Square $a^4$ be equal to the known Quantity multiplying $x\,x$, and by means of this Line reduce all the known Quantities multiplying $x^4$ to the Expreſſion $ab$, all theſe multiplying $x^3$, &c. to the Expreſſion $a\,a\,c$; and then the Equation ſhall be reduced to this Form $x^6 \pm a\,b\,x^4 \pm a\,a\,c\,x^3 + a^4\,x\,x \pm a^4 f\,x \pm a^5 g = o$. And ſubſtituting $a^4\,y\,y$, $a\,a\,x\,y$, and $a\,a\,y$ for $x^6$, $x^4$ and $x^3$, and this Equation of the ſecond Degree will be had $y\,y \pm \frac{b}{a}\,x\,y \pm c\,y + x\,x \pm f\,x \pm a\,g = o$, the Locus of which ſhall be a Circle, * if the Angle $AEB$ be made a right Angle; which may be eaſily done thus. * *Art.* 378. * *Art.* 327, *and* 329.

In the indefinite right Line $AP$ aſſume $AB = a$; and upon this Line, as a Diameter, deſcribe the Semi-circle $AEB$, towards the ſame Parts that $PM$ falls, when $\frac{b}{a}\,x\,y$ is negative, and towards contrary Parts when the ſame is poſitive. In the Diameter of this Semicircle aſſume the Line $BE = \frac{1}{2}\,b$, from $B$ to $E$; and drawing $AE$ (*e*), the Line $PM$ muſt be drawn parallel to $BE$, and the reſt of the Conſtruction will be the ſame as for the Ellipſis, which will here be a Circle; becauſe the Angle $CGM$ ſhall be a right Angle; and $AEB$ being a right-angled Triangle, $e\,e$ is $= a\,a - \frac{1}{4}\,b\,b$, which muſt expreſs the Ratio of the Diameter $LK$ to the Parameter thereof. This Circle $KMM$ is the Locus of this Equation $y\,y - \frac{b}{a}\,x\,y - c\,y + x\,x - f\,x - a\,g = o$, which is the ſame as that in the Problem, only $d$ is $= a$. FIG. 228.

Now if the Line $AB$ be divided into many equal Parts $AP$, $AP$, and Parallels $PM$, $PM$, *&c.* be drawn to $BE$; and if every $PM$ be taken equal to a fourth Proportional to its Correſpondent $AP$, and the given Line $AB$: Then if a Curve $(M\,A\,M)$ be drawn through all the Points $(M)$ thus found; it is plain, that this Curve will be the Locus of the Equation $x^3 = a\,a\,y$; and conſequently by means of the Points of Interſection $(M, M)$ of the Cubick Parabola and Circle, may be found all the Roots $(AP, AP)$ of the propoſed Equation.

## SCHOLIUM III.

413. BEcauſe Parabola's of all Degrees have been often ſpoken of in this Book; and ſince we have ſhewn how to uſe a Cubick Parabola for the Conſtruction of Equations of the fifth and ſixth Degree, it will not be foreign to our Purpoſe to examine the ſeveral Sorts of Lines theſe Parabola's may make. For which End, let $BC$, $DE$ be two indefinite ſtraight Lines, cutting one another in the Point $A$, and let there be a Parabola $AM$, of any Degree (in the Angle $BAD$) whoſe Nature is ſuch, that $MP$ being drawn from any Point $M$ thereof, FIG. 229.

of, parallel to $DE$, and meeting $BC$ in the Point $P$, and the indeterminate Quantities $AP$, $PM$, and the given Quantity $AB$ being called $x$, $y$, and 1; we have always $x^m = y^n$ (where the Letters $m$ and $n$ do denote the Indices of the Powers of $x$ and $y$, and may be any poſitive whole Numbers at pleaſure, and $m$ is ſuppos'd to be greater than $n$) it is evident, 1. When $AP$ $(x)$ is $= o$, $PM$ $(y)$ is ſo likewiſe, and the more $AP$ $(x)$ increaſes, the more does $PM$ $(y)$

*Art. 237. alſo increaſe. 2. The Subtangent $PT\left(\frac{n}{m}\,x\right)$ is * always leſs than $AP$ $(x)$ becauſe $n$ is ſuppos'd to be leſs than $m$. Whence it follows, that the Parabola $AM$, be it of what Degree ſoever, ſhall always paſs through the Point $A$; the ſame recedes infinitely more and more from the right Line $BC$ eſteem'd as being a Diameter thereof; and laſtly, the Convex Side of it is next to that Diameter. But becauſe the Curve $AM$, which falls in the Angle $DAB$, is only a Portion of the Parabola, we muſt next examine in which of the Angles $DAC$, $CAE$, $EAB$, that Parabola is continued; and here there are three Caſes.

*Caſe* 1. When the Index $(m)$ of the Power of $x$ is an even Number, and the Index $(n)$ of the Power of $y$, an odd Number. The Root $(m)$ of $x^m$ ſhall be $\mp x$, and the Root $(n)$ of $y^n$ ſhall be only $+y$; for if $m$ be $= 4$ and $n = 3$, then it is manifeſt, that the fourth Power of $\mp x$ is always $x^4$, but it is not ſo of the Cube of $\mp y$; ſince the
Fig. 229. Cube of $+y$ is $y^3$, and the Cube of $-y$ is $-y^3$. Hence it is evident, that $AP$ $(x)$ may be both poſitive and negative, but $PM$ $(y)$ is always poſitive; and conſequently the Parabola $AM$ muſt be continued in the Angle $DAC$, adjoining to $BAD$, ſo that if a right Line be drawn through any Point $K$ of the Line $AD$ parallel to $BC$, the ſame ſhall meet the Parabola $MAM$ in two Points $M$, $M$, equally diſtant from the Point $K$. And this is the common Parabola, which is the Locus of the Equation $xx = ay$, or $xx = y$, ſuppoſing the Parameter $a = 1$.

*Caſe* 2. When the Indices $m$ and $n$ are odd Numbers: The Root $(m)$ of $x^m$ ſhall be only $+x$, and the Root $n$ only $+y$; but becauſe the Equation $-x^m = -y^n$ is the ſame as $x^m = y^n$, and the Root $(m)$ of $-x^m$ is $-x$, and the Root $(n)$ of $-y^n$ is $-y$; therefore $AP$ $(x)$ may be both poſitive and negative, as alſo $PM$ $(y)$, obſerving that when $AP$ is poſitive, $PM$ is ſo likewiſe, and contrariwiſe. Whence
Fig. 230. it appears, that the Parabola $AM$, in this Caſe, muſt be continued in the Angle $CAE$ vertically oppoſite to $DAB$, in a Poſition altogether the ſame, as in the Angle $DAB$, but inverted; ſo that $AP$ being taken equal to $AP$, and $PM$ drawn making with $AP$ the Angle $APM$ equal to the Angle $APM$; the right Line $PM$ does

meet the Portion $AM$ falling in the Angle $CAE$, in the Point $M$ being ſuch, that $PM$ is equal to $PM$. And ſuch is the principal Cubick Parabola $x^3 = aay$, or $x^3 = y$, ſuppoſing $a = 1$.

*Caſe* 3. When the Index ($m$) of the Power of $x$ is an odd Number, and the Index ($n$) of the Power of $y$ an even Number: The Root ($m$) of $x^m$ ſhall be always $+x$, and the Root ($n$) of $y^n$ will be $\mp y$. For Example; let $AM$ be a Cubick Parabola, which is the Locus of the Equation $x^3 = ayy$, $x^3 = yy$, it is plain that the Cube Root of $x^3$ is only $+x$, and the Cube Root of $yy$ is $\mp y$. Therefore the Parabola $AM$ muſt be continued in the Angle $BAE$, adjacent to the Angle $BAD$; in ſuch manner, that if a right Line be drawn through any Point $P$ of the Line $AB$, parallel to $DE$, it ſhall meet the whole Parabola $MAM$ in two Points $M$, $M$, equally diſtant from the Point $P$. FIG. 231.

Now the general Equation $x^m = y^n$, always appertains to one of the aforeſaid three Caſes; for if $m$ and $n$ be ſuppos'd to be both even Numbers, you muſt extract the ſquare Root of both Sides of the Equation as often as poſſible; by which means it may be reduced to an Equation, wherein one of the Indexes ſhall neceſſarily be odd. And $m$ may be ſuppoſed always to be greater than $n$; for if it ſhould be leſs, and (for Example) $aax = y^3$, then comparing the Points of the Parabola $AM$ with the Points of the Line $DE$, and calling $AK$, $x$; $KM$, $y$; we ſhall have this Equation $x^3 = aay$, which ſhall expreſs likewiſe the Nature of the ſame Parabola $AM$, and wherein the Index of the Power of $x$ is greater than the Index of the Power of $y$; ſo that the ſame Reaſoning would hold with regard to the Line $DE$, as before with regard to the Line $BC$. From whence it is evident, that all Parabola's of any Degree whatſoever, will always have one of the three preceding Figures. FIG. 232.

## PROPOSITION IX.

### Problem.

414. *IT is propoſed to conſtruct the following Equation of the eighth Degree* $x^8 - bx^7 + cx^6 - dx^5 + ex^4 - fx^3 + gxx - hx + l = 0$, *wherein none of the Terms are wanting, by means of two Loci; one of the ſecond, and the other of the fourth Degree.*

Take $xx = ay$ for the Locus of the ſecond Degree, and for $x^8$, $x^7$, $x^6$, $x^5$, $x^4$, $x^3$ and $xx$ ſubſtitute their Values $a^4y^4$, $a^3y^3x$, $a^3y^3$, $aayyx$, $aayy$, $ayx$, $ay$; then if the given Line $a$ be taken for Unity, we ſhall have this Equation, $y^4 - \frac{b}{a}xy^3 + cy^3 - dxyy + aeyy - afxy + aagy - aahx + a^3l = o$, the Locus whereof being one of the fourth Degree, and the moſt ſimple poſſible; becauſe one of the un-

known Quantities $x$ ariſes no higher than to the firſt Degree, all the Points thereof may be determin'd in uſing right Lines and Circles only.

Now if the Parabola being the Locus of the firſt Equation $xx = ay$ be conſtructed, and if there be taken any Number of different Magnitudes for $y$, and the Values of $x$ anſwering to them in the ſecond Equation be determined; then the Locus, which ſhall paſs through the Extremities of all the Values of $y$, and which will be conſequently the Locus of the ſaid Equation, will determine the ſought Values of the Roots of the given Equation, by means of the Points wherein that Locus cuts the Parabola. This is evident, becauſe ſubſtituting $\frac{xx}{a}$ the Value of $y$ in the ſecond Equation, and the Powers of that Value for the Powers of $y$, and there will ariſe the given Equation $x^5 - bx^7$ &c. $= o$.

### COROLLARY I.

415. BEcauſe Unity $a$ is arbitrary, it may be ſuppos'd given; and ſo the Parabola which is the Locus of the firſt Equation $xx = ay$, is given. Now by means of this Equation it is evident, that any Equation of the ſeventh or eighth Degree may be always tranſform'd into an Equation of the fourth Degree, wherein the unknown Quantity $x$ is only of one Dimenſion; therefore any Equation of the ſeventh or eighth Degree having all the Terms, or wanting ſome of them, may be conſtructed always by means of a given Parabola, and a Locus of the fourth Degree, in which one of the unknown Quantities has only one Dimenſion, without any other Preparation than taking the Parameter $a$ of the given Parabola for Unity, that ſo the known Quantities multiplying $x^6$, may be reduced to the Expreſſion $a\,c$, thoſe that multiply $x^5$ to the Expreſſion $a\,a\,d$, &c.

### COROLLARY II.

416. IT may be prov'd after the ſame manner, that any Equation of the ninth or tenth Degree may always be conſtructed by means of a given Parabola, and a Locus of the fifth Degree, wherein one of the unknown Quantities hath only one Dimenſion: Alſo Equations of the eleventh and twelfth Degree may be conſtructed by means of a given Parabola, and a Locus of the ſixth Degree; and ſo of others to Infinity.

PROPO-

## PROPOSITION X.

### Problem.

417. *TO conſtruct the following Equation of the ninth Degree* $x^9 - bx^7 + cx^6$, &c. $= o$, *wanting only the ſecond Term, by means of two Loci, each of the third Degree.*

Take $x^3 = aay$ for one of the Loci of the third Degree, and inſtead of $x^9$, $x^7$, $x^6$, &c. ſubſtitute their Values $a^6 y^3$, $a^4 xyy$, $a^4 yy$, &c. then if $a$ be taken for Unity, we ſhall get the other following Equation of the third Degree, *viz.* $y^3 - \frac{b}{a} xyy + cyy$, &c. $= o$, wherein the unknown Quantity $x$ can riſe no higher than the ſecond Degree, becauſe every where wherein $x^3$ happens in the propoſed Equation, $aay$ is ſubſtituted for the ſame.

And it is plain, that if this Locus be conſtructed with a Cubick Parabola, which is the Locus of the other Equation $x^3 = aay$; then the Interſections of the two Loci ſhall determine the Roots of the given Equation.

### COROLLARY.

418. ANY Equation of the ſixth, eighth, or ninth Degree being given, it is manifeſt, after the ſecond Term be gotten out, and it is afterwards multiply d by the Root $x$, (when the Equation is of the eighth Degree) and by $xx$ (when it is one of the ſeventh) that it ſhall be always transformed into a Locus of the third Degree, in ſubſtituting, as above, by means of the Equation $x^3 = aay$, whoſe Locus is a given Cubick Parabola; ſo that this is a general Way for all Equations of the ſeventh, eighth, and ninth Degree. After the ſame manner it will be found, that any Equation of the twelfth Degree, not having the ſecond Term, may be transform'd into a Locus of the fourth Degree by means of the ſaid Equation $x^3 = aay$; as likewiſe Equations of the tenth and eleventh Degrees, by raiſing them to the twelfth.

But if an Equation of the ſixteenth Degree, only wanting the ſecond Term, be propos'd; then by means of the Locus of the fourth Degree $x^4 = a^3 y$, that Equation may be transform'd into one of the fifth Degree. After the ſame manner you will find, that an Equation of the twentieth Degree may be transform'd into a Locus of the ſixth Degree, by means of the ſaid Locus of the fourth Degree $x^4 a^3 y$; as alſo Equations of the ſeventeenth, eighteenth, and nineteenth Degrees. Moreover, Equations of the 25th only wanting the ſecond Term, may be transform'd into a Locus of the ſixth Degree, by means of the Locus of the fifth Degree $x^5 = a^4 y$; as likewiſe all Equations

of the 21ſt, 23d and 24th Degrees. And this Enquiry may be continued on further at pleaſure.

## SCHOLIUM I.

419. HENCE it is neceſſary to obſerve, that if an Equation of the ſixteenth Degree wants the third and ſixth Terms, as well as the ſecond; the Locus of the fifth Degree, which conjointly with that of the fourth Degree $x^4 = a^3 y$ ſerves to conſtruct the Equation, may be transform'd into one of the fourth Degree; and the ſame Obſervations may be made upon Equations of higher Degrees. But although it be true, that an Equation of the ſixteenth Degree, only wanting the ſecond Term, can be transform'd only to a Locus of the fifth Degree, if for this Effect the Locus of the 4th Degree $x^4 = a^3 y$ (having but two Terms) be uſed; yet it muſt not be generelly concluded from thence, that the moſt ſimple Loci for reſolving a compleat Equation of the ſixteenth Degree, muſt be one of the fourth and the other of the fifth Degree. For, on the contrary, it appears evident, that if a Locus of the fourth Degree, conſiſting of ſeveral Terms, be uſed inſtead of $x^4 = a^3 y$, having but two Terms, that Locus may be ſo choſen, as to ſerve to transform the compleat Equation of the ſixth Degree to another Locus of the fourth. The Reaſon of which is this; if two Loci of the fourth Degree be aſſum'd, in one of which the unknown Quantity $x$ ariſes to the fourth Degree, and in the other the unknown Quantity $y$; then it is manifeſt from the Rules of Algebra, that if the unknown Quantity $y$ be gotten out by means of thoſe two Equations, that an Equation will be had, wherein the unknown Quantity $x$ ariſes to the ſixteenth Degree. And becauſe two Loci of the fourth Degree may both together have more than ſixteen Terms, ſince each of them may have fifteen different ones; therefore thoſe Loci may contain all the known Quantities of the given Equation: This is enough to ſhew the Poſſibility of conſtructing a compleat Equation of the ſixteenth Degree by two Lines of the fourth.

Further, it ought to be ſuppoſed that the two moſt ſimple Loci for conſtructing a compleat Equation of the 20th, 19th, and 17th Degrees, ſhall be, the one of the fourth, and the other of the fifth Degrees, becauſe the Product of theſe two Loci ariſes to the 20th Degree, and they may contain both together more Terms than the propoſed Equation, and ſo do contain all the known Quantities happening therein. And if the unknown Quantity in a propoſed Equation has 21, 22, 23, 24, or 25 Dimenſions, then the two moſt ſimple Loci that will conſtruct them, will be thoſe of five Degrees each. From hence

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ariſes the following Rule by which may be found the two moſt ſimple Loci that can ſolve a propoſed Equation.

Extract the ſquare Root of the higheſt Power or Degree of the unknown Quantity. If there be no remainder, then each of the two Loci muſt have the ſame Number of Degrees as there are Unities contain'd in that ſquare Root; but if there be a remainder, then the ſame is equal, leſs or greater than the ſquare Root, if it be equal or leſs, the Degree for one of the Loci will be the Root itſelf; and for the other that Root plus Unity: If the Remainder be greater than the Root, then the Degree of both the Loci ſhall be the Root plus Unity.

For Example, it is required to find the two moſt ſimple Loci that can reſolve an Equation of the 37th Degree. Becauſe the ſquare Root of 37 is 6, and the Remainder 1 is leſs than 6, one of the Loci muſt be of the ſixth, and the other of the ſeventh Degree, which Loci will do alſo for Equations of the 38th, 39th, 40th, 41ſt, and 42d Degrees. Again, if an Equation be of the 43d Degree, becauſe the ſquare Root of 43 is 6, and the Remainder 7 is greater than 6, the two Loci muſt be each of the 7th Degree: Which Loci alſo ſerve for Equations of the 44th, 45th, 46th, 47th, 48th and 49th Degrees.

## SCHOLIUM II.

420. IT ſometimes happens that a given Equation may be conſtructed by means of one Curve only put into two different Poſitions, as will appear in the following Example.

It is requir'd to conſtruct this Equation of the 9th Degree, $x^9 + a^8 x - a^8 b = 0$, wanting all the middle Terms, except the laſt but one. Take the Equation $x^3 = aay$, the Locus whereof is the Cubick Parabola $MAM$, having the given Line $AB = a$ for a Parameter, and right Lines, as $PM$ ($y$) making an aſſum'd Angle $APM$ with the correſpondent Parts $AP$ ($x$) of the Axis or Diameter for Ordinates; then cube both Sides of this laſt Equation, and we ſhall have $x^9 = a^6 y^3$, which (by Subſtitution) ſhall change the propoſed Equation into this here $y^3 = aab - aax$, the Locus whereof may be thus conſtructed.

In $AP$ produced aſſume $AC = b$, and through the Point $C$ draw the indefinite right Line $CK$ parallel to $PM$; then with $CK$ as an Axis, whoſe Parameter is $CD = a$, and Ordinates right Lines as $KM$ parallel to $AP$, deſcribe another Cubick Parabola $MCM$. I ſay this ſhall be the Locus requir'd. FIG. 233.

For by Conſtruction $MK$ or $CP = b - x$, and by the Property of the Curve $\overline{CK}^3 = MK \times \overline{CD}^2$, that is, in analytick Terms, $y^3 = aab - aax$. And it is evident, 1. That if right Lines as $PM$ be drawn from Points ($M$) wherein this latter Cubick Parabola $MCM$ meets the other $MAM$;

$MAM$; then the Parts $AP$ ſhall expreſs the Roots $x$ of the propos'd Equation $x^9 + a^8 x - a^8 b = o$. 2. That the Cubick Parabola's $MCM$, $MAM$, are the ſame in all reſpects, becauſe their Parameters $AB$, $CD$, are equal, and the Angles $APM$, $CKM$, that are form'd by the Ordinates, are ſo likewiſe.

The Situation of the two Cubick Parabola's $MAM$, $MCM$, ſhews that the propoſed Equation $x^9 + a^8 x - a^8 b = o$, has but one real Root $AP$ ($x$), which is always affirmative, and leſs than $AC$ ($b$); and ſo all the other eight Roots are imaginary.

## PROPOSITION XI.

### Problem.

421. *TO conſtruct an Equation of whatſoever Degree, by means of a ſtraight Line, and a Locus of the ſame Degree, all the Parts of which Locus may always be determin'd by uſing ſtraight Lines only.*

The laſt Term of the propoſed Equation muſt be brought to ſtand alone on one Side, and the whole Equation muſt be afterwards divided by the Line repreſenting Unity, repeated as often as neceſſary, that ſo each of the Terms may expreſs right Lines only: As ſuppoſe this Equation was propoſed $x^5 - b x^4 + a c x^3 - a a d x x + a^3 e x - a^4 f = o$, then $f$ muſt be $= \frac{x^5}{a^4} - \frac{bx^4}{a^4} + \frac{cx^3}{a^3} - \frac{dxx}{aa} + \frac{ex}{a}$.

FIG. 334. Now in the indefinite ſtrait Line $AB$, whoſe fixed Origin let be in the Point $A$, aſſume any Part $AP$ for the Value of $x$, and draw a right Line $PM = \frac{x^5}{a^4} - \frac{bx^4}{a^4} + \frac{cx^3}{a^3} - \frac{dxx}{aa} + \frac{ex}{a}$ (which may be done *Art. 376. * always by the uſe of ſtrait Lines only) parallel to the Line $AC$ given in poſition; then the Extremity $M$ of this Line ſhall be one Point of the Curve $ADEM$, whoſe Points of Interſection $M$, $M$, $M$, &c. made by the right Line $KM$, drawn parallel to $AB$ through the Point $K$, ſo that $AK$ be $= f$, ſhall determine the Parts $KM$, $KM$, $KM$, &c. which will be the ſought Values of the unknown Quantity $x$ in the given Equation.

For if the right Lines $MP$, $MP$, $MP$, &c. be drawn parallel to $AC$, and the indeterminate Quantities $AP$, $PM$, be called $x$ and $y$; then by the Property of the Curve, $ADEM$, we ſhall have this Equation $PM\ (y) = \frac{x^5}{a^4} - \frac{bx^4}{a^4} + \frac{cx^3}{a^3} - \frac{dxx}{aa} + \frac{ex}{a}$ which is a Locus of the fifth Degree; and by the Property of the right Line $KM$ this other Equation $y = f$, and by ſubſtituting $x$ for $y$, and multiplying by $a^4$, and there will ariſe the propoſed Equation $x^5 - bx^4 + acx^3 - aadx + a^3 ex - a^4 f = o$. Theſe

Theſe kind of Conſtructions may be very uſeful for finding the Limits of Equations. For Example, let us ſuppoſe that we have a way to determine the Parts $AF$, $AG$, in the Line $AC$, being ſuch that the right Lines $FD$, $GE$, (parallel to $AB$) do touch the Curve in the Points $D$, $E$; then it is manifeſt, 1. That if $AK$ ($f$) be leſs than $AF$, and greater than $AG$, as is ſuppoſed in this Figure, the propos'd Equation will have three affirmative Roots $KM$, $KM$, $KM$, and two imaginary Roots; becauſe the Figure of the Curve is ſuch, that the Line $KM$ ſhall meet it in three Points, and not more. 2. If $AK$ ($f$) be leſs than $AG$, the Line $KM$ ſhall cut the Curve in five Points, that is, the propoſed Equation will have five affirmative Roots. 3. If $AK$ ($f$) be greater than $AF$, the Equation will have but one affirmative Root, and four imaginary ones. 4. If $AK$ be $= AF$, the Equation ſhall have three affirmative Roots, two whereof will be equal to one another, *viz.* $FD$, $FD$. 5 If $AK$ be $= AG$, the Equation ſhall have five affirmative Roots, two whereof will be equal, *viz.* $GE$, $GE$.

The ſame Curve $ADEM$ being continued from the Point $A$, ſhall determine the Roots of this Equation $x^5 - bx^4 + acx^3 - aadxx + a^3ex + a^4f = o$, only differing from that above in this, that the laſt Term has the Sign + prefixed to it; which ſhews that the Line $KM$ muſt then be drawn below $AB$, becauſe the Locus thereof muſt be $y = -f$.

### SCHOLIUM.

422. THE precceding Conſtruction may be varied divers Ways; for inſtead of making the laſt Term equal to all the others, you may do the ſame of any other Term, or even any two being next to each other, and afterwards divide them ſo, that when they be made equal to $y$, the Locus of the Equation may be one of the firſt Degree. For Example, Let there be an Equation of the 3d Degree, $x^3 - abx - aac = o$; make $\frac{bx}{a} + c = \frac{x^3}{aa}$, then we ſhall have the two following Equations $x^3 = aay$, and $y = \frac{bx}{a} + c$, the Loci of which being ſeparately conſtructed after the following Manner, will determine the Roots of the propoſed Equation.

Aſſume two unknown and indeterminate ſtraight Lines $AP$ ($x$), $PM$, ($y$) making any Angle $APM$ with one another, and deſcribe the principal Cubick Parabola $MAM$, which let be the Locus of the firſt Equation $x^3 = aay$. Through the Point $A$, the Origin of the $x$', draw a right Line parallel to $PM$, in which take $AC = b$, $AD = c$, both tending towards the ſame Parts as $PM$; alſo in $AP$ produc'd on this Side $A$, aſſume $AB = a$, and through the Point $D$ draw an indefinite FIG. 235.

finite right Line parallel to $BC$: I ſay, if right Lines, as $MP$, be drawn parallel to $AC$ from the Points $(M)$, wherein this Parallel meets the principal Cubick Parabola $MAM$, the Parts $AP$ ſhall be the Roots of the given Equation $x^3 - abx - aac = o$.

For if $DE$ be drawn parallel to $AP$, the ſimilar Triangles $BAC$ $DEM$, will give this Proportion, $BA\,(a) : AC\,(b) :: DE\,(x) : EM = \frac{bx}{a}$, and conſequently $PM\,(y) = \frac{bx}{a} + c$. And ſince $MAM$ is a Cubick Parabola, we have $x^3 = aay$. If then $\frac{bx}{a} + c$ be ſubſtituted for $y$, there will ariſe the given Equation $x^3 - abx - aac = o$.

If $b$ in the given Equation had been affirmative, then $AC$ muſt have been taken tending the contrary way to that which $PM$ tends, as alſo $AD$ if $C$ had been affirmative. From whence it is manifeſt, that this Conſtruction is general for any given Equation of the third Degree. For when the ſecond Term is gotten out, the Equation may be always reduced to one of the ſaid Forms.

It is plain that a given Cubick Parabola may be uſed, ſince you need only aſſume the arbitrary Quantity $a$, (taken for Unity) for the Parameter thereof.

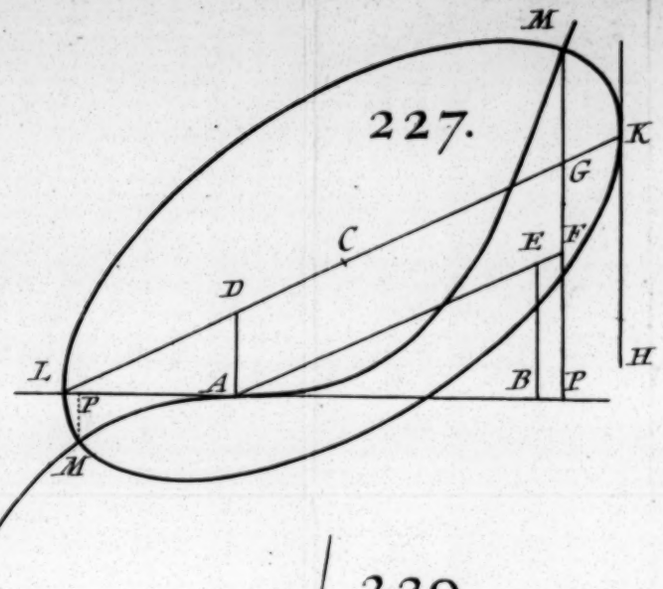
## PROPOSITION XII.

### Problem.

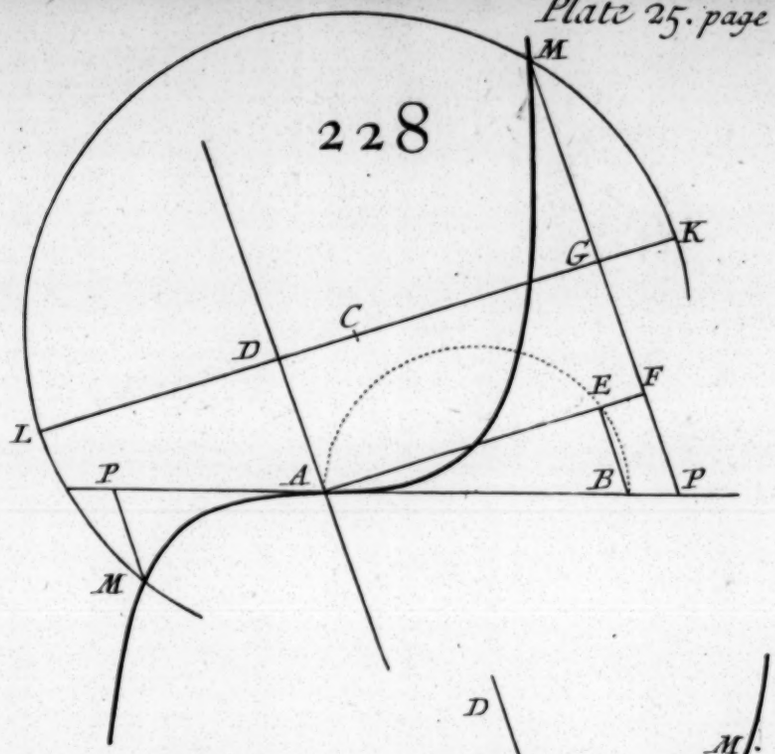
423. *TO find the Value of the Roots of any Equation of the third and fourth Degree, or of higher Equations that have but two Terms, to any required Exactneſs by Approximation; by means of right Lines and Circles only.*

Let a given Equation of the third Degree be $x^3 \mp 2apx - aaq = o$; multiply this Equation by $x$, that ſo it may be rais'd to the fourth Degree, and tranſpoſing the Term $aaqx$, and we have $x^4 \mp 2apxx = aaqx$; then add $aapp$ to both Sides, that ſo one Side of the Equation be a Square, and we get $x^4 \mp 2apxx + aapp = aapp + aaqx$; and extracting the ſquare Root of both Sides, and there comes out $xx \mp ap = a\sqrt{pp + qx}$; laſtly, tranſpoſing $ap$, and extracting the ſquare Root again, and then $x$ is $= \sqrt{\mp ap + a\sqrt{pp + qx}}$. This being done, I find that if inſtead of the exact Value of the affirmative Root $x$, there be a Magnitude taken greater than it, as $c$; then it follows, 1. That $c$ is greater than $\sqrt{\mp ap + a\sqrt{pp + qc}}$. 2. That $\sqrt{\mp ap + a\sqrt{pp + qc}}$ ſhall be yet greater than the exact Value of $x$. This ſecond Propoſition is plain, but for the firſt it may be proved thus.

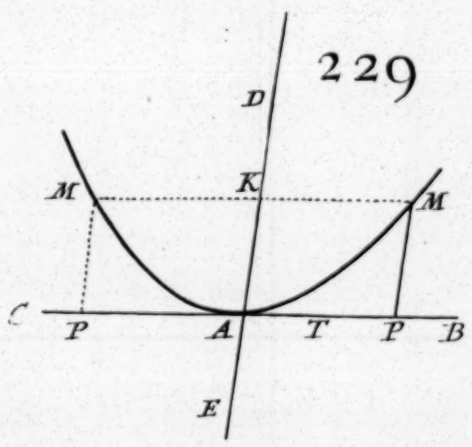
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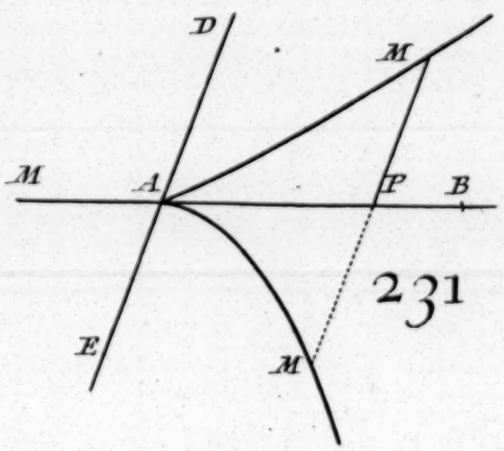



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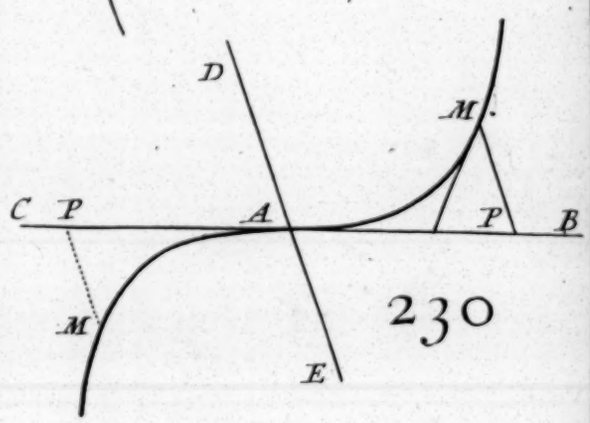



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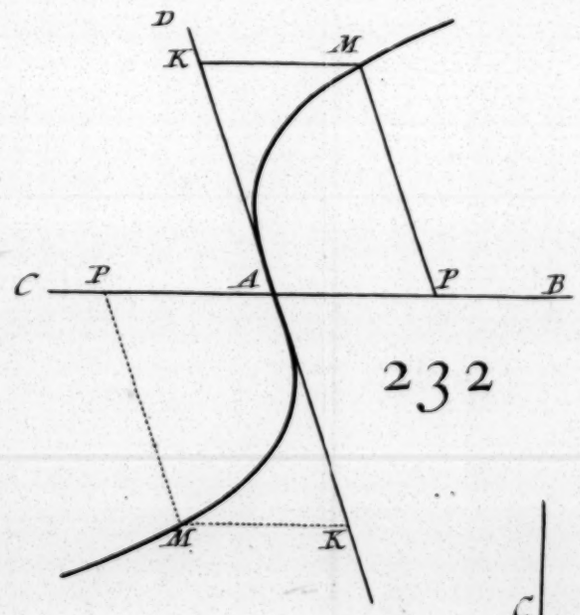



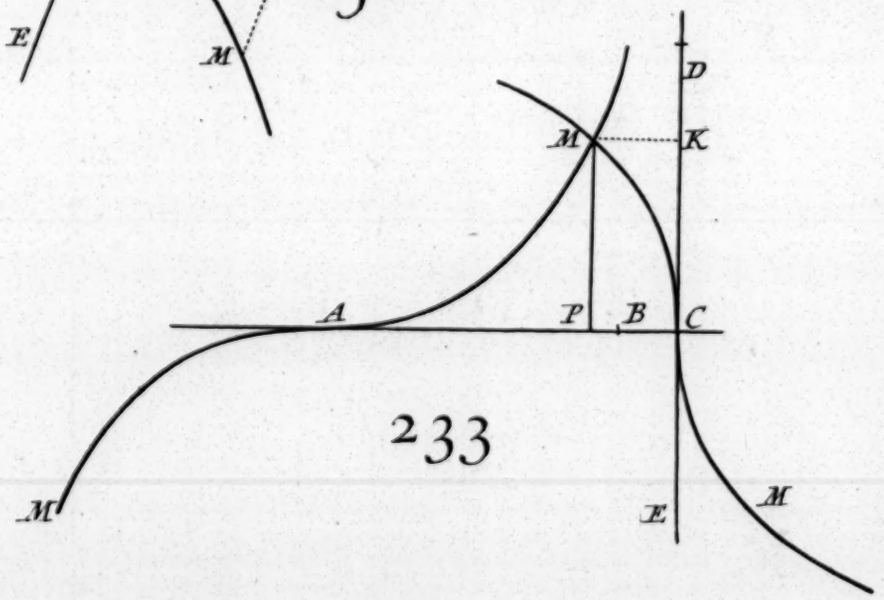

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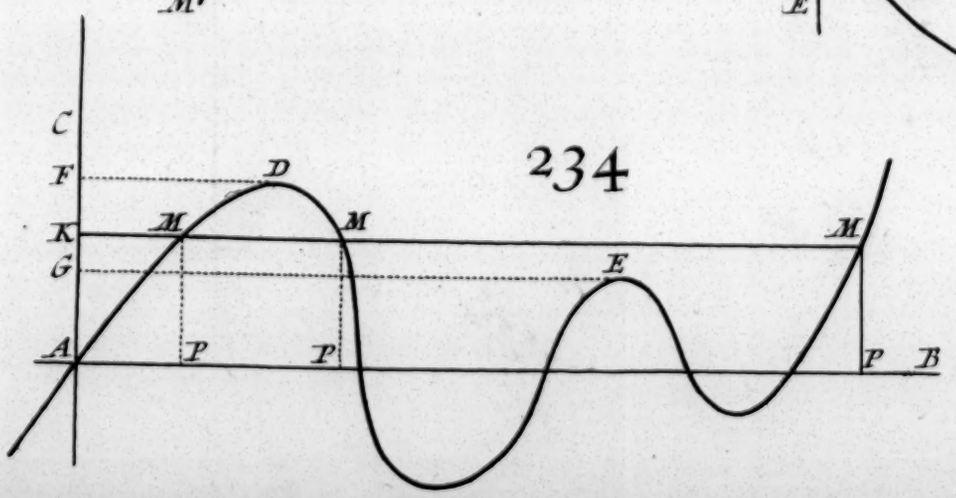

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If $2apx$ in the Equation of the third Degree be affirmative, it is manifeſt that $c^4 + 2apcc \rhd {}^{*}aagc$; from whence adding the Square $aapp$ to both Sides, and compleating the Calculus as above, there comes out $c \rhd \sqrt{-ap - a\sqrt{pp+qc}}$. But when $2apx$ is negative, by tranſpoſing $2apx$ and dividing by $x$, we ſhall have $xx = 2ap + \frac{aaq}{x}$, and ſo if in $\frac{aaq}{x}$ you put for $x$ a Value $c$ greater than the affirmative Root of the Equation $x^3 - 2apx - aaq = o$, the Quantity $2ap + \frac{aaq}{c}$ ſhall be leſs than the Square $xx$ (becauſe $\frac{aaq}{c}$ is leſs than $\frac{aaq}{x}$) and ſo, much leſs than the Square $cc$. Therefore we have $cc \rhd 2ap + \frac{aaq}{c}$, and multiplying by $cc$, there ariſes $c^4 - 2apcc \rhd aaqc$, from whence (working as has been directed) and we get $c \rhd \sqrt{ap + a\sqrt{pp+qc}}$. This being ſuppos'd, I form this Series, $\sqrt{\pm ap + a\sqrt{pp+qc}}$, $\sqrt{\pm ap + a\sqrt{pp+qf}}$, $\sqrt{\pm ap + a\sqrt{pp+qg}}$, &c. wherein $f$ expreſſes the Term $\sqrt{\pm ap + a\sqrt{pp+qc}}$ immediately going before, and moreover $g$ the Term $\sqrt{\pm ap + a\sqrt{pp+qf}}$, &c.

* *This Sign* $\rhd$ *is one of Majority.*

Now from what has been demonſtrated, it is evident that all the Terms of this Series will be greater than the exact Value of the affirmative Root $x$, and that they always approach nearer and nearer to it: So that if the Series be continued on *ad infinitum*, the laſt Term of the Series ſhall be exactly equal to the ſought Value of the unknown Quantity $x$. For let $z$ be the laſt Term, then by the Nature of the Series it is certain, that $z$ ſhall approach the neareſt to the unknown Quantity $x$ of all the other Terms, and conſequently the Term $\sqrt{\pm ap + a\sqrt{pp+qz}}$, which would follow it immediately if it was not the laſt, cannot be leſs than it; becauſe if it was leſs, it would approach the neareſt to the unknown Quantity $x$, and conſequently would be the laſt Term, which is contrary to the Hypotheſis. And it cannot be greater, becauſe it has been ſhewn that all the Terms of the Series go on diminiſhing: Therefore the ſame ſhall neceſſarily be equal to it, and ſo $z$ is $= \sqrt{\pm ap + a\sqrt{pp+qz}}$, that is, (clearing the Equation of Surds) $z^3 \pm 2apz - aaq = o$, whence it appears that $z = x$. *W. W. D.*

After the ſame way of reaſoning it may be prov'd, that if a Magnitude $c$ be taken leſs than the exact Value of $x$, all the Terms of the

ſaid Series will go on increaſing to the laſt Term, which ſhall be exactly equal to the Value of $x$. We next proceed to ſhew the manner of conſtructing the ſaid Series geometrically by right Lines and Circles.

FIG. 236, 237. Draw two indefinite right Lines $BD$, $CP$, cutting one another at right Angles in the Point $A$; and in one of them take $AB=a$, $AD=p$, both on the ſame Side the Point $A$, when $2apx$ is affirmative; and one on one Side, and the other on the other, when the ſame is negative (as is ſuppos'd in the two Figures) and in the other, take $AC=q$, $AP=c$, always on both Sides the Point $A$. This being done, on the Diameter $CP$ deſcribe a Semicircle cutting $AD$ in $E$, and in $AC$ aſſume $AF$ equal to $AE$, and in $AD$ from the Point $D$ towards the Point $A$, in the firſt Caſe, and from $D$ the contrary way, in the ſecond, lay off $DG$ equal to $DF$. Laſtly, Upon the Diameter $BG$ deſcribe a Semicircle cutting $AP$ in $Q$, I ſay $AQ=\sqrt{ap+a\sqrt{pp+qc}}$. For from the Nature of the Semicircle $CEP$, the Line $AE$ or $AF$ is $=\sqrt{qc}$; and ſince $FAD$ is a right-angl'd Triangle, the Hypothenuſe $FD$ or $DG=\sqrt{pp+qc}$, and conſequently $AG=p+\sqrt{pp+qc}$; and becauſe $BQG$ is a Semicircle, the Line $AQ$ is $=\sqrt{ap+a\sqrt{pp+qc}}$. Now calling $AQ$, $f$; and repeating the ſame Operation, uſing $AQ$ inſtead of $AP$, we ſhall find $AR=\sqrt{ap+a\sqrt{pp+qf}}$, and afterwards by means of $AR$, which call $g$, we ſhall find $AS=\sqrt{ap+a\sqrt{pp+qg}}$ repeating the ſame Operation again: So that continuing on the ſame Operation at pleaſure, you will find Lines, as $AP$, $AQ$, $AR$, $AS$, &c. approaching nearer and nearer to the exact Value of $x$, the Root of the propoſed Equation $x^3-2apx-aaq=o$.

Here you muſt obſerve, that $AP(c)$ may be taken firſt of any Length at pleaſure; for if this Length be found greater than the Root $x$, the other Lines $AQ$, $AR$, $AS$, &c. do go on always diminiſhing; and contrariwiſe, if $AP$ be leſs than $x$, thoſe Lines will go on increaſing; ſo that the true Root is contained between $AP$ of one of the two Figures, and $AP$ of the other, and $AQ$ and $AQ$, $AR$ and $AR$, $AS$ and $AS$. Whence if there be form'd two converging Series, wherein the firſt Term of one of them is greater than the true Root, and of the other, leſs; then by taking the correſpondent Terms of theſe two Series, we ſhall have always the Limits between which the Root muſt be found; ſo that the Difference of theſe Limits diminiſhes more and more *ad infinitum*.

If the two other Roots of the propoſed Equation $x^3-2apx-aaq=o$, had been demanded. Call $m$ the Root found as above by Approximation, and ſuppoſe the ſame to be the true Root; therefore if that Equation be divided by $x-m$, the Remainder will be equal to nothing

(ſince the ſame is $m^3 - 2apm - aaq = o$, and $x$ is ſuppos'd $= m$) and the Quotient ſhall be $xx + mx + mm - 2ap = o$, and if this Quotient or Equation be reſolv'd, we ſhall have the two Roots requir'd.

All Equations of the third Degree may be reduced to one or the other of thoſe two Forms; for if the ſecond Term be gotten out, and $aaq$ is affirmative; then putting $aaq$ negative will only change the affirmative Roots into negative ones, and the negative ones into affirmative ones. From whence it appears, that the aforeſaid Conſtructions are ſufficient for approximating the Roots of any given Equation of the third Degree. We now paſs on to thoſe of the fourth.

It is requir'd to approximate the Roots of the following Equation of the fourth Degree $x^4 - 3apxx - aaqx - a^3r = o$. Firſt find the Roots of this Equation of the third Degree nearly by Approximation, according to the manner before preſcrib'd, *viz.*

$$\begin{array}{r} y^3 - 3ppy + 2p^3 = o. \\ +4ary + 8apr \\ -aqq \end{array}$$

wherein you muſt obſerve to write $-2p^3$ when $+3apxx$ is in the propoſed Equation; $-4ar$ when $+a^3r$ is in the ſame; and laſtly, $-8apr$ when the Signs of the Terms $3apxx$ and $a^3r$ are different. Now one of the approximating Roots $y$ muſt be ſuppoſed to be exact, and having found a Line $v = \sqrt{ay \mp 2ap}$, viz. $+2ap$ when $3apxx$ is negative, and $-2ap$ when the ſame is affirmative; and then for the four approximating Roots of the propoſed Equation, we ſhall have thoſe of theſe two Equations of the 2d Degree $xx - vx + \frac{ay \mp ap}{2} - \frac{aaq}{2v} = o$ and $xx + vx + \frac{ay \mp ap}{2} + \frac{aaq}{2v} = o$ (obſerving to take $-ap$ when $3apxx$ is negative, and $+ap$ when the ſame is affirmative, in the propoſed Equation) which may be eaſily conſtructed by means of ſtrait Lines and Circles. All this is only a Continuation of the Rule laid down by *Deſcartes*, in the Third Book of his Geometry, for reducing any Equation of the fourth Degree into one of the third, from whence knowing one of the Roots of the propoſed Equation, we have all four; and ſince this depends upon pure Algebra, it may be here ſuppoſed as demonſtrated; yet the Demonſtration being ſhort take it as follows.

The Equation of the fourth Degree $x^4 - 3apxx - aaqx - a^3r = o$, muſt be eſteem'd as being the Product of two Planes $xx - vx + ab - ac = o$, and $xx + vx + ab + ac = o$, wherein the Letters $v$, $a$, $b$, $c$, do denote unknown Quantities that muſt be determin'd hereafter ſo, that the Product of theſe two Equations which is $x^4 \begin{matrix} -vvxx \\ +2abxx \end{matrix} - 2acvx \begin{matrix} +aabb \\ -aacc \end{matrix} = o$,

may be equal to the propoſed Equation. For this End, if the correspondent Terms be compared, we ſhall have 1. $c = \frac{aq}{2v}$. 2. $b = \frac{vv - 3ap}{2a}$. 3. $bb - c\,c = -ar$, or $bb - cc + ar = 0$; that is (putting for $b$ and $c$ the Values that we have thus found) the Equation $v^6 - 6apv^4 \begin{matrix} +9aappvv \\ +4a^3rvv \end{matrix} - a^4qq = o$. And if you make $vv = ay + 2ap$, then (by Subſtitution), will be found this Equation of the third Degree

$y^3 - 3ppy \begin{matrix} +2p^3 \\ +8apr \\ +4ary \\ -aqq \end{matrix} = o = o$, from whence knowing one Root $y$, we ſhall have the Value of $v$, by taking the ſquare Root of $ay + 2ap$, and then the Values of $b$ and $c$, which being put in the two plane Equations firſt ſuppoſed, and there will be found two other plane Equations, whoſe Product ſhall be equal to the propoſed Equation, and the Reſolution thereof ſhall conſequently furniſh the four Roots ſought. If it be required only to find one affirmative Root of an Equation of the fourth Degree, this may be done immediately by a Series, after the following Manner.

Let $x^4 \mp 2apxx - aaqx - a^3r$ be $= o$, then working after the ſame manner as for an Equation of the third Degree, and $x$ will be $= \sqrt{\mp ap + a\sqrt{qx + pp + ar}}$ from whence (making $pp + ar = nn$, for brevity's ſake) we ſhall have this converging Series $c, \sqrt{\mp ap + a\sqrt{nn + qc}}$, $\sqrt{\mp ap + a\sqrt{nn + qf}}$, $\sqrt{\mp ap + a\sqrt{nn + qg}}$, &c. whoſe Conſtruction is the ſame as that of the Series before given, only $AF$ muſt be $= n$, and $DG = FE$.

If $a^3r$ had been affirmative, then will $x$ be $= \sqrt{\mp ap + a\sqrt{qx + pp - ar}}$, and when $pp$ exceeds $ar$, the ſame converging Series as above will be found (in making $pp - ar = nn$). But it muſt be obſerved, when $2apx$ is affirmative in the given Equation, then $\sqrt{qx + pp - ar}$ muſt be greater than $p$, that ſo $\sqrt{-ap + a\sqrt{qx + pp - ar}}$ may not include a Contradiction; from whence $x$ is $> \frac{ar}{q}$ and conſequently $c$ muſt be taken greater than $\frac{ar}{q}$.

If $pp$ be leſs than $ar$, then making $ar - pp = qn$, we ſhall have this converging Series $c$, $\sqrt{\mp ap + a\sqrt{qc - qn}}$, $\sqrt{\mp ap + a\sqrt{qf - qn}}$, $\sqrt{\mp ap + a\sqrt{qg - qn}}$, &c. wherein it muſt be obſerved, that when $2apxx$ in the given Equation is negative, $x$ muſt be greater than $n$ or $\frac{ar - pp}{q}$

that

that ſo $\surd ap + a\surd\overline{qx+pp-ar}$ the Value of $x$ may not include a Contradiction, whence $c$ muſt be taken greater than $n$.

When $r$ is affirmative in the given Equation, then it may happen that all the four Roots are imaginary, in which Caſe we ſhall certainly fall into ſome Contradiction in the Conſtruction of the Series: For the Demonſtration of the converging of the Series depends upon the Suppoſition of the given Equation's having one affirmative Root. Finally, the Conſtruction of the laſt Series is ſomething different from the Conſtruction of the others, but becauſe it is not more difficult, I ſhall omit the ſame.

This Method becomes troubleſome when extended to compleat Equations, exceeding the fourth Degree; for which reaſon I ſhall content my ſelf with applying the ſame to an Equation of the fifth Degree, having only two Terms, from whence may be known how to extend the ſame to others more compound that have likewiſe but two Terms.

Let $x^5 - a^4 b$ be $= o$, then multiplying by $x$, and tranſpoſing, and there will ariſe $x^6 = a^4 bx$, and extracting the ſquare Root, we have $x^3 = aa\surd bx$, or $x^4 = aax\surd bx$, and continually extracting the ſquare Root twice more, $x$ will be $= \surd a\surd\overline{x\surd bx}$, from whence we get the following converging Series $c$, $\surd a\surd\overline{c\surd bc}$, $\surd a\surd\overline{f\surd bf}$, $\surd a\surd\overline{g\surd bg}$, &c. whoſe geometrical Conſtruction is thus.

Draw two indefinite ſtrait Lines $BD$, $CP$, cutting one another at right Angles in the Point $A$, and in one of them aſſume $AB = a$, and in the other $AC = b$, $AP = c$, on both Sides the Point $A$. This being done, upon the Diameter $PC$ deſcribe a Semicircle, cutting $BA$ in $D$, and in $AC$ take $AF = AD$, and then upon the Diameter $PF$ deſcribe another Semicircle cutting $AD$ in $E$. Laſtly, on the Diameter $BE$ deſcribe a Semicircle cutting $AP$ in $Q$; then it is plain that $AQ$ is $= \surd a\surd\overline{c\surd bc}$. Now calling $AQ$, $f$; and repeating the ſame Operation, uſing $AQ$ inſtead of $AP$, and you will find $AR = \surd a\surd\overline{f\surd bf}$, and alſo $AS = \surd a\surd\overline{g\surd bg}$. And the right Lines $AP$, $AQ$, $AR$, &c. do come nearer and nearer to the exact Value of the unknown Quantity ($x$) of the given Equation $x^5 - a^4 b = o$. This may be prov'd after the ſame manner as before, for the Equations of the third Degree. FIG. 238.

*M. Bernoulli* invented theſe Series, as may be ſeen in Page 455. of the *Acta Eruditorum*, for the Year 1689.

PROPO-

## PROPOSITION XIII.

### Problem.

424. *A Portion of a Conick Section being given, to find the Roots of a given Equation of the third and fourth Degree by means thereof.*

It appears in the preceding Problem, that an Equation of the fourth Degree being given, we can by means thereof find always one of the third, from whence knowing one Root, we have all the four of the propoſed Equation, by ſtraight Lines and Circles only. Alſo it is well known, that any Equation of the third Degree may be brought to this Form $x^3 \mp 2apx - aaq = o$, whereof one of the Roots is affirmative, and the two others negative or imaginary. This being laid down; let $x^3 \mp 2apx - aaq = o$ be an Equation of third Degree, whoſe Roots are requir'd to be found by means of the given Portion $(B\,D)$ of a Parabola, having the Line $CH$ for the Axis, and the Point $C$ the Origin thereof. From the Points $B$, $D$, the Extremities of the given Portion, draw the Perpendiculars $B\,G$, $D\,H$, to the Axis; then it is manifeſt, that if the affirmative Root was greater than $BG$, and leſs than $D\,H$, the Circle deſcrib'd about the Centre $E$, (found as directed at the End of *Art.* 387. for Equations wanting the ſecond Term) with the Radius $E\,C$, would certainly cut the Portion $B\,D$ in ſome Point $M$, from whence drawing the Perpendicular $M\,Q$ to the Axis, this Line $M\,Q$ would be the affirmative Root thereof. Now when the ſaid Circle does not cut the Portion $B\,D$, the Equation muſt be tranſform'd into another, whereof the Root may be contain'd between the Limits $BG$, $D\,H$. In order for this, call the given Lines $B\,G, f$; $D\,H, g$; and ſuppoſe we have two Limits $m$, $n$, (where $m$ is leſs than $n$, and $f$ than $g$) between which the true Root $x$ is contain'd. Then $x$ will be greater than $m$, and leſs than $n$, and multiplying every Term by $f$, and dividing by $m$, and there comes out $\frac{fx}{m}$ greater than $f$, and leſs than $\frac{fn}{m}$. Now if $z$ be made $= \frac{fx}{m}$, and inſtead of $x$ you put its Value $\frac{mz}{f}$ in the Equation $x^3 \mp 2apx - aaq = o$, this Equation will be transform'd into this here $z^3 \mp \frac{2apff}{mm} - \frac{aaqf^3}{m^3} = o$, whoſe Root $z = \frac{fx}{m}$ ſhall be greater than $f$, and leſs than $\frac{fn}{m}$. Therefore, if the Limits $m$ and $n$ be ſuch, that $\frac{fn}{m}$ be equal to, or leſs than $g$, then you need

Fig. 239.

need but conſtruct this laſt Equation by means of *Art.* 387, for having the affirmative Root $MQ$ ($z$) thereof, by means of the given Portion $BC$. From hence may be gathered the following Conſtruction.

By the laſt Problem make two converging Series approaching to the true Root $x$, of the given Equation $x^3 \mp 2apx - aag = o$, the firſt Term of one of them being leſs than $x$, and of the other greater. Chuſe two correſpondent Terms $m$, $n$, in thoſe Series being ſuch that $\frac{fn}{m}$ may be equal to or leſs than $g$: Which may be done always, becauſe $f$ is leſs than $g$, and the Difference between $m$ and $n$ does leſſen continually. This being done, transform the given Equation into another, as $z^3 + \frac{2apff}{mm}z - \frac{aaqf^3}{m^3} = o$; the unknown Quantity of which ſhall be $z = \frac{fx}{m}$; and conſtructing this by the Directions at the End of Article 387. the Circle will cut always the given Portion $BC$ in ſome Point, as $M$, from which drawing the Line $MQ$ perpendicular to the Axis, this Line ſhall be the affirmative Root $z$ of the laſt mention'd Equation; and afterwards making $x = \frac{mz}{f}$, that Line $x$ ſhall be the affirmative Root of the Equation $x^3 \mp 2apx - aaq = o$.

If you want the two other Roots of this Equation, when they are not imaginary; the Equation need only be divided by the unknown Quantity $x$ *minus*, that Quantity (which we have found) that will bring down the Equation to the ſecond Degree, the two Roots of which may be found by means of a Circle, according to *Article* 380.

All this is ſo plain, that there need be no more ſaid about it; only obſerve, that if the given Portion $BD$ were an Ellipſis or Hyperbola, the 398th or 403d Articles muſt have been uſed; and then all the Difficulty would be in transforming the given Equation into another, wherein the affirmative Root ſhould have given Limits; and this might be done as above in the Parabola.

*The End of the Ninth Book.*

BOOK

# BOOK X.

## *Of Determinate* PROBLEMS.

### *A General Propoſition.*

425. *A Determinate Geometrical Problem being propos'd, to find the Solution thereof.*

The propoſed Problem muſt be eſteem'd firſt as if it was reſolv'd, and ſuch Lines as is judg'd moſt convenient for the Solution thereof muſt be drawn. Then name all theſe Lines, (which commonly do form right-angl'd, or ſimilar Triangles) with Letters of the Alphabet, *viz.* the known Lines with the initial Letters, and the unknown ones with the ultimate ones; and go thro' all the Conditions of the Problem, comparing thoſe Lines to one another in the moſt ſimple and natural Order poſſible; that thereby the ſame Number of Equations may be form'd, as there are unknown Quantities. Laſtly, By common Algebra reduce theſe different Equations to one only, wherein there is but one unknown Quantity, and, if poſſible, bring it down to a lower Degree; and ſolve the ſame by the Directions in the laſt Book, and then the Problem will be anſwered. All this will appear by the following Examples.

### EXAMPLE I.

Fig. 240. 426. THE right Line $AB$ being given, to find the Point $C$, out of that Line, ſuch that the right Lines $AC$, $CB$ being drawn; (1.) The Sum of their Squares may be to the Triangle $ABC$ in the given Ratio of $f$ to $g$, (2.) And the Angle $ACB$ contain'd under thoſe Lines may be equal to the given Angle $GDK$.

Suppoſe the Point $C$ to be that ſought, and draw $CH$ perpendicular to $AB$, which biſect in the Point $E$. Now call the given Line $AE$ or $EB$, $a$; and the unknown ones $BH$, $x$; $HC$, $y$; and then $AH$ is $= a - x$, and $BH = a + x$. Then becauſe $AHC$, $BHC$, are right-angl'd Triangles $\overline{AC}^2$ ſhall be $= aa - 2ax + xx + yy$, and $\overline{BC}^2 = aa + 2ax$

$2ax+xx+yy$; and conſequently $\overline{AC}^2+\overline{BC}^2=2aa+2xx+2yy$. But ſince the Triangle $ACB=AE\times CH\,(ay)$; therefore by the firſt Condition of the Problem it follows that $2aa+2xx+2yy : ay :: f : g$; from whence multiplying the Extremes and Means, and dividing by $2g$, we get this Equation $aa+xx+yy=\frac{af}{2g}y=2my$, in taking a Line $m=\frac{af}{4g}$ that ſo the Equation may be cleared of Fractions.

The next thing is to accompliſh the ſecond Condition, *viz.* That the Angle $ACB$ be equal to the given Angle $GDG$. To do which, from the Point $G$ taken at pleaſure in the right Line $GD$, draw the right Line $GF$ perpendicular to the Side $DK$, produced, if neceſſary, and from the Point $A$ the Perpendicular $AL$ to the Side $BC$ (alſo produced, if neceſſary) that ſo we may have two right-angl'd ſimilar Triangles $ACL$, $GDF$, whereof $GDF$ is given. This being done, call the given Lines $DF, b$; $FG, c$; and make $BC=n$, for Brevity's Sake; then becauſe the right-angl'd Triangles $BCH$, $BAL$, are ſimilar, therefore $BC\,(n) : CH\,(y) :: BA\,(2a) : AL=\frac{2ay}{n}$. And $BC\,(n) : BH\,(a+x) :: BA\,(2a) : BL=\frac{2aa+2ax}{n}$. And conſequently $CL$ or $BL-BC=\frac{2aa+2ax-nn}{n}$. Whence becauſe the Angle $ACL$ muſt be equal to the Angle $GDF$, therefore the following Quantities muſt be Proportionals, *viz.* $CL\left(\frac{2aa+2ax-nn}{n}\right) : AL\left(\frac{2ay}{n}\right) :: DF\,(b) : FG\,(c)$; and ſo multiplying the Means and Extremes, and then $2aac+2acx-cnn=2aby$ that is (ſubſtituting $aa+2ax+xx+yy$ for $nn$) $aac-cxx-cyy=2aby$, which includes the ſecond Condition of the Problem. Now becauſe we have found the ſame Number of Equations as there are unknown Quantities, and ſince all the Conditions of the Problem are taken in; the next thing to be done muſt be to reduce theſe Equations by the common Rules of Algebra to one Equation only, wherein there may be but one of the unknown Quantities $x$ or $y$. This may be done thus: One of the Equations is $aa+xx+yy=2my$, and the other $aac-cxx-cyy=2aby$, or $aa-xx-yy=\frac{2aby}{c}$; which being added together, and we get $2aa=\frac{2aby}{c}+2my$, and ſo $y$ is $=\frac{aa}{m+f}$ taking $f=\frac{ab}{c}$. And ſubſtituting this Value for $y$, and its Square for $yy$ in the ſaid Equations, and then $xx$ will be $=\frac{aamm}{}$

$\frac{aamm-aaff-a^4}{mm+2mf+ff}$ and $x = \frac{a\sqrt{mm-ff-aa}}{m+f}$; from whence it appears, if $mm$ be leſs than $aa+ff$, the Problem ſhall be impoſſible. Now the Conſtruction of the Problem is after the following manner.

FIG. 241. Through $E$ the Middle of $AB$ draw the indefinite right Line $ON$ perpendicular to $AB$, and through the Point $A$ draw the Line $AM$ making with $AB$ the Angle $EAM$ equal to the Angle $DGF$, which is given. About the Point $M$, wherein $AM$ meets the Perpendicular $ON$, as a Centre, with the Radius $MA$, deſcribe the Arc of a Circle $ACB$. Then in $EM$ produced towards $M$, aſſume $MN = m$; and join $NA$, and draw $AO$ perpendicular thereto, meeting $NO$ in the Point $O$, through which draw a right Line parallel to $AB$. I ſay, this Parallel ſhall meet the Arc of the Circle $ACB$ in the ſought Point $C$.

For drawing $CH$ perpendicular to $AB$, then will $CH = EO$ be $= \frac{aa}{m+f}$, becauſe the right-angl'd Triangles $NEA$, $AEO$, being ſimilar, we have this Proportion, *viz.* $NE\ (m+f) : AE\ (a) :: AE\ (a) : EO = \frac{aa}{m+f}$. And (by the Nature of the Circle) $\overline{CM}^2 = \overline{AM}^2 = aa+ff$; and ſo ſince $MO$ is $= f + \frac{aa}{m+f}$, and the Triangle $MCO$ is right-angl'd, therefore $\overline{CO}^2$ or $\overline{EH}^2$ $(xx)$ is $= a\ a + ff - ff - \frac{2aaf}{m+f} - \frac{a^4}{mm+2mf+ff} = \frac{aamm-aaff-a^4}{mm+2mf+ff}$. Therefore, &c.

## SCHOLIUM.

427. IF after all the Conditions of a Problem be taken in, there ariſes two Equations, in each of which are contain'd both the unknown Quantities; there is no Neceſſity of reducing both theſe Equations to one, wherein is only one unknown Quantity, as is preſcrib'd in the general Propoſition; but the Problem may be reſolv'd by conſtructing ſeparately the Loci of theſe two Equations, for by means of the Interſections of the two Loci ſhall the Values of the two unknown Quantities be found. This appears evident in this Example, wherein the right Lines $EH\ (x)$, $HC\ (y)$ forming the right Angle $EHC$, are taken for the unknown Quantities; and by the Conditions of the Problem are gotten theſe two Equations, $aa+xx+yy=2my$, and $aa-xx-yy=2fy$; for the Interſections of the Circles which are the Loci thereof, will be the Points ſought. Now theſe Circles may be deſcrib'd thus.

The

The circular Arc $ACB$ being deſcrib'd as in the former Conſtruction, about the Centre $A$ with a Radius $AP = m$, deſcribe an Arc, cutting the Perpendicular $EM$ in $P$. In this Perpendicular aſſume $E\mathcal{Q} = m$ from $E$ towards the Arc $ACB$, and about the Centre $\mathcal{Q}$ with the Radius $\mathcal{Q}C = EP$, deſcribe a Circle which ſhall cut the Arc $ACB$ in Points $(C)$, which will be thoſe requir'd.

For from the Nature of this laſt Circle $\overline{\mathcal{Q}C}^2$ or $\overline{EP}^2$ $(mm - aa)$ is $= \overline{\mathcal{Q}O}^2$ $(mm - 2my + yy) + \overline{OC}^2 (xx)$ that is, the firſt Equation $aa + xx + yy = 2my$; and from the Nature of the other Circle $ACB$, we get $\overline{MC}^2$ or $\overline{MA}^2$ $(ff + aa) = \overline{MO}^2 (ff + 2fy + yy) + \overline{OC}^2 (xx)$, that is, the ſecond Equation $aa - xx - yy = 2fy$. Therefore the ſought Point $C$ will be in both theſe two Arcs, that is, it will coincide with the Point of Interſection of them.

Hence it appears that there are two different Points $(C)$ that will anſwer the Problem, when the two Circles cut one another, as in the Square: When they touch one another there will be but one; and when they neither cut or touch, then the Problem will be impoſſible.

In ſolving a Problem by two Loci, care muſt be taken that one does not fall into a more compound Conſtruction, than that of one Equation having only one unknown Quantity $x$ therein. As in this Example, ſuppoſe it be requir'd to ſolve a Problem, (which is the third following) whoſe Conditions are contain'd in theſe two Equations, $y = \frac{cd - cx}{b}$, and $\frac{bb}{ff} yy = aa + xx$: If the Loci of theſe two Equations be uſed for determining the Problem, then it is plain that a ſtrait Line muſt be drawn * for the Locus of the firſt Equation, and an Hyperbola * for the Locus of the ſecond, that ſo the Values of the unknown Quantities $x$ and $y$, may be determin'd by their Interſections. But becauſe in bringing theſe two Equations into one, this Equation of the ſecond Degree will be had, *viz.* $xx - \frac{2ccd}{cc + ff} x + \frac{ccdd - aaff}{cc - ff} = 0$, which may be conſtructed by ſtraight Lines and Circles only; therefore it will be a conſiderable Fault to uſe an Hyperbola.

* *Art.* 306.

* *Art.* 330, *and* 332.

## EXAMPLE II.

428. THE Square $ABCD$ being given; it is requir'd to draw the right Line $AE$ from the Angle $A$ thereof, ſo that $FE$ the Part of this Line contain'd between the Side $BC$, and the Side $DC$ (continu'd out) be equal to a given Line $b$. FIG. 242.

Let us ſuppoſe the Point $E$ taken upon the Side $DC$ (produced) be ſuch that $FE$ the Part of the Line $AE$ be equal to a given Line $b$, that is, let us ſuppoſe the Queſtion to be ſolved, and call the given Lines $AB$, or $AD$, or $DC$, or $CB$, $a$; and the unknown Line $DE$, $x$. This being done, becauſe the Triangles $EDA$, $ECF$, are ſimilar, therefore $ED(x) : DA(a) :: EC(x-a) : CF = \frac{ax-aa}{x}$, and ſince $ECF$ is a right angl'd Triangle, therefore $\overline{FE}^2 = \overline{EC}^2 + \overline{CF}^2 = xx - 2ax + aa + \frac{aaxx-2a^3x+a^4}{xx}$. But becauſe (by the Conditions of the Problem) $EF$, muſt be equal to $b$, therefore $xx - 2ax + aa + \frac{aaxx-2a^3x+a^4}{xx} = bb$, or $x^4 - 2a^3x + 2aaxx - bbxx - 2a^3x + a^4 = 0$. Whence if this Equation be ſolved, the Value of $DE$ $(x)$ ſhall be ſuch, that if the right Line $AE$ be drawn, the Part $(FE)$ thereof comprehended between the Side of the Square $CB$, and the Side $CD$ continued out, will be equal to a given Line $b$.

Becauſe the Equation here found is one of the fourth Degree, a Conick Section muſt be uſed in the Conſtruction thereof. Therefore it will be firſt neceſſary to try whether the Equation cannot be brought down to a lower Degree, by the Rules of Algebra; for this End we find that if $cc$ be taken $= aa + bb$, the ſame ſhall be the Product of theſe two Equations of the ſecond Degree $xx + aa - ax - cx = 0$, and $xx + aa - ax + cx = 0$; ſo that the four Roots of the Equation of the fourth Degree $x^4 - 2ax^3$, &c. may be had by finding the Roots of each of theſe two Equations, I ſhall not here ſhew how to find the Roots of the Equation $xx + aa - ax + cx = 0$; becauſe $c$ being greater than $a$, the Diſpoſition of the Signs ſhews that the Roots are both negative: But the Roots of the other Equation $xx + aa - ax - cx = 0$ being both affirmative, may be determin'd thus.

In the Side $AB$ produced, aſſume $BG = c$, and upon the Diameter $AG$ deſcribe a Semicircle cutting the Side $DC$ (produced) in $E$. I ſay this Point ſhall be that ſought.

For calling $DE$, $x$; and drawing the Perpendicular $EH$, then we ſhall have $HG = a + c - x$, and by the Property of the Circle $AH \times HG\ (ax + cx - xx) = \overline{EH}^2 = aa$.

## SCHOLIUM I.

429. IF after having brought a Problem to an Equation, the ſame is a compound one having ſeveral real Roots, then it is plain that there is but one of thoſe Roots that expreſſes the Value of the unknown Quantity ſought: But it muſt be well obſerv'd, that all the other Roots may ſerve in ſomewiſe likewiſe to ſolve the Problem; neither

neither will they be different from the Root ſuppoſed, but only in ſome particular Circumſtances. So in this Example the little affirmative Root $DL$ $(x)$ of the Equation $xx-ax-cx+aa=o$, gives the Point $L$, upon the Side $DC$, ſuch that the right Line $AL$ being drawn, meeting the Side $BC$ in $K$, the Part $AL$ thereof is equal to the given Line $b$. Moreover, if you take $Bg=c$ in the Side $BA$ produced towards $A$, then if a Semicircle be deſcrib'd upon the Diameter $Ag$, this will cut the Side $CD$ produced towards $D$ in the Points $e$, $l$, ſo that $De$, $Dl$, ſhall be the two negative Roots of the Equation $xx+cx-ax+aa=o$: And if the right Lines $Ae$, $Al$, be drawn meeting the Side $CB$ produced in the Points $f$, $k$, then each of the right Lines $ef$, $lk$, ſhall be yet equal to the given Line $b$. From whence it appears, that notwithſtanding the Value of $DE$ only was what we requir'd in the Problem, yet we have found an Equation whoſe Roots have furniſh'd us with other Values $DL$, $De$, $Dl$, all of which will in ſome wiſe anſwer the Problem.

## SCHOLIUM II.

430. IF there be a Suſpicion that an Equation taking in the Conditions of a Problem may be brought down to a lower Degree, it will be proper to try other ways different from theſe we have followed, even when they appear leſs natural; becauſe it may often happen, that they lead us to more ſimple Equations; and otherwiſe, ſince it is difficult to bring down Compound Equations to lower Degrees. We ſhall now proceed to ſolve the aforeſaid Problem two other ways, that ſo what we have here ſaid may be underſtood.

Suppoſe the Problem to be reſolved, and draw $EG$ perpendicular to $AE$ meeting the Side $AB$ produced in $G$, and let $AF=y$ and $BG=z$, be the unknown Quantities. This being done, becauſe the right-angl'd Triangles $ABF$, $AEG$, are ſimilar, therefore $AB$ $(a)$: $AF$ $(y)$:: $AE$ $(y+b)$ : $AG$ $(a+z)$. And therefore $yy+by=aa+az$. And ſince there are two unknown Quantities, and the Problem is determinate, there muſt be another Equation ſought. To find which, we muſt conſider that $EG$ is $=AF$ $(y)$; for drawing $EH$ perpendicular to $AG$, the right-angl'd Triangle $EHG$ is ſimilar and equal to the right-angl'd Triangle $ABF$, becauſe the Homologous Sides $AB$, $EH$, are equal to one another. Therefore (becauſe $AEG$ is a right-angl'd Triangle) $aa+2az+zz$ is $=yy+2by+yy=2yy+2by+bb$, in which Equation ſubſtituting, for $2yy+2by$, its Value $2aa+2az$, found by means of the firſt Equation, and there will ariſe $aa+2az+zz=2aa+2az+bb$, which may be reduced to this very ſimple Equation, *viz.* $z=\sqrt{aa+bb}$, which firſt furniſhes the ſame Conſtruction as above. FIG. 242.

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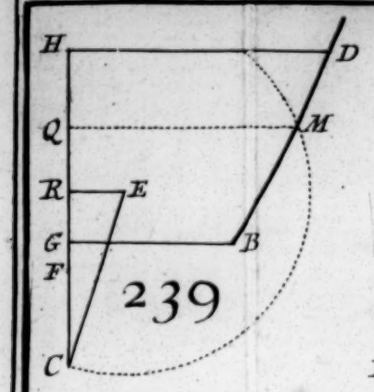
*Another Way.*

FIG. 243. The following Way is particular in this, *viz.* That it ſucceeds equally whether the Figure $ABCD$ be a Square, or a Rhumbus. Through the ſought Point $F$, as given, draw the Line $FG$ making the Angle $AFG$ with $AF$ equal to the given Angle $ACE$, and meeting the Diagonal $AC$ produced in the Point $C$; then we ſhall have three Triangles $ACE$, $AFG$, $GCF$, which will be ſimilar. For 1. The Angle at $A$ being common to both the Triangles $ACE$, $AFG$, and the Angles $ACE$, $AFG$, being equal by the Hypotheſis, it is manifeſt that theſe two Triangles ſhall be ſimilar. 2. The Triangle $ADC$ being Iſoſcelles, the Angle $DCA$, or $ECG$, ſhall be equal to the Angle $DAC$, or $ACF$; and adding the ſame Angle $FCE$ to both Sides, then the Angle $FCG$ ſhall be equal to the Angle $ACE$ or $AFG$; and therefore becauſe the Angle at $G$ is common, the two Triangles $AGF$, $FGC$, ſhall be ſimilar. Now let the unknown Quantities $CE$ and $AG$ be equal to $x$ and $z$, and the given Quantities $DC=a$, $FE=b$, $AC=c$; then ſince $AD$, $CF$, are parallel, therefore $CE\,(x) : FE\,(b) :: CD\,(a) : AF=\frac{ab}{x}$. And becauſe the Triangles $ACE$, $AFG$, $GCF$, are ſimilar, therefore $AC\,(c) : CE\,(x) :: AF\left(\frac{ab}{x}\right) : FG=\frac{ab}{c}$. And $AG\,(z) : FG\left(\frac{ab}{c}\right) :: \frac{ab}{c} : CG\,(z-c)$. Whence multiplying the Means and Extremes, and we ſhall get this Equation $zz-cz=\frac{aabb}{cc}$ and by this means the following Conſtruction.

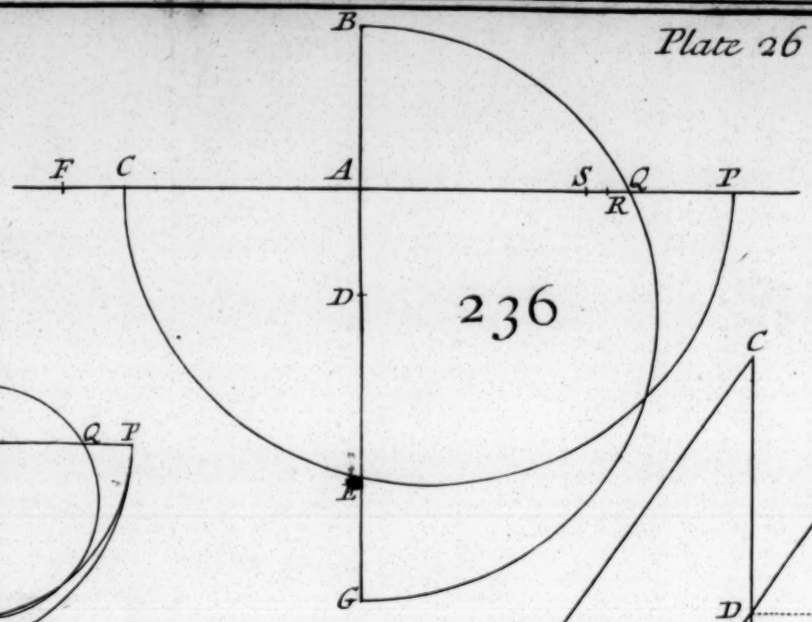
FIG. 243. From the Point $A$ draw the Line $AH=\frac{ab}{c}$, perpendicular to $AC$, and through $L$ the middle of the Diagonal $AC$ draw the Line $HL$, and in the Diagonal produced beyond $C$, take $LG$ equal to $LH$. Then if an Arc of a Circle be deſcrib'd about the Centre $G$, with the Radius $GF=AH$, this ſhall cut the Side $BC$ in the Point $F$ ſought. This is evident ſince (by Conſtruction) $zz-cz$ is $=\frac{aabb}{cc}$, and $GF=\frac{ab}{c}$.

## EXAMPLE III.

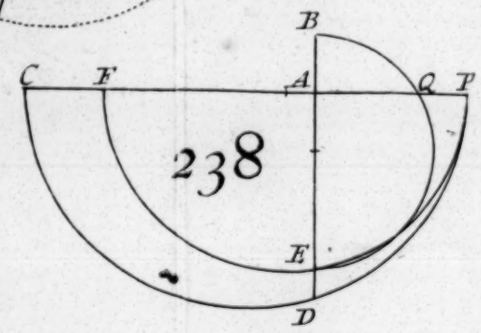
FIG. 244. 431. UPON the indefinite right Line $DE$ given in Poſition, to find two Points $D$, $E$; from which the right Lines $DO$, $OE$, $DC$, $CE$, being drawn to the two Points $O$, $C$, given without that Line, the Angle $DOE$ may be a right Angle, and the Angle $DCE$ equal to a given Angle $TPS$.



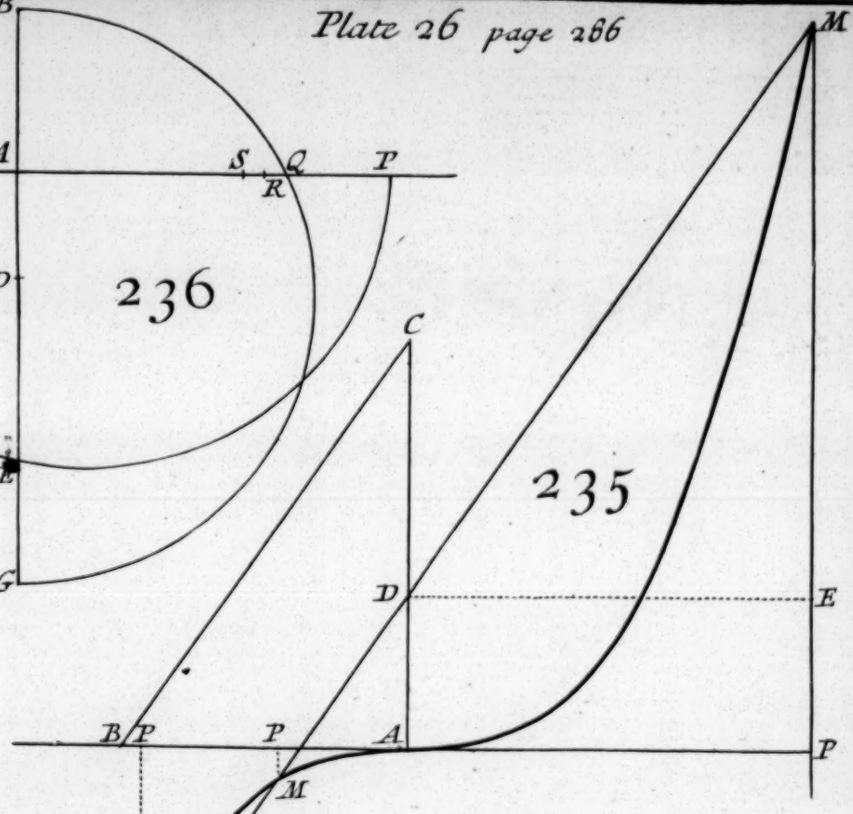
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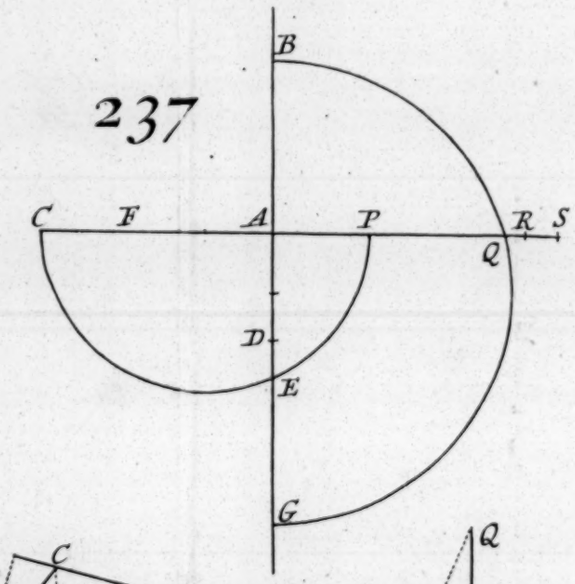
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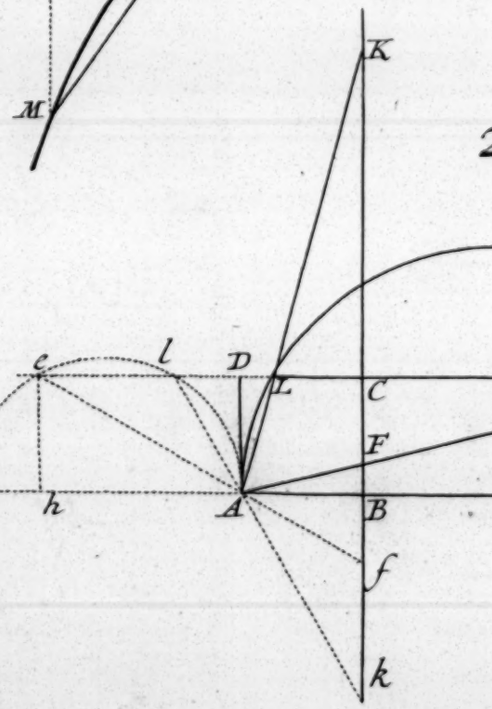
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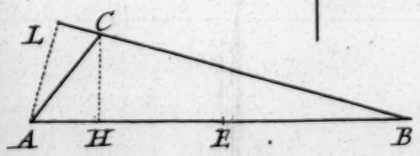
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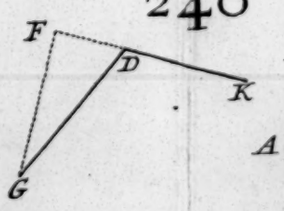
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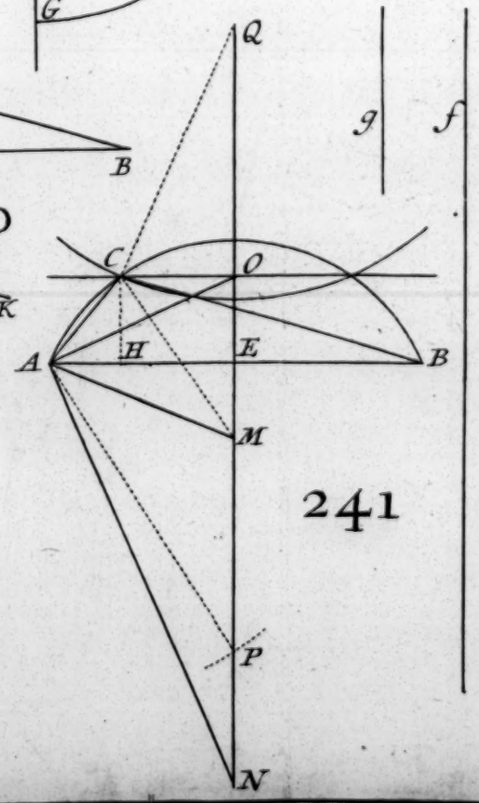


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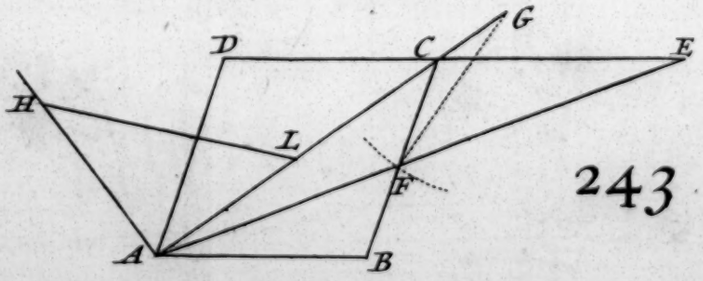


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Let us ſuppoſe the thing done, and upon the Diameter $DE$ deſcribe a Semicircle paſſing through the Point $O$, becauſe the Angle $DOE$ is a right Angle; and upon $DE$, as a Chord, deſcribe the Segment of a Circle containing an Angle equal to the given Angle, whoſe Arc ſhall neceſſarily paſs through the Point $C$. From the Point $H$, the Centre of a Circle, whereof this Segment is a Part, as likewiſe from the Points $O, C$, draw the right Lines $HK$, $OA$, $CB$, perpendicular to $DE$, and call the given Lines $OA, a$; $CB, b$; $AB, c$; and the unknown ones $AK, x$; $KH, y$. Now by the Elements of Geometry it is manifeſt, 1. That the Point $K$ is the Middle of the Line $DE$, and conſequently is the Centre of the Semicircle $DOE$. 2. That if the Line $P\mathcal{Q}$ be drawn from $P$, the Centre of the given Angle $TPS$ perpendicular to one of the Sides $PT$, the Angle $\mathcal{Q}PS$ form'd by that Perpendicular and the other Side $PS$, ſhall be equal to the Angle $KEH$. And becauſe the Triangles $KAO$, $HKE$ are right-angl'd, therefore the Square $\overline{KO}^2$ or $\overline{KE}^2$ is $= aa + xx$, and $\overline{HE}^2 = aa + xx + yy$: But producing $HK$ until it meets the right Line $CR$ drawn parallel to $DE$ in the Point $R$, then becauſe $CRH$ is a right-angl'd Triangle, the Square $\overline{CH}^2$ will be $= bb + 2by + yy + cc + 2cx + xx$. Therefore, ſince the Lines $HE, HC$, are Radii of the ſame Circle, by making an Equality between their Values, and we ſhall have this Equation, $aa + xx + yy = bb + 2by + yy + cc + 2cx + xx$, then ſtriking out $yy + xx$ from both Sides, and making $\frac{aa - bb - cc}{2c} = d$, (for Brevity's Sake) and $y$ will $= \frac{cd - cx}{b}$.

If Regard be had to the Way we have taken for getting the aforeſaid Equation, it will appear that the Circles deſcrib'd about the Centres $K, H$, with the Radii $KO, HC$, do meet each other upon the Line $DE$ in the ſame Points $D, E$; ſo that what remains to be done, is only to order it ſo, that the Angle $KEH$ be equal to the Angle $\mathcal{Q}PS$. Whence

In the Line $P\mathcal{Q}$ aſſume $P\mathcal{Q}$ equal to $CB$, and draw $\mathcal{Q}S$ parallel to the Side $PT$, and bounded by the other Side $PS$; then it is evident that the right-angl'd Triangle $EKH$ ought to be ſimilar to the right-angl'd Triangle $P\mathcal{Q}S$, and ſo calling the given Line $\mathcal{Q}S, f$, and we ſhall get this Proportion, $EK\ (\sqrt{aa + xx}) : KH\ (y) :: P\mathcal{Q}\ (b) : \mathcal{Q}S\ (f)$, therefore $y$ is $= \frac{f}{b}\sqrt{aa + xx} = \frac{cd - cx}{b}$. Now ſquaring both Sides to get out the Surds, and ordering the Equation, and then we ſhall have this Equation, $xx - \frac{2ccd}{cc - ff}x + \frac{ccdd - aaff}{cc - ff} = o$, the Value of one Root of which will determine $AK\ (x)$; ſo that if a Circle

Circle be deſcrib'd about the Centre $K$, with the Radius $KO$, it will cut the Line $DE$ in the two Points ſought $D, E$.

The Roots of the ſaid Equation may be found by the 380th or 382d Articles: But though the Methods laid down in them are very ſimple, conſidering their being general, yet there may be very often more eaſy Conſtructions found by a due Conſideration of the Nature of a Problem. For Example: It may be here obſerv'd, 1. That if through $F$, the Middle of the Line $OC$ joining the two given Points, there be drawn the Perpendicular $FG$, meeting the Line $DE$ given in Poſition in the Point $G$, then will $AG$ be $= d$; for calling $AG, z$; the right-angl'd Triangles $GAO$, $GBC$, will give $GO = zz + aa$, and $\overline{GC}^2 = zz + 2cz + cc + bb$, and making an Equality between theſe two Values, becauſe they are equal, ſince the Point $G$ is in the Perpendicular $FG$, which does biſect the Line $OC$, therefore $zz + aa$ is $= zz + 2cz + cc + bb$, and ſo $AG$ $(z)$ is $= \frac{aa - bb - cc}{2c} = d$. 2. That the Equation $f\sqrt{aa + xx} = cd - cx$ taking in the Conditions of the Problem, may be reduced to this Proportion, $GK\ (d - x) : KO\ (\sqrt{aa + xx}) :: \mathfrak{Q}S\ (f) : AB\ (c)$: So that if the Locus of all the Points $K$ be deſcrib'd * ſuch, that the right Lines $KG$, $KO$, being drawn to the given Points $G, O$, may be always to one another in the given Ratio of $\mathfrak{Q}S$ to $AB$; then ſhall this Locus cut the Line $DE$ in the Point $K$ ſought. From whence may be gotten the following very eaſy Conſtruction.

* *Art.* 350.

Through $F$ the Middle of the Line $OC$, which joins the two given Points, draw the Perpendicular $FG$, meeting the Line $DE$ given in Poſition in the Point $M$, ſo that $GM : MO :: \mathfrak{Q}S : AB$. And produce $GM$ to the Point $N$, ſo that $GN : NO :: \mathfrak{Q}S : AB$. This being done, with $MN$ as a Diameter deſcribe a Circle, which ſhall cut the Line $DE$ in the Point $K$, about which (as a Centre) if a Circle be deſcrib'd, this Circle will meet the Line $DE$ in the ſought Points $D, E$.

Becauſe the Circle, whoſe Diameter is $MN$, does cut the Line $DE$ not only in the Point $K$, but likewiſe in the Point $L$; therefore the ſame Uſe may be made of the Point $L$, as of $K$, for finding two other Points upon the Line $DE$, which will anſwer the Problem: Whence it appears, that the Problem may have two Solutions.

If the Angle $DCE$ be a right Angle as well as $DOE$, then it is manifeſt that $\mathfrak{Q}S\ (f)$ will become equal to nothing, and ſo the Equation $f\sqrt{aa + xx} = cd - cx$ will be chang'd into this here, $cd - cx = o$, and ſo $x$ is $= d$, that is, the Centre $K$ falls then in the Point $G$. And if the Point $B$ be ſuppos'd to fall in the Point $A$, the Equation

$f\sqrt{}$

$f\sqrt{aa+xx} = cd - cx$ will become this $f\sqrt{aa+xx} = \frac{aa-bb}{2}$, by ſubſtituting $\frac{aa-bb-cc}{2}$ for $cd$ its Value, and afterwards ſtriking out all the Terms affected with $c$ (which in this Caſe become equal to nothing) and ſo in this Caſe, if an Arc of a Circle be deſcrib'd about the Point $O$, as a Centre, with the Radius $OK = \frac{aa-bb}{2f}$, it ſhall cut the Line $DE$ in the ſought Point $K$. This agrees exactly with the 66th, 67th, and 68th Articles, and the general Conſtruction may ſerve without any more to do in an Ellipſis, two of whoſe Conjugate Diameters are given, to find two other Conjugate Diameters that may make a given Angle with each other; and this is what was referr'd in the 65th Article to this Place.

## EXAMPLE IV.

432. THREE Points $A$, $B$, $C$, being given, to find ſome fourth Point $M$; from which, if the three right Lines $MA$, $MB$, $MC$, be drawn to thoſe three Points, the Differences between one of them and each of the two others ſhall be given. FIG. 245.

This Problem is capable of three Caſes. For either all the three Lines $MA$, $MB$, $MC$, are equal between themſelves, or only two of them are equal; or elſe they are all three unequal.

*Caſe* 1. When the three Lines $MA$, $MB$, $MC$, are all equal to one another; or, which is all one, when the two Differences are nothing; then it is manifeſt, that the Point $M$ ſought ſhall be the Centre of a Circle paſſing through the given Points $A$, $B$, $C$.

*Caſe* 2. When two of the given Lines $MA$, $MB$, $MC$, as $MA$, $MB$, muſt be equal to one another; or (which is the ſame thing) when one of the Differences is nothing, then through the given Point $C$ draw the right Line $CO$ perpendicular to the Line $AB$ joining the two other given Points $A$, $B$; and from the Point $M$ (ſuppos'd to be that ſought) draw the right Lines $MP$, $MQ$, parallel to $CO$, $OB$; then it is plain that $AP$ ſhall be equal to $PB$, becauſe $AM$ muſt be equal to $MB$. Now calling the given Quantities $AP$ or $PB$, $a$; $OP$, $b$; $Oc$, $c$; $AM - MC$, $f$; and the unknown Quantities $AM$, $PM$, $z$, $y$; then the right-angl'd Triangles $APM$, $MQC$, will give theſe two Equations, $zz = aa + yy$, and $zz - 2fz + ff = cc - 2cy + yy + bb$; and orderly ſubſtracting each Side of the latter Equation from each Side of the former, and then will $2fz - ff = aa - cc + 2cy - bb$, and from hence we get this Proportion, $z : y + \frac{aa-bb-cc+ff}{2c} :: c : f$; and ſo the following Conſtruction. FIG. 246.

Through $P$ the Middle of the Line $AB$ draw the Perpendicular $PD = \frac{aa-bb-cc+ff}{2c}$. Divide the Hypothenuſe $AD$, produced towards that Part as is neceſſary, in the Points $E, F$; ſo that $AE : ED :: c : f$, and $AF : FD :: c : f$. Then if a Circle be deſcrib'd with $EF$ for a Diameter, the ſame ſhall cut the Line $PD$ in the Point $M$ ſought.

*Art. 350. For drawing the right Line $MA$, then it is plain * (by the Nature of the Circle $EMF$) that $AM\,(z) : MD\,(y + \frac{aa-bb-cc+ff}{2c} :: c : f$; and $zz$ is $= aa + yy$, ſince $PM$ is perpendicular to $AB$. And becauſe theſe two Equations take in the Conditions of the Problem; therefore, *&c.*

If through the other Point $N$, wherein the Line $DP$ meets the Circumference, there be drawn the right Lines $NA$, $NB$, $NC$; then ſhall $NA$, $NB$, be equal to one another, and the Difference between each of theſe Lines and the third Line $NC$, ſhall be equal to the given Quantity $f$; ſo that the Point $N$ ſhall alſo anſwer the Problem, but with this Difference, that $NC$ is the greateſt of the three right Lines $NA$, $NB$, $NC$, whereas $MC$ is the leaſt of the three Lines $MA$, $MB$, $MC$.

FIG. 247. This ſecond Caſe may be reſolv'd yet without any Calculus, thus. Suppoſe (as before) that $M$ be the Point ſought, and draw the right Lines $MA$, $MB$, $MC$; and about the Centre $C$, with the Radius $CD = MA - MC$, deſcribe a Circle $DEKFH$. From the Point $D$, wherein the Line $MC$ meets this Circle, draw the right Lines $DA$, $DB$, to the two given Points $A$, $B$, meeting the Circle in the Points $E, F$, through which Points draw the Radii $EC$, $CF$, and the Chord $EF$. This being done, becauſe $MC + CD$, or $MD$ is $= MA$, and ſince the Lines $CD$, $CE$, are Radii of the ſame Circle; therefore the Triangles $DMA$, $DCE$, ſhall be Iſoſcelles; and conſequently ſimilar, ſince the Angle at $D$ is common; therefore the Lines $CE$, $MA$, ſhall be parallel. After the ſame manner we prove, that the Lines $CF$, $MB$, ſhall be alſo parallel; and ſo $DA : DE :: DM : DC :: DB : DF$. Hence it appears, that the whole Difficulty is brought to this, *viz.* to find the Point $D$ in the Circumference of the Circle $DEKFH$ being ſuch, that if the right Lines $DA$, $DB$ be drawn meeting the Circumference in the Points $E, F$; the Chord $EF$ may be parallel to the Line $AB$. And this may be done thus.

From the Point $C$ deſcribe a Circle, having a Line $CD = AM - MC$ for a Radius, and draw $AC$ meeting this Circle in the Points $K$, $H$; in the Line $AB$ aſſume $AG$ a fourth proportional to $AB$, $AH$, $AK$; and from the Point $G$ draw the Tangent $GE$ to the Circle $EDHFK$. Then if the right Line $AE$ be drawn through the Point of Contact $E$, meeting the Circle in the Point $D$, and the Line $DC$ be

be drawn, and if the Point $M$ be ſo taken upon $DC$, that $DM:DC::DA:DE$. I ſay the Point $M$ will be that ſought.

For by the Nature of the Circle $DEKFH$, the Rectangle $HA\times AK$ is $=DA\times AE$: And conſequently $BA:AD::AE:AG$: therefore the Triangles $DAB$, $GAE$, having the Angle $A$ common, and the Sides about this Angle reciprocally proportional, ſhall be ſimilar. And ſo the Angle $AEG$ ſhall be equal to the Angle $ABD$; but this Angle $AEG$ being form'd by the Tangent $EG$, and the Chord $DE$ produced, is meaſur'd by half the Arc $DE$. Whence, drawing the Chord $EF$ through the Point $F$ wherein the Line $BD$ meets the Circumference, that Angle ſhall be equal to $DFE$; and conſequently the Lines $FE$, $AB$, ſhall be parallel to one another. But by Conſtruction $DC:DM::DE:EA::DF:FB$. Therefore the Triangles $DMA$, $DMB$, ſhall be Iſoſcelles; becauſe the Triangles $DCE$, $DCF$, being ſimilar to them, are Iſoſcelles. Whence the Lines $AM$, $MB$, ſhall be each equal to $DM$, and ſo equal to one another; and beſides $AM$, or $DM$ ſhall be greater than $MC$ by the given Length $CD$. And this is what was propoſed.

*Caſe* 3. When the three Lines $MA$, $MB$, $MC$, are unequal. From the given Point $C$ draw the Line $CO$ perpendicular to the Line $AB$, which does join the two other given Points; and from the Point $M$, which ſuppoſe to be that requir'd, draw the right Lines $MP$, $MQ$, perpendicular to the Lines $AB$, $CO$. This being done, call the given Quantities $AO$, $a$; $OB$, $b$; $CO$, $c$; $AM-MB$, $d$; $AM-MC$, $f$; and the unknown Quantities $OP$, $x$; $PM$, $y$; $AM$, $z$; then will $AP$ be $=a+x$, $BP=b-x$, $CQ=c-y$, $BM=z-d$, $CM=z-f$. Now by means of the right-angl'd Triangles $APM$, $BPM$, $CQM$, we get the three following Equations, 1. $zz=aa+2ax+xx+yy$, 2. $zz-2dz+dd=bb-2bx+xx+yy$. 3. $zz-2fz+ff=cc-2cy+yy+xx$; and orderly ſubſtracting each Side of the two latter Equations from each Side of the firſt Equation, and we ſhall get a fourth and fifth Equation $2dz-dd=aa-bb+2ax+2bx$, and $2fz-ff=aa-cc+2ax+2cy$. Now inſtead of $yy$ in the firſt Equation, ſubſtitute the Square of the Value of $y$ found by means of the fifth Equation; and then inſtead of $x$ and $xx$ their Values found by means of the fourth Equation: By doing thus we ſhall get an Equation, having only the unknown Quantity $z$ therein ariſing no higher than a Square. And ſo the ſame may be reſolved by ſtraight Lines and Circles, according to the Directions in the 380th or 382d Articles. But when the Value of $z$ is found, it will be afterwards very eaſy to find the ſought Point $M$; for the ſame will be the Interſection of two circular Arcs, whereof one ſhall have the Point $A$ for the Centre, and the Line $AM$ ($z$) for a Radius; and the other the Point $B$ for the Centre, and the Line $BM$ ($z-d$) for a Radius.

FIG. 248.

Becauſe the Equation of the ſecond Degree will be found, by compleating the ſame, to have very compound Terms; ſo that it will require a great Number of Operations to reduce them to ſimple Expreſſions, as the 308th and 382d Articles require, which will render the Conſtruction very tedious. For this reaſon we ſhall lay down the following Conſtruction which reduces this Caſe to the ſecond Caſe.

FIG. 249. The right Lines $AB$, $AC$, which join the given Points, being biſected in the Points $D$, $F$, and the right Lines $MP$, $MQ$, being drawn from the Point $M$ (ſuppoſed to be that ſought) perpendicular to $AB$, $AC$; call the given Quantities $AB$, $2a$; $EC$, $2b$; $AM-MB$, $2c$; $AM-MC$, $2d$; and the unknown Quantities $DP$, $x$; $FQ$, $y$. This being laid down, if the Sum of the right Lines $AM$, $BM$, be named $2t$, the greater of them, *viz.* $AM$ ſhall be $t+c$, and the leſſer $BM$ ſhall be $t-c$. Now becauſe $APM$, $BPM$, are right-angl'd Triangles, therefore $\overline{PM}^2 = \overline{AM}^2 - \overline{AP}^2 = \overline{BM}^2 - \overline{BP}^2$, that is, $tt+2ct+cc-aa-2ax-xx = tt-2ct+cc-aa+2ax-xx$, whence $t$ will be $=\frac{ax}{c}$; and conſequently $AM\ (t+c) = \frac{ax}{c}+c$. In like manner, by means of the right-angl'd Triangles $AQM$, $CMQ$, we ſhall get $AM=\frac{by}{d}+d$; and therefore making an Equality between theſe two Values of $AM$, then will $\frac{ax}{c}+c$ be $=\frac{by}{d}+d$, or $\frac{ax}{c}=\frac{by}{d}+d-c=\frac{by}{d}+f$, making $d-c=f$ for Brevity's Sake. Hence the ſought Point $M$ muſt be ſuch, that the right Lines $MP$, $MQ$, being drawn perpendicular to $AB$, $AC$, we have this Equation $\frac{ax}{c}=\frac{by}{d}+f$, or (which is the ſame) this Proportion, $x : y+\frac{df}{b} :: b : \frac{ad}{c}$. And this does ſuffice for finding the following Conſtruction.

Join the given Points by two right Lines $AB$, $AC$, and biſect theſe Lines in the Points $D$, $F$; then in $AC$ aſſume $FK=\frac{df}{b}$ towards $A$, and draw the right Lines $DO$, $KS$, perpendicular to $AB$, $AC$, meeting one another in the Point $H$, alſo in the Angle $OHS$ draw the right Line $HM$ the Locus of the Points $M$, being ſuch that the right Lines $MO$, $MR$, drawn from any one of them perpendicular to the Sides $HO$, $HS$; the right Line $MO$ may be to the right Line



$MR$,

$MR$ always in the given Ratio of $b$ to $\frac{ad}{c}$. This being done, draw $AE$ perpendicular to $HM$, and having produced the ſame to $G$, ſo that $EG$ be equal to $AE$; by the ſecond Caſe, find the Point $M$ being ſuch, that if the right Lines $MA$, $MG$, $MC$, be drawn; $MA$, $MG$, may be equal to one another, and the Difference between $MA$, $MC$, may be equal to the given Quantity $2d$. I ſay the Point $M$ will be that ſought.

For by the Property of the right Line $HM$, we have always this Proportion, $MO$ or $DP(x) : MR$ or $QK\left(y + \frac{df}{b}\right) :: b : \frac{df}{b}$ and conſequently the Point $M$ muſt needs be in the Line $HM$. It will be equally diſtant from the Points $A$, $G$; but beſides the Difference between $AM$, $MC$, muſt be the given Difference $2d$. Therefore, &c.

### SCHOLIUM.

433. IF the two Sums, made by adding one of the three right Lines $MA$, $MB$, $MC$, to each of the other two, had been given, inſtead of the two Differences, or elſe if the Sum of two of them, together with the Difference between one of them, and the third had been given: Then the Point $M$ might have been found by the ſame Methods as eaſy as when the Differences are given. But I ſhall leave this to the Induſtry of the Learner.

### COROLLARY I.

434. HENCE ariſes the manner of deſcribing a Circle, that ſhall touch three other given Circles.

For let the Points $A$, $B$, $C$, be the Centres of the three given Circles, and the Point $M$ the Centre of the ſought Circle, which touches the given Circles in the Points $D$, $E$, $F$, as *per* Figure. Let the Radii of the given Circles $AD$ be $= a$, $BE = b$, $CF = c$; and the Radius of the Circle ſought $MD$ or $ME$, or $MF = z$. Then will $AM$ be $= z + a$, $MB = z + b$, $MC = z - c$; and therefore $AM - MB = a - b$, $MB - MC = b + c$, $AM - MC = a + c$. Whence it appears that the Queſtion is brought to this, *viz.* to find a Point $M$, from which if the three right Lines $MA$, $MB$, $MC$, are drawn to the three given Points $A$, $B$, $C$, they may have given Differences. Fig. 250.

### COROLLARY II.

435. HENCE we have a way of deſcribing a Conick Section (having the given Point $F$ for a Focus) through two given Points $B$, $C$, which ſhall touch a right Line $DE$ given in Poſition. Fig. 251, 252.

This.

This Problem has two Caſes, the firſt whereof is, when the three given Points *F*, *B*, *C*, do fall on the ſame Side the indefinite right Line *D E*; and the ſecond, when they fall on both Sides.

FIG 251. *Caſe* 1. Draw *F D* perpendicular to *D E*, and produce the ſame to *A* ſo that *D A* be equal to *D F*, and draw the right Lines *F B*, *F C*. Then find the Point *M*, ſo that the Difference between *A M* and *B M*, be equal to *F B*, and the Difference between *A M* and *M C*, be equal to *F C*. This being done, deſcribe * a Conick Section with the Points *F*, *M*, as Foci, and a Line equal to *A M* paſſing through the Foci, for an Axis. I ſay this Section will be that ſought.

* *Def.* I. II. *and* I. III.

For 1. The Point *E* wherein the Line *A M* meets the Line *D E*, is in the Section, ſince F *E* being equal to *A E*, the Sum of the right Lines *F E*, *E M*, in the Ellipſis, and their Difference in the Hyperbola, will be equal to the Axis paſſing through the Foci; and by the ſame reaſon the Points *B*, *C*, ſhall be alſo in the Section. 2. The Angles *F E D*, *D E A*, by Conſtruction are equal to one another; and conſequently * the Line *E D* touches the Section in *E*.

* *Art.* 60. *and* 123.

Here it muſt be obſerved that when the Focus *M* be ſought on the ſame Side the given Focus *F* with regard to the Line *D E*, the Section found is an Ellipſis; and when that Point is ſought on the other Side, it will be an Hyperbola or two oppoſite Sections.

FIG. 252. *Caſe* 2. Here it is manifeſt that the Section anſwering the Problem will not be an Ellipſis, but two oppoſite Hyperbola's, which may be found thus. Draw (as in the firſt Caſe) the right Line *F D* perpendicular to *D E*, and produce the ſame to *A*, ſo that *D A* be equal to *D F*; then find the Point *M* ſuch, that *A M* + *B M* be equal to the given Line *F B*, and *M C* — *A M* equal to the given Line *F C*. This being done, deſcribe two oppoſite Hyperbola's, with the Points *F*, *M*, as Foci, and having the firſt Axis equal to the Line *A M*. I ſay theſe Hyperbola's ſhall have the requiſite Conditions.

For 1. The Point *E*, wherein the Line *A M* meets the Line *D E* ſhall be in one of the oppoſite Hyperbola's, ſince *F E* being equal to *A E*, the Difference between the right Lines *F E*, *M E*, ſhall be equal to *A M* the Value of the firſt Axis. And for the ſame reaſon the Points *B*, *C*, ſhall be in the ſaid oppoſite Hyperbola's. 2. The Line *D E* ſhall * touch the Section in *E*, becauſe by Conſtruction the Angles *A E D*, *D E F*, are equal to one another.

* *Art.* 123.

If the Point *C* ſhould fall on the ſame Side *B* with regard to *D E*, then the Sum of the right Lines *A M* and *M C* would be equal to the given Line *F C*; as is their Difference when the Points *B*, *C*, do fall on both Sides the Line *D E*, as is ſuppoſed in the Figure.

If it be requir'd to deſcribe a Conick Section, which ſhall paſs thro' a given Point, having a given Point for a Focus, and two right Lines given in Poſition for Tangents to it; then you muſt find two Points

(after

(after the ſame manner as the Point *A* was) by means of theſe two Lines, from which if two right Lines be drawn to terminate in the Focus ſought, they may be equal to one another; and moreover, the Difference between either of them, and the Line drawn from the Point thro' which the Section muſt paſs to the ſought Focus, or elſe the Sum of either of them, and this laſt mention'd Line, ſhall be always given: So that the Problem may be ſolv'd always by the precedent Example, and the Scholium belonging thereto. Laſtly, If it be requir'd to deſcribe a Conick Section that ſhall touch three right Lines given in Poſition, and have a given Point for the Focus thereof; then you muſt find three Points by means of theſe three Lines, as the Point *A* in the laſt two Caſes was determin'd by help of the Line *D E*, and the Centre of the Circle paſſing thro' theſe three Points ſhall be the other Focus of the Section, whoſe firſt Axis will be a Line equal in Length to the Radius of this Circle. In all the aforeſaid Caſes it muſt be obſerv'd, that if the ſought Point *M* be infinitely diſtant from the Point *F*; then the Section ſhall be a Parabola, whoſe Diameters ſhall be parallel to Lines, which being infinitely continued do terminate in the Point ſought.

## EXAMPLE V.

436. A Parabola *NCS* being given, together with the Arc *MN* thereof; to find another Arc *RS* of the ſame, which ſhall be to the Arc *MN*, in the given Ratio of Number to Number. FIG. 253.

Produce the Axis of the Parabola beyond (*C*) its Origin, ſo that *CA* be equal to ¼ the Parameter, and deſcribe an equilateral Hyperbola *EAF*, with the Point *C* as a Centre, and the Line *CA* as a Semi-firſt Axis; then draw the right Lines *MB*, *NE*, *RD*, *SF*, parallel to the Axis *CA*, meeting the ſecond Axis in the Points *H*, *L*, *K*, *O*, and the Hyperbola in the Points *B*, *E*, *D*, *F*, from which draw the right Lines *BP*, *EQ*, *DG*, *FI*, perpendicular to the Aſymptotes. This being done, it is manifeſt * that the Rectangle $AC \times MN$, or * the Hyperbolick Trapezium *HBEL* is equal to the Hyperbolick Sector *CBE plus* the Triangle *CLE minus* the Triangle *CHB*; and alſo $AC \times RS = CDF + COF - CKD$. And ſuppoſing the given Ratio of the Arc *MN* to the Arc *RS* be as *m* is to *n* (where the Letters *m* and *n* denote any whole Numbers at pleaſure); then, by the Conditions of the Problem, we ſhall have this Proportion $AC \times MN$, or $CBE + CLE - CHB : AC \times RS$, or $CDF + COF - CKD :: m : n$, and conſequently $n\,CBE + n\,CLE - n\,CHB = m\,CDF + m\,COF - m\,CKD$. Now if you make the given Quantities $CP = b$; $CQ = c$, the unknown Quantity $CG = x$, and take $CI = x \sqrt[m]{\frac{c^n}{b^n}}$, then * *Art.* 246.

will

* *Art.* 223. will * the Hyperbolick Sector $CBE : CDF :: m : n$; and ſo $n\,CBE$ is $= m\,CDF$; whence the preceding Equation will be chang'd into this, $n\,CLE - n\,CHB = m\,COF - m\,CKD$, wherein are Triangles only, whoſe Analytick Values we now proceed to determine.

The right Lines $CP$, $HB$, do form two right-angl'd Triangles $VHC$, $VPB$, which are ſimilar, becauſe the vertical Angles at $V$ are equal, therefore $HV : CV :: VP : VB$, and multiplying the Extremes and Means $HV \times VB = CV \times VP$. Again, becauſe $EAF$ is an Equila-
* *Def.* 16. III. teral Hyperbola, the Angle $VCA$ or $CVH$ is * half a right Angle, and conſequently the right-angl'd Triangle $CVH$ is Iſoſcelles, as well as $VPB$ being ſimilar to it; whence $VP = PB$, $CH = HV$, and $\overline{CV}^2 = \overline{CH}^2 + \overline{HV}^2 = 2\,\overline{HV}^2$. Therefore four times the right-angl'd Triangle $CHB$, that is, $2\,CH \times HB$ is $= 2\,HV \times \overline{HV + VB} = 2\,\overline{HV}^2 + 2\,HV \times VB = \overline{CV}^2 + 2\,CV \times VP = \overline{CV}^2 + 2\,CV \times VP + \overline{VP}^2 - \overline{VP}^2 = \overline{CP}^2 - \overline{PB}^2$, becauſe $\overline{CV}^2 + 2\,CV \times VP + \overline{VP}^2$ is the Square of $CV + VP$ or $CP$. And conſequently the Triangle $CHB$ is $= \frac{1}{4}\overline{CP}^2 - \frac{1}{4}\overline{PB}^2$. After the ſame manner we prove, that the Triangle $CLE$ is $= \frac{1}{4}\overline{C\mathcal{Q}}^2 - \frac{1}{4}\overline{\mathcal{Q}E}^2$, and $CKD = \frac{1}{4}\overline{CG}^2 - \frac{1}{4}\overline{GD}^2$, and $COF = \frac{1}{4}\overline{CI}^2 - \frac{1}{4}\overline{IF}^2$. Whence, if the Power of the Hyperbola be call'd $aa$, then will the Triangle $CHB = \frac{1}{4}bb - \frac{a^4}{4bb}$, theTriangle $CLE = \frac{1}{4}cc - \frac{a^4}{4cc}$, the Triangle $CKD = \frac{1}{4}xx - \frac{a^4}{4xx}$, the Triangle $COF = \frac{xx}{4}\sqrt[m]{\frac{c^{2n}}{b^{2n}}} - \frac{a^4}{4xx}\sqrt[m]{\frac{b^{2n}}{c^{2n}}}$, becauſe $PB^2$ is $= \frac{aa}{b}$, $\mathcal{Q}E = \frac{aa}{c}$, $IF = \frac{aa}{x}\sqrt[m]{\frac{b^n}{c^n}}$: Now if if theſe Values be ſubſtituted for the Triangles they expreſs in the Equation $n\,CLE - n\,CHB = m\,COF - m\,CKD$, we ſhall have this Equation $\frac{1}{4}n \times cc - \frac{a^4}{cc} - bb + \frac{a^4}{bb} = \frac{1}{4}m \times xx\sqrt[m]{\frac{b^{2n}}{c^{2n}}} - \frac{a^4}{xx}\sqrt[m]{\frac{b^{2n}}{c^{2n}}} - xx + \frac{a^4}{4xx}$, which by due ordering, will be brought to the following Equation of the 2d Degree, $x^4 - \frac{na^4 - nbbcc \times cc - bb \times \sqrt[m]{b^{2n}}}{mbbcc \times \sqrt[m]{c^{2n}} - \sqrt[m]{b^{2n}}}\,xx + a^4\sqrt[m]{\frac{b^{2n}}{c^{2n}}} = 0$. Now if the Value of $CG\,(x)$ be found, and $CI$ be taken equal to $x\sqrt[m]{\frac{c^n}{b^n}}$, and if the Perpendiculars $GD$, $IF$, be drawn, meeting the Equilateral Hyperbola in the Points D, $F$; then ſhall

ſhall the Arc (*R S*) of the Parabola intercepted between the Parallels (*D R*, *F S*) to the Axis, be to the Arc *M N* in the given Ratio of *n* to *m*.

Here it is neceſſary to obſerve, (1.) That the ſecond Term of the laſt Equation is always negative, becauſe *C Q* (*c*) is greater than *C P* (*b*); and ſo both the Roots thereof will be affirmative or imaginary, according as the known Quantity in the ſecond Term is greater, equal to, or leſs than $a\,a\,\sqrt[m]{\frac{b^n}{c^n}}$ the ſquare Root of the laſt Term. (2.) That when $\overline{CG}^2$ $(x\,x)$ is one Root of the Equation, then will $\overline{IF}^2$ be the other, becauſe *C I* being $= x\sqrt[m]{\frac{c^n}{b^n}}$, therefore * $IF = \frac{a\,a}{x}\sqrt[m]{\frac{b^n}{c^n}}$. But the laſt Term of an Equation, is the Product of its Roots. Therefore if $a^4\sqrt[m]{\frac{b^{2n}}{c^{2n}}}$ the laſt Term of the preceding Equation be divided by $\overline{CG}^2$ $(x\,x)$, which is ſuppos'd to be one of its Roots, then will the other Root be $\frac{a^4}{x\,x}\sqrt[m]{\frac{b^{2n}}{c^{2n}}}$, which is the Square of *F I*. Whence if *C G*, *C T* be taken in the Aſymptotes equal to the Roots of the Equation; and if the Parallels *G D*, *T F* be drawn to the Aſymptotes, as alſo the right Lines *D R*, *E S*, through the Points *D*, *F*, (wherein *G D*, *TF*, meet the Hyperbola) parallel to the Axis, then ſhall theſe latter Parallels intercept the Arc (*R S*) of the Parabola ſought.

* *Art.* 101.

If *m* be $= n$, then the general Equation ſhall become this $x^4 - \frac{bbcc - a^4}{cc}\,x\,x + \frac{a^4bb}{cc} = o$, whoſe two Roots will give us $CG\ (x) = b = CP$, and $CT\ (x) = \frac{a\,a}{c} = QE$; whence it follows, that the Arc *R S* is found by means of them, ſimilarly ſituate on the other Side the Axis with regard to the Arc *M N*. And ſince it otherwiſe appears, * that the two Arcs *R S*, *M N*, being alike ſituate on each Side the Axis, are equal to one another; therefore that ſerves to confirm the Reaſonings here laid down. Hence, if an Arc (*M N*) of a Parabola be given, it may be eaſily concluded, that no other Arc (*R S*) thereof being nearer, or farther from *C* the Origin of the Axis, and equal to *M N*, can be found, without ſuppoſing the Quadrature of ſome Hyperbolick Sector, or (which amounts to the ſame thing) the Rectification of ſome Parabolick Arc.

* * *Art.* 86.

If *m* be $= 1$ and $n = 2$, then will $x^4 - \frac{2a^4bb - 2ccb^4}{c^4 - bbcc}\,x\,x + \frac{a^4b^4}{c^4} = o$; and if *m* be $= 2$ and $n = 3$, or, which is the ſame thing, if the Arc

$RS$ must be to the Arc $MN$ as 3 is to 2; then ſhall $x^4 - \frac{3a^4 - 3bbcc \times bcc - b^3}{2c^5 - 2ccb^3} xx + \frac{a^4b^3}{c^3} = o$; and the Roots of theſe two Equations will ſolve the Problem. Underſtand the ſame of other Values of $m$ and $n$.

M. *Bernoulli*, a Celebrated Profeſſor of Mathematicks at *Groningen* was the firſt that reſolv'd this Problem; as may be ſeen in the *Acta Eruditorum* for the Year 1698. *page* 261.

EXAMPLE VI.

FIG. 254. 437. AN Angle $BAC$ being given, together with a Point $D$ within the ſame; to deſcribe a Circle that ſhall paſs through the given Point $D$, touch the Side $AB$ in ſome Point $P$, and intercept the Part $OC$ of the other Side $AC$ equal to a given Line, as $2a$.

Suppoſe the thing done, and from the Point $D$ draw the Line $DA$ to $A$ the Vertex of the given Angle, and the Line $DP$ paſſing through the Point of Contact $P$, meeting the Side $AC$ continu'd out in $H$; alſo draw the Line $DE$ parallel to $AC$, and the Line $DB$ perpendicular to the Side $AB$; then biſecting the intercepted Part $OC$ ($2a$) in $Q$, call the unknown Quantities $AP$, $x$; $AQ$, $z$; $DH$, $t$; and the given Quantities $AE$, $m$; $AB$, $g$; $BD$, $b$; $DE$, $f$; $AD$, $n$. This being done, by the Nature of the Circle, we have $\overline{AP}^2 (xx) = CA \times AO$, or $\overline{AQ}^2 (zz) - \overline{QO}^2 (aa)$, and therefore $zz = xx + aa$. Moreover, becauſe the Triangles $PED$, $PAH$, are ſimilar, therefore $AE (m) : AP (x) :: DH (t) : HP = \frac{tx}{m}$. And $PE (m - x) : ED (f) :: AP (x) : AH = \frac{fx}{m-x}$. Whence $HQ = z + \frac{fx}{m-x}$, and $CH \times HO$, or $\overline{HQ}^2 - \overline{QO}^2 = zz + \frac{2fxz}{m-x} + \frac{ffxx}{\overline{m-x}^2} - aa = xx + \frac{2fxz}{m-x} + \frac{ffxx}{\overline{m-x}^2}$ (ſubſtituting $xx + aa$ for $zz$) $= DH \times HP \left(\frac{ttx}{m}\right)$ from the Nature of the Circle; and ſo (dividing by $x$) we ſhall have $x + \frac{2fz}{m-x} + \frac{ffx}{\overline{m-x}^2} = \frac{tt}{m}$. But $PD$ or $DH - HP$ is $= \frac{mt - tx}{m}$; and (becauſe $DBP$ is a right-angl'd Triangle) the Square thereof, *viz.* $tt - \frac{2mtx + ttxx}{mm}$ is $= xx - 2gx + gg + bb = xx - 2gx + nn$ by ſubſtituting $nn$ for its Value $bb + gg$; whence $\frac{tt}{m} = \frac{mxx - 2gmx + mnn}{\overline{m-x}^2} = x + \frac{2fz}{m-x} +$



$ffx$

$\frac{ffx}{m-x}$ & multiplying by $m-x$, & tranſpoſing the Term $\frac{ffx}{m-x}$ then will $mx-xx+2fz=\frac{mxx-2gmx-ffx+mnn}{m-x}=\frac{mxx-mmx-nnx+mnn}{m-x}$ (becauſe the Triangle $DEB$ being right-angl'd, $ff$ is $=bb+gg-2gm+mm=nn-2gm+mm$) : that is, becauſe the two Parts of the Equation may be exactly divided by $m-x$, $mx-xx+2fz=-mx+nn$, or $2fz=xx-2mx+nn$. Laſtly, Squaring both Sides, and ſubſtituting $xx+aa$ for $zz$, and this Equation of the fourth Degree will be had,

$$\begin{array}{l} x^4-4mx^3+4mmxx-4mnnx+n^4 \\ \quad -4ff \qquad\quad -4aaff \\ \quad +2nn \end{array} = 0$$

whoſe Roots, which may be found by means of a Circle and a given Parabola, or other Conick Section, will give us the Values of $AP$ $(x)$ being ſuch, that if $PM$ be drawn perpendicular to $AP$, and $PD$ join'd, and if the Angle $PDM$ be made equal to the Angle $DPM$, the Point $M$ wherein $DM$, $PM$, the Sides of the Iſoſcelles Triangle $DPM$ do cut one another, will be the Centre of the Circle ſought, the Radius thereof being $MP$ or $MD$. Or elſe if $AP=x$ be taken in the Side $AB$, and $AQ=\sqrt{xx+aa}$ in the other Side $AC$, and the Lines $PM$, $QN$, be drawn perpendicular to the ſaid Sides; the Point $M$ wherein theſe Perpendiculars meet, ſhall be the Centre of the Circle required.

Becauſe nothing is more proper to clear the Underſtanding, than ſhewing the different ways by which the Knowledge of the ſame Truth may be acquired: I therefore ſhall here reſolve this Problem after another way, which in my Opinion is more natural than the former.

Let us ſuppoſe the Point $M$ to be the Centre of the Circle ſought. Draw the Lines $MP$, $MQ$, perpendicular, and the Lines $MF$, $MG$, parallel to the Sides of the given Angle $BAC$, and from the given Point $D$, draw the right Lines $DB$, $DE$, $DK$, parallel to $MP$, $MF$, $AB$. Now call the given Quantities $DB$, $b$; $BE$, $c$; $DE$, $f$; $AB$, $g$; $AE$, $m$; $AD$, $n$; and the unknown Quantities $AP$, $x$; $PM$ or $MD$, $y$; then will $PB$ or $DK=g-x$, $MK=y-b$: Whence (becauſe $MKD$ is a right-angl'd Triangle) $yy$ is $=gg-2gx+xx+yy-2by+bb$, and conſequently $y=\frac{xx-2gx+bb+gg}{2b}$ $=\frac{xx-2gx+nn}{2b}$ ſubſtituting $nn$ for it Value, $bb+gg$. And becauſe the Triangles $DBE$, $MPF$, are ſimilar, therefore $DB\,(b):BE\,(c)::PM\,(y):PF=\frac{cy}{b}$, and ſo $AF$ or $MG=\frac{bx-cy}{b}$; and becauſe the

Triangles $DBE$, $M\mathfrak{Q}G$, are ſimilar, we ſhall have this proportion $DE(f):BD(b)::MG\left(\frac{bx-cy}{b}\right):M\mathfrak{Q}=\frac{bx-cy}{f}$; whence becauſe the Conditions of the Problem require that $\mathfrak{Q}C$ the half of the intercepted Part $OC$ be equal to the given Line $a$, and that the right Lines $MC$ and $MP$ be Radii of the Circle ſought; therefore $\overline{MC}^2=\frac{bbxx-2bcxy+ccyy}{ff}+aa=\overline{MP}^2(yy)$, now multiplying by $ff$, and we ſhall have $bbxx-2bcxy+ccyy+aaff=ffyy=bbyy+ccyy$, by putting $bb+cc$ for its Value $ff$, that is, $ffxx+aaff=ccxx+2bcxy+bbyy$, by ſtriking out $ccyy$ from both Sides, and ſubſtituting $ffxx-ccxx$ for $bbxx$; whence, by the Extraction of the ſquare Root, $f\sqrt{xx+aa}$ is $=cx+by=\frac{2cx-2gx+xx+nn}{2}$, by ſubſtituting $\frac{xx-2gx+nn}{2b}$ for its Value $y$, and laſtly, putting $m$ for $g-c$, and then will $2f\sqrt{xx+aa}=xx-2mx+nn$. Which is the ſame Equation as above.

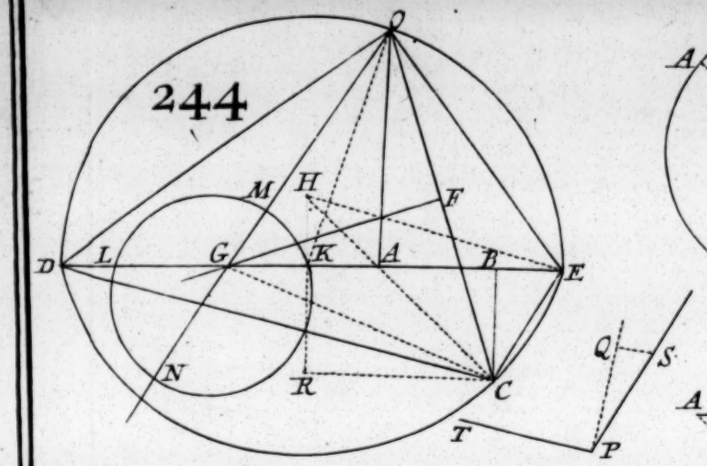
Now we can ſolve this Problem after a new way which firſt gives a very eaſy Conſtruction, but it requires the Deſcription of two Parabola's. 1. I ſeek the Locus of Points as $M$, being ſuch, that if a right Line $MD$ be drawn from every of them to the given Point $D$, and the right Line $MP$ perpendicular to the Line $AB$ given in Poſition; theſe two Lines $MD$, $MP$, may be always equal to one another. But this Locus is * a Parabola, the Point $D$ being the Focus, and the Line $AB$ the Directrix. 2. I find the Locus of Points as $M$, being ſuch, that if a Circle be deſcribed about every of them as Centres, and paſſing through the given Point $D$; this Circle ſhall intercept the Part $OC$, of the Line $AL$ given in poſition, equal to a given Line $2a$. For this End, draw the right Line $DL$ from the given Point $D$ perpendicular to $AL$, and the right Lines $MR$, $M\mathfrak{Q}$, from one of the ſought Points $M$, (ſuppoſed to be given) perpendicular to $DL$, $AL$: Then if the unknown and indeterminate Quantities $DR$, $RM$ (forming a right Angle $DRM$) be called $x$, $y$; the Square $\overline{MD}^2$ is $=xx+yy$, and $\overline{MC}^2=\overline{M\mathfrak{Q}}^2(bb-2bx+xx)+\overline{\mathfrak{Q}C}^2(aa)$ becauſe $MRD$, $M\mathfrak{Q}C$, are right-angl'd Triangles. But the Lines $MD$, $MC$, being Radii of the ſame Circle, are equal to one another, and conſequently $xx+yy$ is $=bb-2bx+xx+aa$, or $yy=bb+aa-2bx$. Therefore if the Parabola being the Locus of this Equation be drawn, it is plain that this Parabola ſhall paſs through $M$ the Centre of the Circle ſought. But the Parabola whoſe Focus is $D$, and Directrix the Line $AB$ alſo does paſs through this Centre; whence

* *Art.* 1.

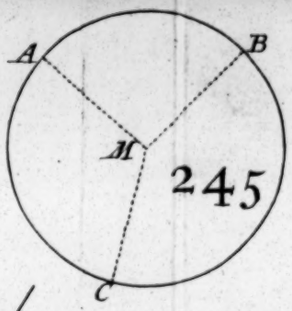
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Plate 27. page 300.

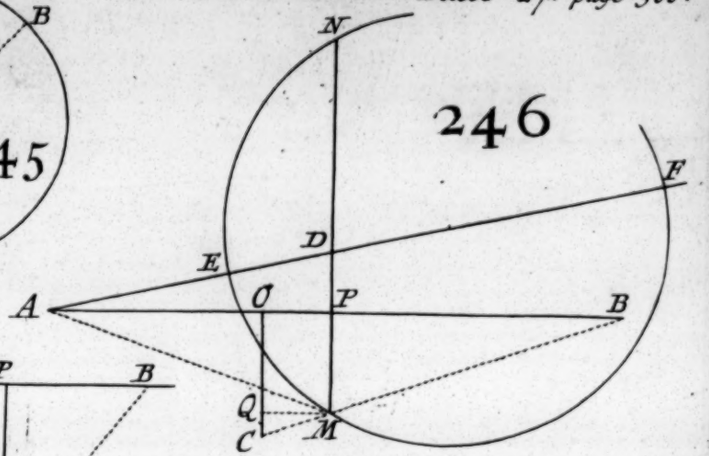
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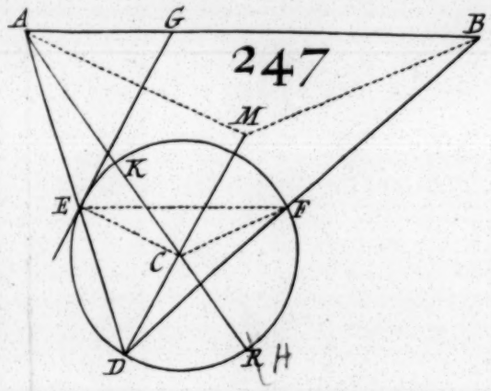

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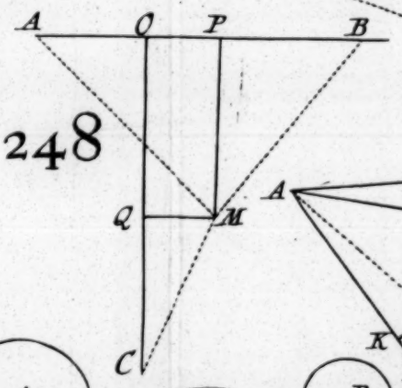

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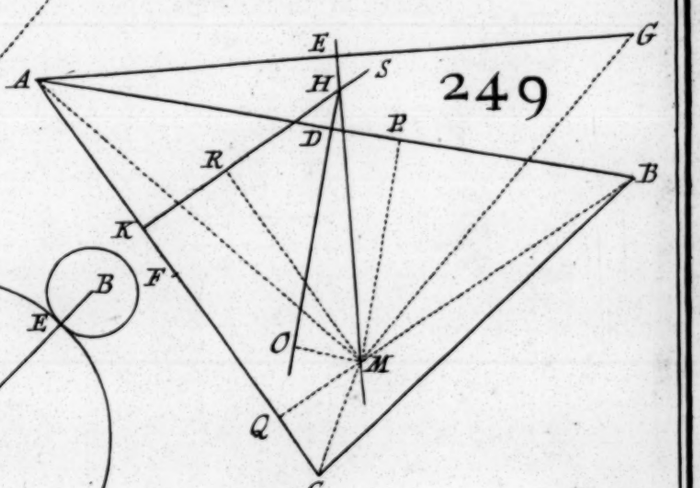

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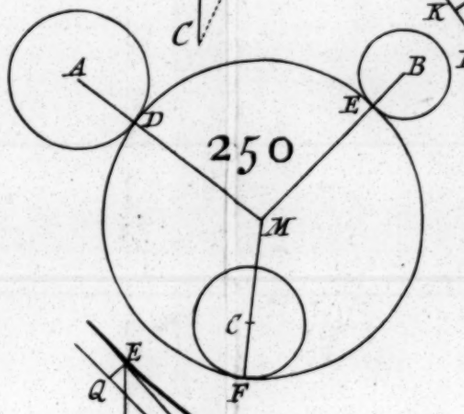

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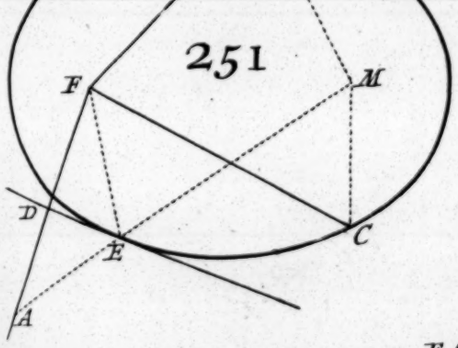

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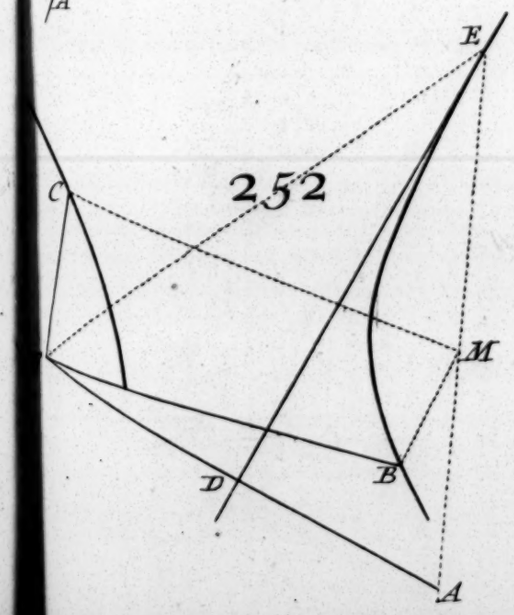

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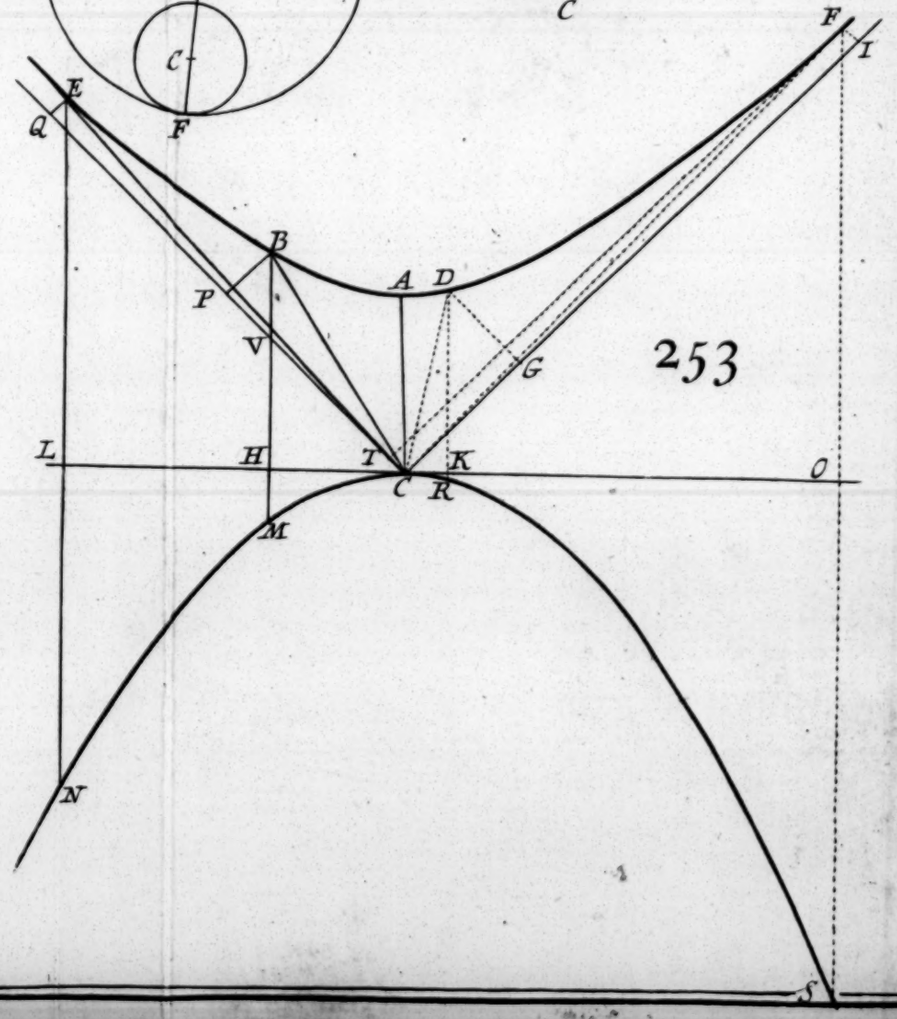

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the Centre of the Circle ſought ſhall be in the Interſection of the two Parabola's.

## EXAMPLE VII.

438. A Circle having the Point *A* for the Centre, and the right Line *A M* for a Radius, being given, together with two Points *E*, *F*, in the Plane of that Circle; to find the Point *M* in the Circumference within the Angle *E A F*, being ſuch, that if the right Lines *A M*, *E M*, *F M*, be drawn; the Angles *A M E*, *A M F* may be equal to one another. FIG. 255.

If the Lines *A E*, *A F*, ſhould be equal, then it is manifeſt that that Line which ſhould biſect the Angle *E A F*, would cut the Circumference in the Point requir'd. Therefore we ſuppoſe thoſe two Lines to be unequal, and alſo (for avoiding Confuſion) that the Line *A E* is leſs than *A F*. And this being premiſed, we ſhall ſolve the Problem two different ways.

Suppoſe the Point *M* to be that requir'd, and draw the right Lines *M B*, *M D*, making the Angles *M B A*, *M D A*, with *A F*, *A E*, equal to the Angles *A M F*, *A M E*, and conſequently equal to one another; then ſince the Triangles *A F M*, *A M B*, and *A E M*, *A M D*, are ſimilar, therefore *A F* : *A M* :: *A M* : *A B*. And *A E* : *A M* :: *A M* : *A D*. Whence becauſe the Lines *A F*, *A E*, are given, together with the Radius *A M*; the Parts (*A B*, *A D*) of the right Lines *A F*, *A E*, ſhall be given likewiſe. Now if the right Lines *M P*, *M Q*, be drawn parallel to *A E*, *A F*; the Triangles *B P M*, *D Q M* ſhall be ſimilar, becauſe the Angles *A P M*, *A Q M*, are equal; and alſo the Angles *P B M*, *Q D M*, ſince theſe are the Complements of the equal Angles *M B A*, *M D A*, to two right Angles; and therefore calling the given Quantities *A B*, *a*; *A D*, *b*; and the unknown ones *A P* or *Q M*, *x*; *P M* or *A Q*, *y*; then will *B P* (*x*—*a*) : *P M* (*y*) :: *D Q* (*y*—*b*) : *Q M* (*x*) : And ſo (by multiplying the Means and Extremes) we ſhall get this Equation $x\,x - a\,x = y\,y - b\,y$, or $y\,y - b\,y - x\,x + a\,x = o$, whoſe Locus being * an Hyperbola may be thus conſtructed. * *Art.* 336.

In the Lines *A F*, *A E*, aſſume *A B*, *A D*, third Proportionals to *A F*, *A M*, and to *A E*, *A M*, and through (*C*) the middle of *B D* draw the indefinite right Line *C H* parallel to *A B*, in which take $C\,K = \sqrt{\frac{1}{4}\,b\,b - \frac{1}{4}\,a\,a}$ (the Line *A D* (*b*) ſhall be greater then *B A* (*a*) becauſe *A E* is ſuppoſed to be leſs than *A F*) and then deſcribe an equilateral Hyperbola, having the Point *C* as a Centre, and the right Line *C K* for a Semi-ſecond Diameter, whoſe Ordinates (*H M*) are parallel to *A D*. I ſay this Hyperbola ſhall meet the Circumference of the given Circle in the ſought Point *M*.

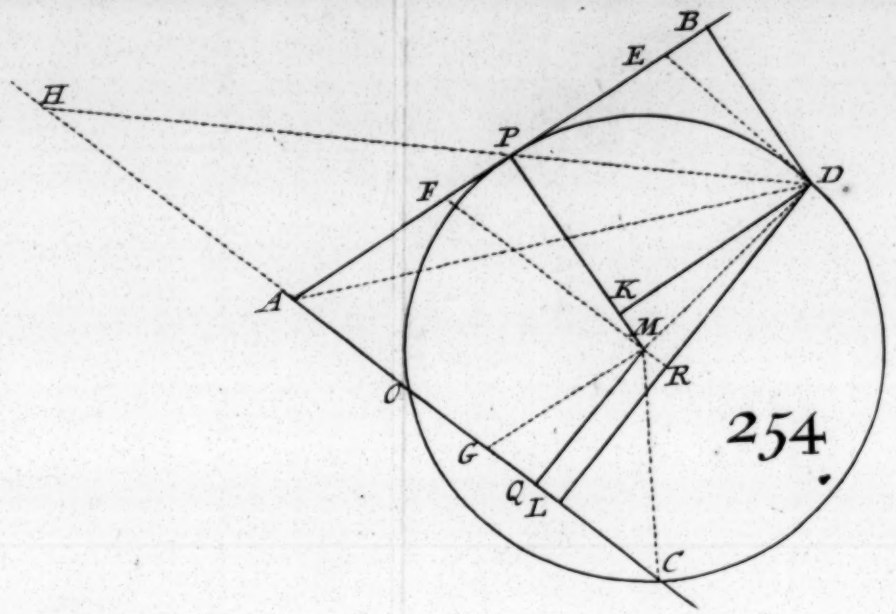
For drawing $CL$ parallel to $AD$, then ſhall the right Lines $CH$, $CL$, biſect the right Lines $AD$, $AB$, in the Points $O$, $L$; becauſe the Point $C$ does biſect the Line $BD$, and ſo $CH$ or $AP-AL$ is $=x-\frac{1}{2}a$, $HM$ or $PM-AO=y-\frac{1}{2}b$. But by the Nature of the equilateral Hyperbola, $\overline{HM}^2$ is $=\overline{CH}^2+\overline{CK}^2$, that is $yy-by+\frac{1}{4}bb=xx-ax+\frac{1}{4}aa+\frac{1}{4}bb-\frac{1}{4}aa$, whence we get this Equation, $yy-by=xx-ax$, which being turned into a Proportion, and then $BP\,(x-a):PM\,(y)::DQ\,(y-b):QM\,(x)$. Whence becauſe the Angles $BPM$, $DQM$, are equal, and the Sides about theſe Angles are proportional; the Triangles $BPM$, $DQM$, ſhall be ſimilar, and conſequently the Angle $MBP$ ſhall be equal to the Angle $MDQ$, and their Complements to two right Angles $ABM$, $ADM$, ſhall be equal. But becauſe $AB:AM::AM:AF$, and $AD:AM::AM:AE$, the Triangles $ABM$, $AMF$, and $ADM$: $AME$ ſhall be ſimilar. Therefore the Angle $ABM$ ſhall be equal to the Angle $AMF$, and the Angle $ADM$ to the Angle $AME$; and conſequently the Angles $AMF$, $AME$, ſhall be equal to one another, becauſe we have prov'd the Angles $ABM$, $ADM$, to be equal.

After the ſame Manner we prove that the Hyperbola oppoſite to this, ſhall cut the Circumference (within the Angle, vertical to $EAF$,) in the Point $M$ being ſuch, that if the right Lines $AM$, $ME$, $MF$, be drawn, the Angles $AME$, $AMF$, ſhall be equal to one another: As likewiſe that theſe two oppoſite Hyperbola's ſhall cut the Circumference (within the Angles adjoining to thoſe vertical Angles) each in one Point $M$ being ſuch, that if the right Lines $MA$, $ME$, $MF$, be drawn; the Angle $AME$ ſhall be equal to the Complement of the Angle $AMF$, to two right Angles.

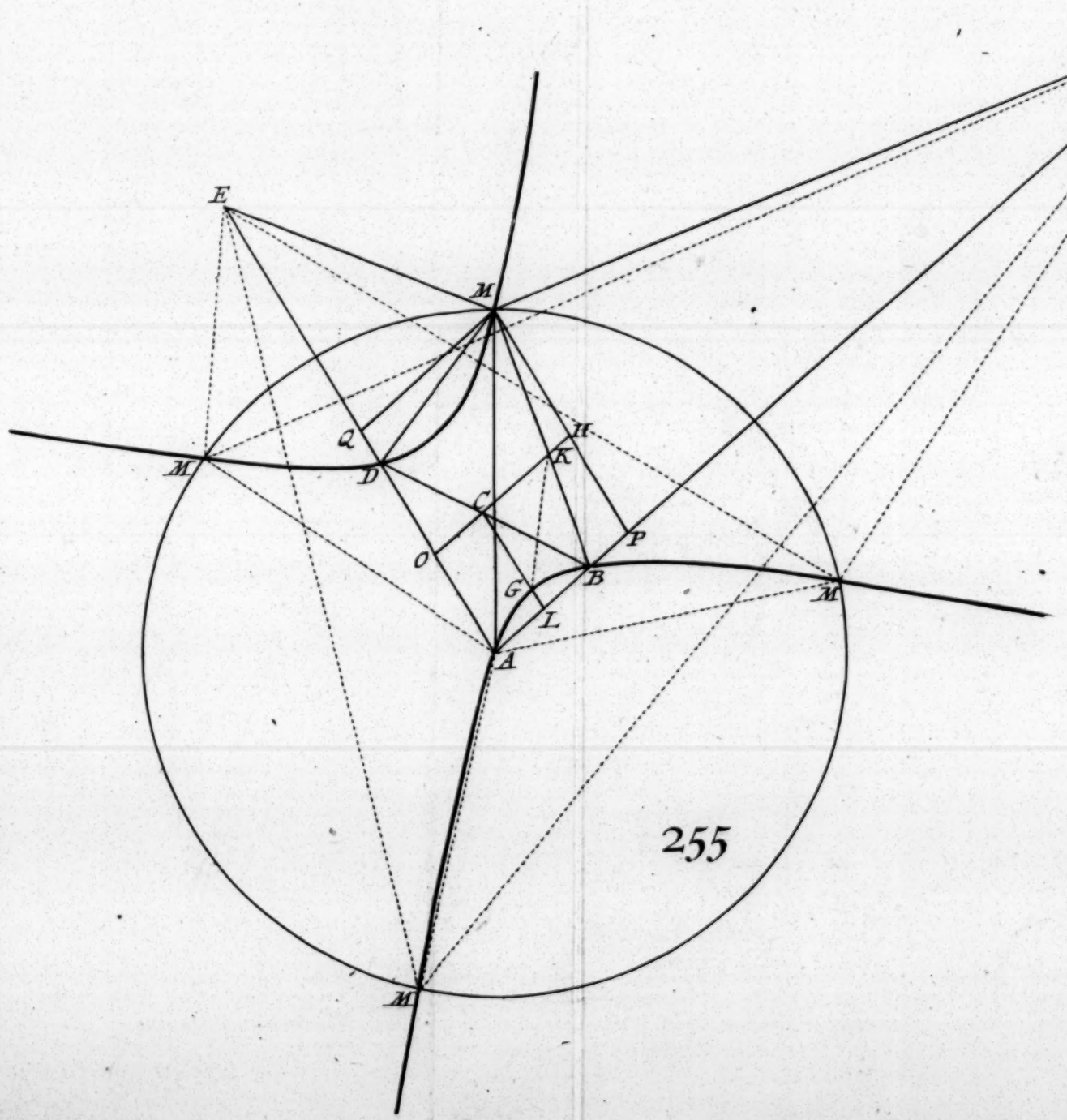
* *Def.* 16. III. If $CG$ be taken in $CL$ equal to $CK$, then $CG$ ſhall be * equal to $\frac{1}{2}$ of the firſt Diameter, being the conjugate to $CK$, and ſo * one of the Aſymptotes of the Hyperbola's ſhall be parallel to $KG$. But in the Iſoſceles Triangle $GCK$, the external Angle $GCO$ (or $BAD$ equal to it) is equal to the two internal and oppoſite Angles, that is, equal to twice the Angle $CGK$. Therefore, becauſe the Lines $CG$, $AD$, are parallel, the Line $KG$, and conſequently one of the Aſymptotes ſhall be parallel to the Lines biſecting the Angle $DAE$. Moreover, it is evident that the Line $AD$ is a double Ordinate to the ſecond Diameter $CK$, becauſe $\overline{OD}^2$ or $\overline{OA}^2\,(\frac{1}{4}bb)=\overline{CO}^2\,(\frac{1}{4}aa)+\overline{CK}^2\,(\frac{1}{4}bb-\frac{1}{4}aa)$; and ſo one of the oppoſite equilateral Hyperbola's does paſs through the Point $D$, and the other through the Point $A$. Now theſe two Obſervations do open the way to the following Conſtruction, which is more eaſy than the precedent one.

* *Art.* 114.

FIG. 256. In the Lines $AF$, $AE$, take $AB$, $AD$, third Proportionals to $AF$, $AM$, and to $AE$, $AM$; and through $C$ the Middle of the Line $BD$ draw




254.




255

draw two indefinite Lines *CH*, *CK*, the one parallel, and the other perpendicular to the Line *AP*, which does biſect the given Angle *EAF*. Then between theſe two Lines, as Aſymptotes, deſcribe two oppoſite Hyperbola's through the Points *D*, *A*, which ſhall cut the Circumference of the given Circle in Points, as *M*, being ſuch that if the right Lines *MA*, *ME*, *MF*, be drawn; the two Angles *AME*, *AMF*, ſhall be equal to one another, when the Point of Interſection *M* does fall in the Angle *EAF*, or in the oppoſite Angle to it; and the Angle *AME* equal to the Complement of the Angle *AMF* to two right Angles, when the Point *M* does fall in either of the Angles adjoining to *EAF* or its oppoſite Angle.

This latter Conſtruction may be eaſily and ſpeedily demonſtrated by a Property of the Equilateral Hyperbola, to be found in the 361ſt Article, which Property may be otherwiſe eaſily prov'd. For if from the Point *M*, wherein the Equilateral Hyperbola *DM* does meet the Circumference of the given Circle, there be drawn the right Lines *BM*, *DM* to *B* and *D*, the Extremities of the firſt Diameter *BD* meeting the Aſymptote *CH* in the Points *O*, *L*, and the Line *AP*, parallel to it in the Points *S*, *R*; then by that Article it is evident, that *MO* is equal to *ML*, and ſo the Angle *MOL* or *MSR* or *BSA* is equal to the Angle *MLO* or *DRA*. But the Angle *BAS* is equal to *DAR* by Conſtruction, becauſe the Line *AP* does biſect the Angle *EAF*. Therefore the remaining Angles *ABM*, *ADM*, in the two Triangles *ABS*, *ADR*, ſhall be equal to one another; whence the Angles *AMF*, *AME*, ſhall be equal; which was to be prov'd.

By means of the latter Conſtruction, we can eaſily find a very ſimple Equation having but one unknown Quantity therein, which Equation being conſtructed by any Conick Section at pleaſure, according to the Rules preſcrib'd in Book VIII. the Value of the unknown Quantity will determine the Solution of the Problem. For Example: From the Point *M* draw the Line *MP* parallel to the Aſymptote *CK*, and meeting the other Aſymptote *CH* in the Point *H*; then call the given Quantities *AM*, *a*; *AK*, *b*; *CK*, *c*; and the unknown ones *AP*, *x*; *PM*, *y*. This being done, by the Nature of the Circle, we ſhall have this Equation $xx+yy=aa$, and by * the Nature of the oppoſite Hyperbola's $CH \times HM\,(xy - cx - by + bc) = CK \times KA\,(bc)$; whence $xy-cx-by$ is $=o$, and ſo $y$ is $=\frac{cx}{x-b}$. Now if the Square of this Value of $y$ be put for $yy$ in the Equation $xx+yy=aa$, by due working we ſhall have this Equation of the fourth Degree,

* *Art.* 100.

$$x^4 - 2bx^3 + bbxx + 2aabx - aabb = b.$$
$$+cc$$
$$-aa$$

And

And if from (*C*) the Centre of the Hyperbola's, there be drawn the Line *CG* perpendicular to *AC*, meeting the Circumference in the Point *G*; the right-angl'd Triangles *ACG*, *AKC*, will give these Equalities $\overline{CG}^2 = \overline{AG}^2 - \overline{AC}^2 = \overline{AM}^2\ (aa) - \overline{AK}^2\ (bb) - \overline{CK}^2\ (cc)$. Whence if the given Quantity *CG* be called *m*, the preceding Equation may be chang'd into this $x^4 - 2bx^3 - mmxx + 2aabx - aabb = o$, wherein the given Quantities are the Radius *AM* (*a*), the Lines *AK*, (*b*), *CK* (*c*), *CG* (*m*), and the unknown Quantity *x* expreſſes the Values of *AP* being ſuch that drawing the Perpendiculars *PM*, theſe ſhall meet the Circumference in the Points ſought.

In order to know which of the two Points, wherein each Perpendicular *PM* cuts the Circumference of the Circle, is that requir'd in the Problem; you muſt draw *PM* on the ſame Side the Line *AP*, as the Point *M* was ſuppos'd to fall in performing the Proceſs. When $\frac{cx}{x-b}$ the Value thereof, which we have found above, is poſitive, that is, when *x* is affirmative, and greater than *b*; and contrariwiſe, *PM* muſt be drawn on the other Side *AP* when its Value is negative, that is, when *x* is affirmative, but leſs than *b*.

*Another Way.*

FIG. 257. Through the ſought Point *M*, (eſteem'd as given) draw the right Line *MD* perpendicular to the Radius *AM*, and through the Point *D*, wherein it meets *AF*, draw the right Line *GH* parallel to *AM*, meeting the Line *MF* in *H*, and the Line *EM* (produced) in *G*; then becauſe the Triangles *FAM*, *FDH* are ſimilar, therefore *AM* : *DH* :: *AF* : *FD*. And becauſe the Triangles *CAM*, *CDG*, are ſimilar, therefore *AM* : *DG* :: *AC* : *CD*. But the Line *DG* is equal to *DH*, becauſe, according to the Conditions of the Problem, the Angles *AME*, *AMF*, muſt be equal to one another, and ſo the Angles *DMH*, *DMG* are equal. Therefore *AF* : *FD* :: *AC* : *CD* and *AF* + *FD* : *AF* :: *AC* + *CD* or *AD* : *AC*. This being premiſed, draw *EB*, *MP*, perpendicular to *AF*, and *MQ* to *EB*, and call the given Quantities *AM*, *a*; *AB*, *b*; *BE*, *c*; *AF*, *d*; and the unknown ones *AP*, *x*; *PM*, *y*. Now becauſe the right-angl'd Triangles *APM*, *AMD*, are ſimilar, therefore *AP* (*x*) : *AM* (*a*) :: *AM* (*a*) : *AD* = $\frac{aa}{x}$. And ſo *FD* is = $d - \frac{aa}{x}$. Again, becauſe the Triangles *EQM*, *MPC*, are ſimilar, therefore *EQ* or *EB* − *MP* (*c*−*y*) : *QM* or *AP* − *AB* (*x*−*b*) :: *MP* (*y*) : *PC* = $\frac{xy-by}{c-y}$. Whence *AC* or *AP* + *PC* = $\frac{cx-by}{c-y}$; and putting

putting the literal Expreſſions of this proportion $AF + PD : AF :: AD : AC$, multiplying the Means and Extremes, and then we ſhall have this Equation $2cdxx - aacx - 2bdxy + aaby + aady = aadc$, which divided by $2dc$, and making $b + d = f$ for brevities ſake, will become this, $xx - \frac{b}{c}yx - \frac{aa}{2d}x + \frac{aaf}{2ad}y - \frac{1}{2}aa = 0$, the Locus whereof which is an Hyperbola between the Aſymptotes being conſtructed by the 339th Article, ſhall cut the Circumference of the Circle in the ſought Point *M*.

If an Equation be requir'd that has only the unknown Quantity $x$ therein, then this Equation $xx + yy$ appertaining to the Circle muſt be uſed, in which if the Square of the Value of $y$ found by means of the preceeding Equation be ſubſtituted for $yy$, we ſhall get an Equation of the fourth Degree, containing but one unknown Quantity $x$, and one of the Roots thereof ſhall expreſs the Value of $AP$ ſought.

## EXAMPLE VIII.

439. A Circle whoſe Centre is *A*, together with the Points *E*, *F*, being given: To find the Point *M* in the Circumference, being ſuch, that if the right Lines *AM*, *MF*, *ME*, be drawn; the right Sine of the Angle *AMF*, may be to the right Sine of the Angle *AME*, in the given Ratio of *m* to *n*. FIG. 258.

I ſhall reſolve this Problem three different ways.

### *Firſt Way.*

In the given Lines *AF*, *AE*, aſſume *AB*, *AD*, third Proportionals to *AF*, *AM*, and to *AE*, *AM*, and from the ſought Point *M* (ſuppoſed to be given) draw the right Lines *MB*, *MD*, and draw the Perpendiculars *MG*, *MH*, to *AF*, *AE*, and the Parallels *MP*, *MQ*, to *AE*, *AF*. Then in *BM* take *BK* equal to *DM*, and from the Point *K* draw the right Lines *KO*, *KL*, parallel to *MG*, *MP*, and from the given Point *D* the right Line *DC* perpendicular to *AF*. Now the Triangles *BMG*, *BKO*, are ſimilar, therefore *BM* : *BK* or *DM* :: *MG* : *KO*. And by the Nature of the Problem $m : n :: KO : MH$, becauſe if *DM* be taken for the Radius or whole Sine, the right Lines *KO*, *MH*, ſhall be the right Sines of the Angles *MBF*, *MDE*, or of their Complements *MBA*, *MDA*, to two right Angles, equal by Conſtruction to the Angles *AMF*, *AME*. Therefore if the Antecedents and Conſequents of the ſaid two Proportions be orderly multiplyed into one another, we ſhall get $m \times BM : n \times MD :: MG \times KO : KO \times MH :: MG : MH :: MP : MQ$, becauſe the Triangles *MPG*, *MQH*, are ſimilar. This being laid down.

Let us call the given Quantities $AD, a$; $AC, b$; $CD, c$; $AB, d$; $AM, r$; and the unknown Quantities $AP$ or $MQ, x$; $PM$ or $AQ, y$. Then becauſe the Triangles $ADC$, $PMG$, $QMH$, are ſimilar, $PG$ will be $= \frac{by}{a}$, $MG = \frac{cy}{a}$, $QH = \frac{bx}{a}$, $HM = \frac{cx}{a}$, $AG = x + \frac{by}{a}$ $GB$ or $AB - AG = d - x - \frac{by}{a}$, $DH$ or $AQ + QH - AD = y + \frac{bx}{a} - a$. And ſince the Triangles $BGM$, $DHM$, are right-angl'd, therefore $\overline{BM}^2$ or $\overline{BG}^2 + \overline{GM}^2$ ſhall be $= xx + \frac{2b}{a}xy + \frac{bbyy}{aa} - 2dx - \frac{2bd}{a}y + dd + \frac{ccyy}{aa} = xx + \frac{2b}{a}xy + yy - 2dx - \frac{2bd}{a}y + dd$, by ſubſtituting $aa$ for $bb + cc$ the Value thereof, becauſe $ACD$ is a right-angl'd Triangle; and moreover, $\overline{DM}^2 = yy + \frac{2b}{a}xy + xx - 2ay - 2bx + aa$. Now by the Nature of the Circle $\overline{AM}^2 (rr)$ is $= \overline{AG}^2 \left( xx + \frac{2b}{a}xy + \frac{bbyy}{aa} \right) + \overline{GM}^2 \left( \frac{ccyy}{aa} \right)$ $= xx + \frac{2b}{a}xy + yy$, by putting $aa$ for its Value $bb + cc$. Whence if $rr$ be ſubſtituted for $yy + \frac{2b}{a}xy + xx$ in the Values of $\overline{BM}^2$ and $\overline{DM}^2$, and if (for brevities ſake) you make $rr + dd = ff$, and $rr + aa = gg$, then will $BM$ be $= \sqrt{ff - 2dx - \frac{2bd}{a}y}$, and $DM = \sqrt{gg - 2ay - 2bx}$. Laſtly, ſubſtituting theſe Values for $BM$ and $DM$ in the Proportion $m \times BM : n \times DM :: MP(y) : MQ(x)$, found above, and multiplying the Means and Extremes, and then we ſhall get this Equation $mx \sqrt{ff - 2dx - \frac{2db}{a}y} = ny \sqrt{gg - 2ay - 2bx}$, and ſquaring both Sides thereof, and getting out the unknown Quantity $y$ by means of the Equation $xx + \frac{2b}{a}xy + yy = rr$, we ſhall come to an Equation of the ſixth Degree, having but one unknown Quantity $x$ therein, which being conſtructed according to the Preſcriptions in Book 9. gives us the Value of $AP(x)$ being ſuch, that drawing $PM$ parallel to $AE$, the Point $M$ wherein that Line meets the Circumference of the Circle, ſhall be that ſought.

If

If you ſuppoſe $m = n$, then it is plain that the Angles $MBF$, $MDE$, ſhall be equal; and ſo likewiſe ſhall the Angles $ABM$, $ADM$, or $AMF$, $AME$; whence it appears that the preceding Problem is only a particular Caſe of this Problem.

*Second Way.*

Join the two given Points $E$, $F$, by a ſtrait Line, and from the given Centre $A$ draw the right Lines $AD$, $AP$, the one perpendicular, and the other parallel to $EF$; alſo through the ſought Point $M$ (ſuppoſed to be given) draw $P\mathcal{Q}$ parallel to $AD$, and from the Point $M$ draw the Radius $AM$ meeting $EF$ in $O$, and the Lines $EM$, $FM$, upon which let fall the Perpendiculars $OG$, $OH$, and $FC$, $EB$, from the Points $O$, $F$, $E$. This being done, the Triangles $EOG$, $EFC$, and $FEB$, $FOH$, are ſimilar, therefore $EO : EF :: OG : FC$. And $EF : FO :: EB : OH$, and conſequently $EO \times EF : EF \times FO$, or $EO : FO :: OG \times BE : CF \times OH$, that is, in the Ratio, compounded of $OG$ to $OH$, or of $m$ to $n$, (ſince $MO$ being the Radius, the right Lines $OG$, $OH$, are the right Sines of the Angle $EMO$, $FMO$, the Complements of the Angles $AME$, $AMF$, to two right Angles) and of $BE$ to $CF$, or of $EM$ to $MF$, becauſe the right angl'd Triangles $BME$, $CMF$, are ſimilar. Therefore $EO : FO :: m \times EM : n \times MF$. This being laid down, FIG. 259.

Call the given Quantities $AD$ or $P\mathcal{Q}$, $a$; $ED$, $b$; $DF$, $c$; $AM$, $r$; and the unknown Quantities $AP$, $x$; $PM$, $y$. Now the Triangles $APM$, $ADO$, are ſimilar, therefore $MP\,(y) : AP\,(x) :: AD\,(a) : DO = \frac{ax}{y}$. And ſo $EO$ is $= \frac{by+ax}{y}$, $FO = \frac{cy+ax}{y}$. But ſince the Triangles $EM\mathcal{Q}$, $FM\mathcal{Q}$, are ſimilar, therefore $\overline{EM}^2$ is $= \overline{E\mathcal{Q}}^2 (bb+2bx+xx) + \overline{M\mathcal{Q}}^2 (aa-2ay+yy) = ff+2bx-2ay$ (by writing $rr$ for $xx+yy$, becauſe $APM$ is a right-angl'd Triangle, and making $aa+bb+rr=ff$) and $\overline{FM}^2 = \overline{F\mathcal{Q}}^2 (cc-2cx+xx) + \overline{M\mathcal{Q}}^2 (aa-2ay+yy) = gg-2cx-2ay$, by writing $rr$ for $xx+yy$, and making $aa+cc+rr=gg$. Then if the Analytick Values now found be put in the precedent Proportion $EO : FO :: m \times EM : n \times MF$, and afterwards the Means and Extremes be multiply'd, we ſhall get this Equation, $\overline{bny+anx}\sqrt{gg-2cx-2ay} = \overline{mcy-max}\sqrt{ff+2bx-2ay}$, both Sides of which being ſquared, and the unknown Quantity $y$ gotten out by means of this Equation $xx+yy=rr$, and we ſhall get an Equation of the ſixth Degree, which being conſtructed, will give us the Value of $AP\,(x)$ being ſuch, that drawing the Perpendicular $PM$, the ſame ſhall cut the Circumference in the ſought Point $M$.

M. *Deſcartes* has reſolv'd this Problem much after the ſame manner in his 65th Letter, *Tom.* 3 It was propos'd to him by M. *Roberval* in a manner appearing different from that we have propos'd, but in the main it comes to the ſame thing.

### *The Third Way.*

Fig. 260. With $AE$, $AF$, as two Diameters, deſcribe two Arcs $ART$, $AST$, in which lay off any two Chords $AR$, $AS$, from the Point $A$, being always to each other in the given Ratio of $m$ to $n$, and draw the right Lines $ER$, $FS$, interſecting one another in the Point $M$. I ſay, the Curve $AM$, being the Locus of all the Points ($M$) this found, ſhall cut the given Circle (whoſe Centre is $A$) in the ſought Point $M$.

For if $AM$ be drawn, and the ſame be ſuppos'd to be the Radius, then it is plain, that the Chord $AR$ is the right Sine of the Angle $AME$, and the Chord $AS$ the right Line of the Angle $AMF$.

Here it is proper to obſerve, 1. That this Conſtruction is attended with this particular Property, *viz.* that it ſucceeds when the Point $M$ is to be found in any Curve at pleaſure, as well as in the Circumference of a Circle. 2. That when two Points of the ſaid Locus are found as directed, as near to the given Curve as poſſible, we need but draw that Portion of the Locus that joins theſe two Points; which makes the Practice of the Conſtruction very eaſy. 3. That the Locus of all the Points, as $M$, thus found, is one of the fourth Degree, as does eaſily appear by the Proceſs in the ſecond Way of ſolving the Problem, becauſe if $rr$ be not ſubſtituted for $xx+yy$ in the Values of $EM$, $FM$, then the Equation of the Locus will be

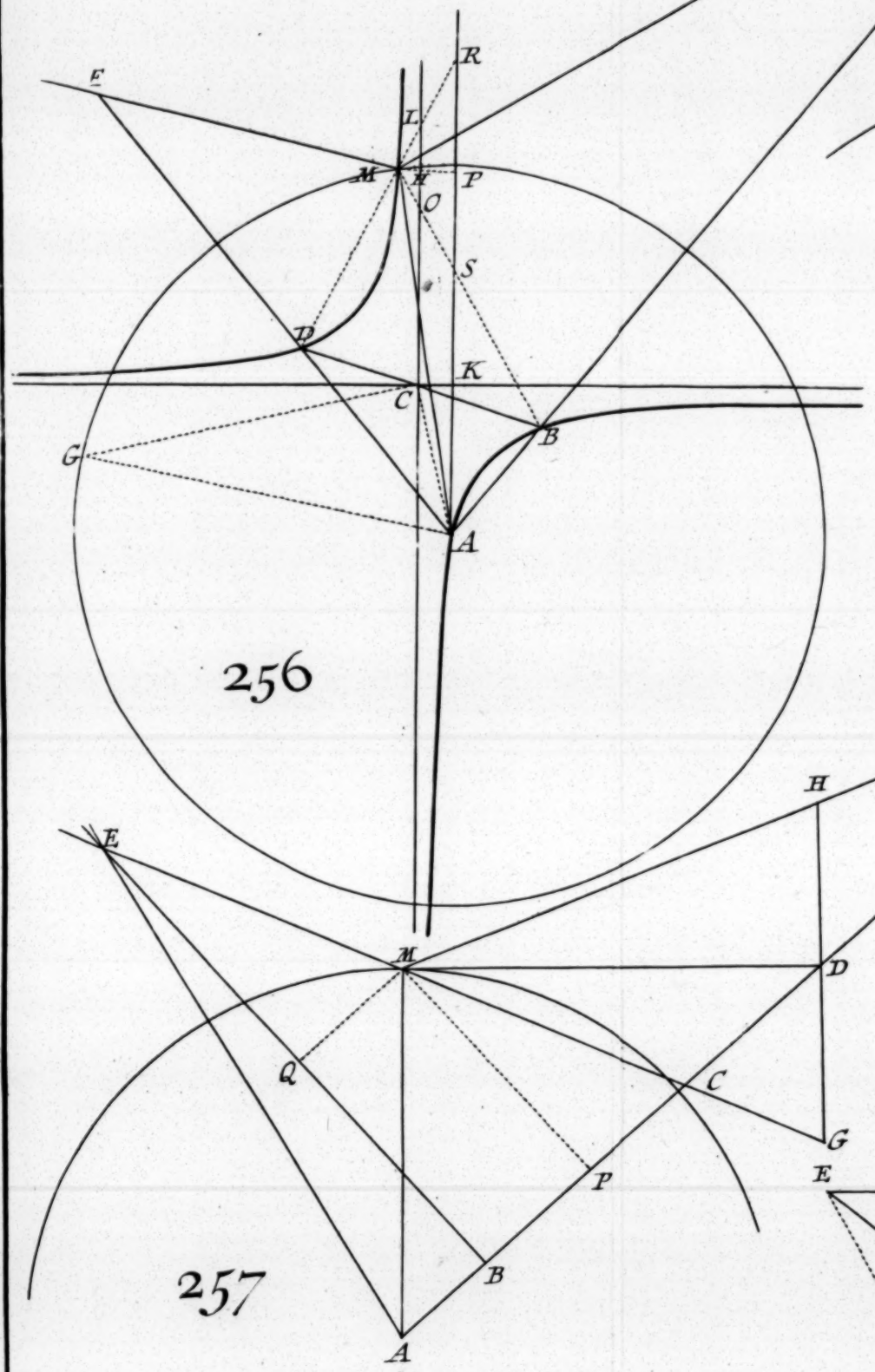
$$nby+nax\sqrt{c-x+a-y}=mcy-max\sqrt{b+x+a-y},$$

wherein the unknown Quantities $x$ and $y$ do ariſe to the fourth Degree, when the ſame is freed from Surds. 4. That according to M. *Deſcartes*, it is a Fault in Geometry to uſe a Curve too much compounded in the Solution of a Problem; ſo that according to him, the two former Ways of ſolving this Problem are to be preferr'd before the latter, becauſe the Problem is determin'd according to thoſe two Ways by two Loci of the third Degree; yet, in my Opinion, the Facility and Simplicity of a Conſtruction will in ſome meaſure recompence this Defect, as will farther appear in the following Example.

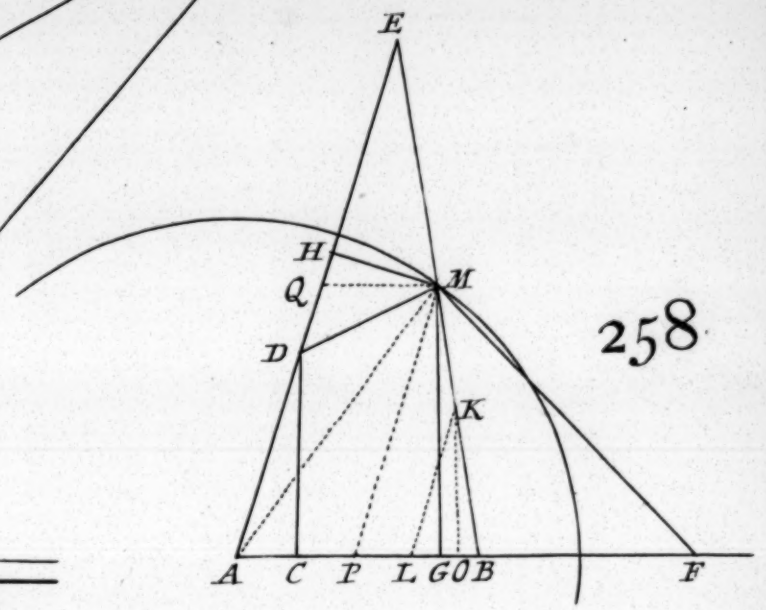
### EXAMPLE IX.

Fig. 261. 440. TO divide a given Scalene Triangle $ABC$ into four equal Parts by two right Lines $DE$, $FG$, interſecting one another at right Angles in the Point $H$.

If



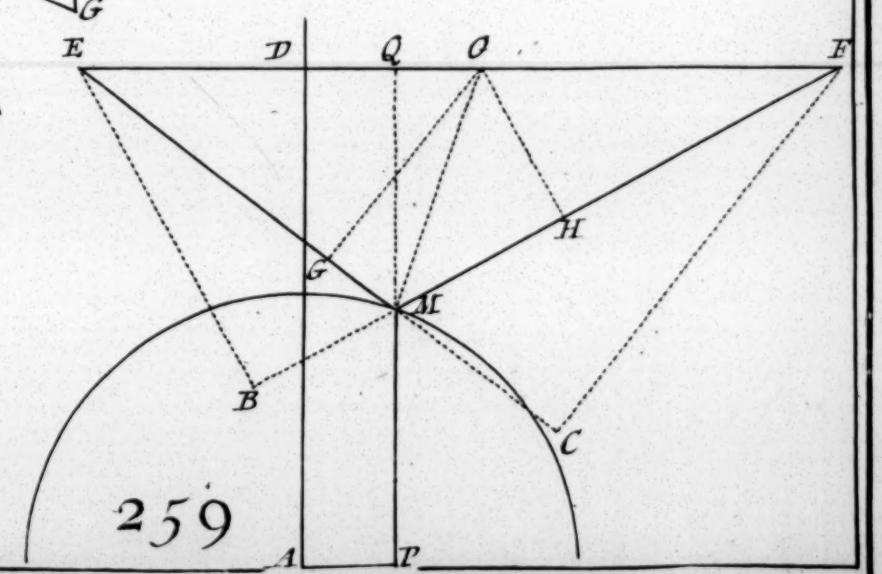

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If Regard be had to the Nature of this Problem, it will appear, 1. That the two Ends $D$, $F$, of the right Lines $DE$, $FG$, muſt be both neceſſarily in the Side $AC$ of the given Triangle, and the two other Extremities $E$, $G$, in the two other Sides $BC$, $BA$. 2. That the two ſought Points $D$, $F$, muſt have theſe two Conditions, *viz.* that the Lines $DE$, $FG$, which do both of them divide the Triangle $ABC$ into two equal Parts, do cut each other at right Angles in the Point $H$; and that they form the Quadrilateral Figure $BGHE$, with the two other Sides of the given Triangle, being the fourth Part of the Triangle $ABC$. This being ſuppos'd,

Draw the right Lines $GI$, $BK$, $EL$, perpendicular on the Side $AC$, and call the given Quantities $AC$, $2a$; $BK$, $b$; $AK$, $c$; $KC$, $d$; and the unknown Quantities $AF$, $x$; $CD$, $y$: Becauſe the Triangle $AGF$, or $GI \times \frac{1}{2} AF$ is to be equal to $\frac{1}{2}$ of the Triangle $ABC$ ($ab$), therefore $GI$ is $= \frac{ab}{x}$; and by the ſame Reaſon $EL = \frac{ab}{y}$. And the ſimilar Triangles $CBK$, $CEL$, and $ABK$, $AGI$, do give theſe Proportions, $BK\ (b) : EL\ \left(\frac{ab}{y}\right) :: CK\ (d) : CL = \frac{ad}{y}$. And $BK\ (b) : GI\ \left(\frac{ab}{x}\right) :: AK\ (c) : AI = \frac{ac}{x}$. And therefore $DL$ or $CD - CL$ is $= y - \frac{ad}{y}$, $FI$ or $AF - AI = x - \frac{ac}{x}$. But the right-angl'd Triangles $DEL$, $FGI$, are ſimilar to one another; becauſe each of them is ſimilar to the Triangle $FDH$, which is right-angl'd at $H$, according to the Import of the Problem requiring the two right Lines $DE$, $FG$, to be at right Angles to each other. Whence $EL\ \left(\frac{ab}{y}\right) : LD\ \left(\frac{yy-ad}{y}\right) :: FI\ \left(\frac{xx-ac}{x}\right) : IG\ \left(\frac{ab}{x}\right)$; therefore multiplying the Means and Extremes, and we ſhall get this Equation $xxyy - acyy - adxx + aacd = aabb$, or $xx - ac \times \overline{yy - ad} = aabb$, which takes in the firſt Condition of the Problem; but now to fulfil the ſecond, *viz.* which requires the Trapezium $BGHE$ to be equal to $\frac{1}{4}$ of the given Triangle $ABC$.

From $H$ the Point wherein the two right Lines $DE$, $FG$, interſect one another, draw the right Lines $HA$, $HC$, $HB$, to the three Angles of the given Triangle; then will $FD\ (x+y-2a) : AF\ (x) :: FHD\ (\frac{1}{4}ab) : FHA = \frac{abx}{4x+4y-8a}$. And therefore the Triangle $AHG$ or $FGA$ *minus* the Triangle $FHA$ is $= \frac{1}{2}ab - \frac{abx}{4x+4y-8a}$. Moreover (2)

$AI$

$AI\left(\frac{ac}{x}\right) : IK\left(\frac{cx-ac}{x}\right) :: AG : GB :: AHG\left(\frac{abx+2aby-4aab}{4x+4y-8a}\right)$ $: GHB = \frac{bxx-5abx+2bxy-2aby+4aab}{4x+4y-8a}$. After the ſame Way of Reaſoning, you will find that the Triangle $HEB$ is $= \frac{byy-5aby+2bxy-2abx+4aab}{4x+4y-8a}$. Now if the Triangles $HGB$, $HEB$, be added together, the Quadrilateral Figure $HGBE$ will be formed, which muſt be equal to $\frac{1}{4}ab$ the fourth Part of the Triangle $ABC$: Whence we ſhall have this ſecond Equation $xx+yy+4xy-8ax-8ay+10baa=o$.

If the unknown Quantity $y$ be gotten out, then we ſhall have an Equation of the eighth Degree, which takes in all the Conditions of the Problem, and has but one unknown Quantity $x$ therein; and ſo the whole Difficulty is brought to this, *viz.* to find the Roots of this Equation, which may be done by means of two Loci of the third Degree each, according to the Directions in the 417th and 418th Articles. But becauſe the Conſtruction of theſe Loci is very long and tedious, upon account of the great Number of Terms in their reſpective Equations, it will be much more natural to conſtruct the Loci of the two Equations before found, though one of them be a Locus of the fourth Degree, and conſequently more compound; for the other being but a Locus of the ſecond Degree, will, together with the Facility of the Conſtruction, make amends for this Inconveniency. Now the Conſtruction of theſe Loci is thus.

FIG. 262. Draw two indefinite right Lines $AB$, $AC$, forming the right Angle $BAC$, and produce $BA$ to $E$, ſo that $AE$ be $= \surd ac$, and $CA$ to $F$, ſo that $AF$ be $= \surd ad$. In $AC$ aſſume the Part $AP$ of any Length, and about the Centre $E$, with the Diſtance $AP$, deſcribe an Arc of a Circle, and take $AH$ ſuch that the Rectangle $HA \times AG$ be equal to the given Triangle $BAC$, and in $AB$ aſſume $AQ = FH$. Again, draw the right Lines $PM$, $QM$, parallel to $AB$, $AC$, interſecting one another in the Point $M$; and an infinite Number of other Points, as $M$, being found after the ſame manner, draw the Curve $KML$ through them. This being done, in $AD$ the Diagonal of the Square $ABDC$ (having the Side $AC$ equal to the Side ($AC$) of the given Triangle $ABC$) aſſume $AT = \frac{1}{3}AD$, and $DS = \frac{1}{6}AD$, and with $TS$, as a firſt Axis, being to its Parameter as 1 to 3, deſcribe the Hyperbola $OSR$. Now, if from the Point $M$, wherein the Hyperbola is ſuppos'd to meet the Curve $KML$ in the Square $ABDC$, the Perpendicular $MP$ be drawn on $AC$, and if in $AC$, the Side of the Triangle $ABC$, you take $AF = AP$, and $CD = PM$; then, I ſay, the Points $F$, $D$, ſhall be ſuch, that two right Lines $FG$, $DE$, being drawn

drawn, ſo as each of them do divide the Triangle $ABC$ into two equal Parts (which is eaſy to do); theſe Lines ſhall cut one another at right Angles, and divide the Triangle into four equal Parts.

For if $AP$ be called $x$; and $PM$, $y$; then ſince the Triangles $EAG$, $FAH$, are right-angl'd at $A$, the Square $\overline{AG}^2$ will be $= \overline{EG}^2$ $(xx) - \overline{AE}^2 (ac)$, and the Square $\overline{AH}^2 = \overline{FH}^2 (yy) - \overline{AF}^2 (ad)$. And becauſe (by Conſtruction) the Rectangle $HA \times AG$ is equal to the given Triangle $BAC$ $(ab)$: Therefore $\overline{HA}^2 \times \overline{AG}^2 (yy - ad \times xx - ac) = aabb$. Whence the Curve $KML$ ſhall be the Locus of the firſt Equation found above; and conſequently the Property thereof ſhall be ſuch, that if from any Point $(M)$ thereof, taken within the Square $ABDC$, there be drawn the Line $MP$ perpendicular to $AC$, and if in $AC$ the Side of the given Triangle $ABC$, you take $AF = AP$, and $CD = PM$; the right Lines $FG$, $DE$, each of which do divide the Triangle $ABC$ into two equal Parts, ſhall interſect one another at right Angles in the Point $H$.

Moreover, if from any Point $(M)$ of the Hyperbola $OSR$, the right Line $MV$ be drawn perpendicular to the firſt Axis $TS$, and if $PM$ be produced, meeting the Diagonal $AD$ in the Point $X$; the right-angl'd Iſoſceles Triangles $APX$, $MVX$, ſhall give theſe Proportions. $1 : \sqrt{2} :: AP$ or $PX (x) : AX = x\sqrt{2}$, and $\sqrt{2} : 1 :: MX (x-y) : MV$ or $VX = \frac{x-y}{\sqrt{2}}$; and therefore $AV$ or $AX - XV = \frac{x+y}{\sqrt{2}}$. But by Conſtruction $AD$ is $= 2a\sqrt{2}$ becauſe $AC$ is $= 2a$, and conſequently $TS$ or $DT - DS = \frac{2}{3} a\sqrt{2}$. Therefore $TV$ or $AV - AT = \frac{x+y-2a}{\sqrt{2}}$, and $VS$ or $TV - TS = \frac{3x+3y-10a}{3\sqrt{2}}$, and by the Nature of the Hyperbola $TV \times VS \left(\frac{3xx+6xy+3yy-16ax-16ay+20aa}{6}\right) : \overline{MV}^2 \left(\frac{xx-2yx+yy}{2}\right) :: 1 : 3$, that is, as the firſt Axis $TS$ to the Parameter thereof: Whence by multiplying the Means and Extremes, we ſhall get this Equation $xx + yy + 4xy - 8ax - 8ay + 10aa = 0$, whoſe Locus ſhall be the Hyperbola $OSR$, being attended with this Property, *viz.* That if from any Point $(M)$ thereof (taken within the Square $ABCD$) you draw $MP$ perpendicular to $AC$, and if in $AC$ the Side of the given Triangle $ABC$, you take $AF = AP$, and $CD = PM$; the right Lines $FG$, $DE$, which do each divide the Triangle $ABC$ into two equal Parts, ſhall alſo divide the ſame Triangle into four equal Parts.

Now becauſe the Point $M$ is both in the Curve $KML$ and in the Hyperbola $OSR$; therefore the Points $D$, $F$, taken in $AC$ the Side of the given Triangle, ſhall have the two Conditions requir'd in the Problem.

If the Curves $OSA$, $KML$, ſhould not meet one another within the Square $ABDC$; then we may be ſure that we have made a falſe Suppoſition, *viz.* In conceiving the two Extremities $D$, $F$, to meet one another in the Side $AC$. Therefore they muſt be ſuppos'd to be in one of the other Sides, and the Proceſs begun again, (by reaſoning after the ſame manner as before) in order to have a Conſtruction relating to this laſt mention'd Side. But if Regard be had to the three Remarks following, you will find it eaſy to take that Side of the given Triangle, wherein the two Extremities $D$, $F$, muſt fall, that ſo you may have no need of beginning the Proceſs again.

The firſt Remark is, that $\overline{CL}^2$ is $= \frac{aabb}{4aa-ac} + ad$, and $\overline{BK}^2 = \frac{aabb}{4aa-ac} + ac$; which appears to be ſo, by ſubſtituting $AC\,(2a)$ for its Value $AP\,(x)$ in the Equation $yy = \frac{aabb}{xx-ac} + ad$, and by putting $AB\,(2a)$ for its Value $AQ\,(y)$ in $xx = \frac{aab}{yy-ad} + ac$. The ſecond Remark conſiſts in this, *viz.* that $CR$ is $= \sqrt{2aa} = BO$; which will be found ſo by firſt ſubſtituting $AC\,(2a)$ for its Value $AP\,(x)$ in the other Equation, $xx+yy+4xy-8ax-8ay+10aa=0$, whoſe Locus is the Hyperbola $OSR$, and afterwards $AB\,(2a)$ for its Value $AQ\,(y)$. The third Remark is taken from hence, *viz.* if $AK\,(c)$ be ſuppos'd leſs than $CK\,(d)$ as it is here, then will $\overline{BK}^2\left(\frac{aabb}{4aa-ad} + ac\right)$ be leſs than $\overline{CL}^2\left(\frac{aabb}{4aa-ad} + ad\right)$. Now this being premis'd, if you require $\overline{BK}^2\left(\frac{aabb}{4aa-ad} + ac\right)$ to be leſs than $\overline{BO}^2\,(2aa)$, then by ſubſtituting $2a-c$ for its Value $d$, and duly working you will find that $bb+cc$ muſt be leſs than $4aa$, that is, the Side $(AB)$ of the given Triangle $ABC$ muſt be leſs than the Side $AC$; and if you require $\overline{CL}^2\left(\frac{aabb}{4aa-ac} + ad\right)$ to be greater than $\overline{CR}^2\,(2aa)$, then by ſubſtituting $2a-d$ for its Value $c$, and duly working you will find that the Side $BC\,(\sqrt{bb+cd})$ muſt be greater than the Side $AC\,(2a)$. But becauſe $BK$ is leſs than $BO$, and $CL$ greater than $CR$; it is evident, that the Curves $KML$, $OMR$, will neceſſarily interſect each other with-

within the Square *A B D C*. Hence, if all the three Angles of the given Triangle *A B C* be acute, and you take for the Side *A C*, in which the two Points *F*, *D*, are ſuppos'd to meet one another, that of the three having a mean Bigneſs between the two others, and the ſhorteſt Side for *A B*, the Problem ſhall always neceſſarily have a Solution; becauſe then the Point *K* (Fig. 261.) will always fall between the Points *A*, *C*; and *A K* is leſs than *A C*, as is ſuppoſed in performing the Calculus upon which all this reaſoning is founded. After the ſame manner if the given Triangle be right or obtuſe-angl'd, and you take the mean Side for the Side *A C*, in which the two Extremities *D*, *F*, muſt fall, the Problem will always have a Solution; ſo that this is a general Remark for all Sorts of Triangles.

It appears in the 262d Figure, that the Hyperbola *O S R*, and the Curve *K M L* do cut one another not only in the Point *M* within the Square *A B D C*, as the Problem requires; but moreover, in another Point *M* without the Square. And by means of this latter Point we can ſolve the following Problem, whereof this here is but a particular Caſe.

To find two Points *F*, *D*, in the Side *A C* of a given Triangle *A B C*, being ſuch, that drawing the right Lines *F G*, *D E*, forming the Triangles *F G A*, *D E C*, with the two other Sides *A B*, *B C*, each equal to $\frac{1}{4}$ of the Triangle *A B C*; the ſaid Lines *F G*, *D E*, ſhall interſect each other at right Angles in the Point *H*, and the Quadrilateral Figure *B G H E* equal to $\frac{1}{4}$ Part of the Triangle *A B C*. Fig. 263.

For when the Point of Interſection (*M*) does fall within the Square *A B D C*, it is manifeſt that each of the Lines *A P*, *P M*, will be leſs than the Side *A C*, and ſo the Points *F*, *D*, determined by them, ſhall both fall between the Points *A*, *C*; and will ſolve the Problem as propoſed at firſt. But when the Point *M* does fall without the Square, becauſe one of the Lines *A P*, *P M*, is leſs than the Side *A C*, and the other greater; therefore one of the Points *F*, *D*, does fall in the Side *A C* of the given Triangle, and the other in the ſaid Side continued out; and by this means we ſhall get a Solution of the Problem as propoſed juſt now. Fig. 262. 263.

## EXAMPLE X.

441. A Conick Section *M A M* being given, together with a Point *S* without the Plane thereof, for the Vertex of the Cone of which it is the Section: It is requir'd to find the Poſition of the Circle *Ma N* which is the Baſe of the Cone.

This Problem may be diſtinguiſhed into two Caſes, the firſt whereof is, when the given Section is a Parabola, and the ſecond, when the ſame is an Ellipſis or Hyperbola.

FIG. 264. *Case* 1. The Problem amounts to this, *viz.* to find the Point $A$ in a Parabola, being ſuch, that drawing the Diameter $AP$ from that Point, together with the Line $AS$; the Line $SD$ from the Point $S$ parallel to $AP$; an Ordinate $PM$ from any Point $P$ to the Diameter $AP$, in the Plane of the Parabola, and a Perpendicular $aD$ to that Ordinate in the Plane of the Triangle $DSA$, meeting the Sides $SA$, $SD$, in the Points $a$, $D$: The Square of $PM$ may be equal to the Rectangle $aP \times PD$. For if a Circle be deſcribed in the Plane $aPM$, having $aD$ for a Diameter, then it is plain that this Circle ſhall paſs through the Point $M$, becauſe the Angle $APM$ is a right Angle, and $\overline{PM}^2$ is $= aP \times PD$, which is an eſſential Property of the Circle; therefore if the Diameter $PA$ be drawn, and if from $D$ the Extremity of the Diameter $Da$, the Line $DS$ be drawn parallel to $PA$, meeting $aA$ drawn from the other Extremity $a$, through $A$ the Origin of the Diameter $AP$, in the Point $S$; the Section of the Cone, having $S$ for its Vertex, and the Circle $MAN$ for the Baſe, made by the Plane $APM$, will be * the given Parabola $MAN$. Now the Point $A$ may be determin'd thus.

* Art. 269.

Call the unknown Parameter of the Diameter $AP$, $v$; then by the Nature of the Parabola $\overline{PM}^2 = AP \times v$: But the Problem requires that $\overline{PM}^2$ be $= aP \times PD$. Whence $aP \times PD = AP \times v$: and ſo $AP : Pa :: PD : v$, and drawing $AO$ parallel to $Da$, then will $SO : AO :: PD$ or $AO : v$, and therefore $SO \times v = \overline{AO}^2$.

Now to find the analytick Values of thoſe Lines, from the given Point $S$, draw the Line $SF$ perpendicular to the Plane of the Parabola, and from the Point $F$ wherein it meets this Plane, draw the Line $FG$ perpendicular to the Axis $BG$, and meeting the Diameter $AP$ in $H$. From the Point $A$ draw the Ordinate $AK$ to the Axis, and the Line $AQ$ perpendicular to the Tangent $AL$, which meet the Line $FQ$ drawn through the Point $F$ parallel to the Axis, in the Points $E$ and $Q$. Laſtly, from the Point $Q$ raiſe $QO$ perpendicular to the Plane of the Parabola, which will meet $SD$ in the ſame Point $O$, as the Line $AO$ (parallel to $AD$) meets it in. For the Tangent $AL$ being parallel to the Ordinate $PM$ which is perpendicular to $aD$, the Angle $LAO$ ſhall be a right Angle, as well as the Angle $LAQ$, and ſo the Plane $QAO$ ſhall be perpendicular to $AL$, and to the Plane of the Parabola paſſing through $AL$; whence the Line $QO$ being perpendicular to this Plane will be in the Plane $QAO$, and conſequently ſhall meet the Line $SD$ in the ſame Point $O$, as the Plane $QAO$ does, that is, in the ſame Point as the Line $AO$ parallel to $aD$ meets it. Here obſerve that all theſe Lines except $FS$, $QO$, are in the Plane of the Parabola. This being premiſed,

Call the given Quantities $SF$ or $QO$, $a$; $FG$ or $KE$, $b$; $GB$, $c$; the Parameter of the Axis, $p$; and the unknown Quantities $BK$, $x$; $KA$ or $GH$, $y$. Now becauſe the Triangles $AKT$, $AEQ$, are ſimilar, therefore $AK\ (y) : KT\ (\frac{1}{2}p) :: AE\ (b+y) : EQ = \frac{bp}{2y} + \frac{1}{2}p$: And ſo ſince the Triangles $AEQ$, $AQO$, are right-angl'd at $E$ and $Q$, $\overline{AO}^2$ or $\overline{AE}^2 + \overline{EQ}^2 + \overline{QO}^2 = \frac{bbpp}{4yy} + \frac{bpp}{2y} + \frac{1}{4}pp + bb + 2by + yy + aa$. And the Parameter of the Diameter $AP$, *viz.* $v$ is $= {}^*p + 4x = p + \frac{4yy}{p}$ by ſubſtituting $\frac{yy}{p}$ for $x$; and $SO$ or $FQ$ or $GB + BK + EQ = c + x + \frac{bp}{2y} + \frac{1}{4}p = c + \frac{yy}{p} + \frac{bp}{2y} + \frac{1}{2}p$; now if theſe literal Values be ſubſtitnted for the Lines they expreſs in the Equation $\overline{AO}^2 = SO \times v$, then will $\frac{bbpp}{4yy} + \frac{bpp}{2y} + \frac{1}{4}pp + bb + 2by + yy + aa = cp + yy + \frac{bpp}{2y} + \frac{1}{2}pp + \frac{4cyy}{p} + \frac{4y^4}{pp} + 2by + 2yy$; and ſtriking out the ſame Quantities from both Sides, ſubſtituting $px$ for $yy$, and afterwards duly ordering the Equation, we ſhall get

$$\begin{array}{l} x^3 + cxx + \frac{1}{4}cpx - \frac{1}{16}bbp = o \\ \quad + \frac{1}{2}p \quad - \frac{1}{4}aa \\ \qquad\quad - \frac{1}{4}bb \\ \qquad\quad + \frac{1}{16}pp \end{array}$$

and the Value of $x$ which may be found * by means of the given Parabola, will give us $BK$, by which we can find the Point $A$ required. * *Art.* 387.

*Caſe* 2. Here the Problem may be expreſſed thus: To find the Point $A$ in the given Hyperbola $MAN$ being ſuch, that if the Diameter $AB$, as alſo the right Lines $SAa$, $BSb$ be drawn; and if thro' any Point $P$ of the Diameter $AB$ an Ordinate $PM$ in the Plane of the Hyperbola, and a Perpendicular $ab$ to this Ordinate in the Plane of the Triangle $aSb$: The Square $\overline{PM}^2$ may be equal to the Rectangle $aP \times Pb$. This is proved like as in the Parabola, and we muſt proceed thus for finding the Point $A$. FIG. 265.

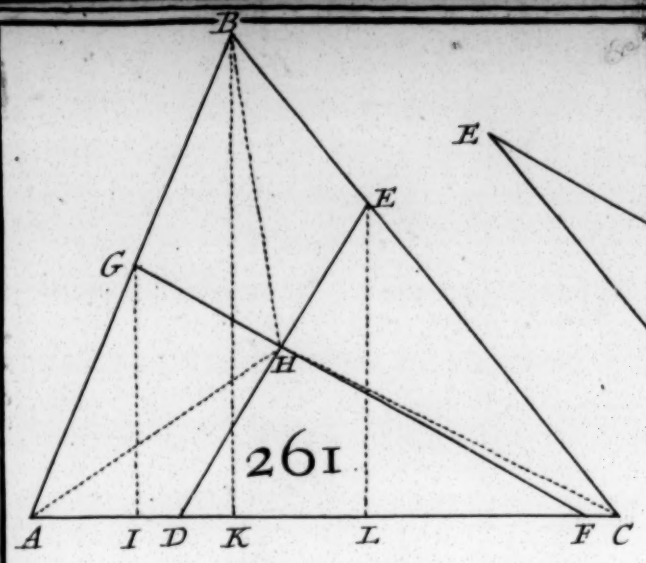
Let $v$ be the Conjugate Diameter to $AB$, and in the Plane of the Triangle $aSb$ draw the Lines $AO$, $OZ$, parallel to $ab$, $AB$, meeting

$SA$ in $Z$; then we ſhall have $AP \times PB$ to $\overline{PM}^2$, or (becauſe of the Circle) to $aP \times Pb$ in the Ratio compounded of $AP$ to $Pa$, or $ZO$ to $OA$, and of $PB$ to $Pb$, or of $BA$ to $AO$; that is, as $ZO \times AB$ to $\overline{AO}^2$. But by the Nature of the Hyperbola $AP \times PB : \overline{PM}^2 :: \overline{AB}^2 : vv$; and therefore $ZO \times AB : \overline{AO}^2 :: \overline{AB}^2 : vv = \frac{AB \times \overline{AO}^2}{ZO}$: Whence $OZ$: $AB$, or $OS : SB :: \overline{AO}^2 : vv$.

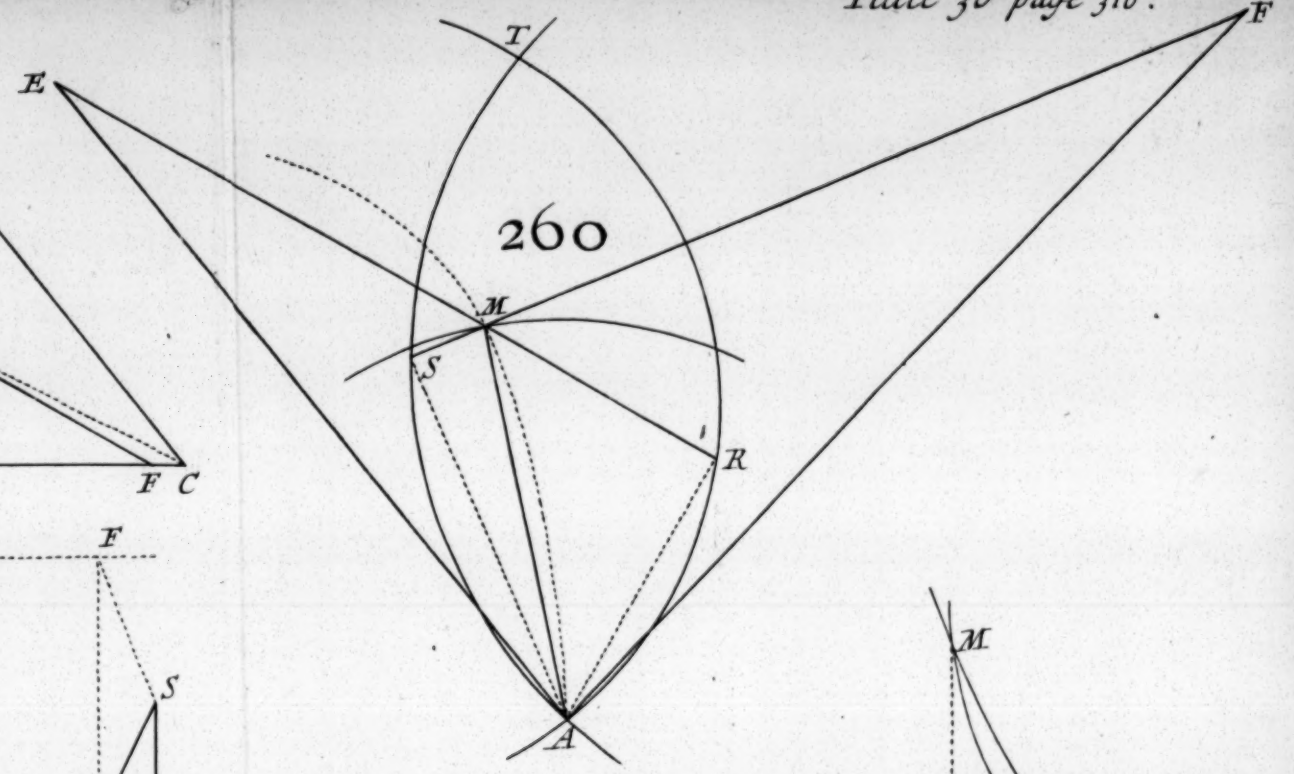
Now to find the literal Values of the Ratio of $OS$ to $SB$, and of $\overline{AO}^2$ to $vv$; from the given Point $S$ draw the Line $SF$ perpendicular to the Plane of the Hyperbola's; and from the Point $F$, wherein it meets this Plane, draw the Line $FG$ perpendicular to the Axis $DK$, which is given both in Poſition and Magnitude, becauſe the Hyperbola's are given. From the ſought Point $A$, draw the Ordinate $AK$ to the Axis, and the Line $AT$ perpendicular to the Tangent $AL$, meeting the Line $BF$ in the Points $E$, $Q$. Laſtly, Draw $BH$ parallel to the Axis meeting $GF$ in $X$, and $TV$ parallel to $BF$ meeting $AE$ in $V$, and $BD$, $QR$, perpendicular to the Axis; then if $QO$ be rais'd perpendicular to the Plane of the Hyperbola, we prove, after the ſame way as in the Parabola, that it will meet the Line $BS$ in the ſame Point $O$, as the Line $AO$ parallel to $ab$ meets it in. Here obſerve, that all the Lines, except $FS$, $QO$, are in the Plane of the Hyperbola's. This being premis'd, let $SF$ be $= a$, $FG = b$, $CG = c$, the firſt Axis $= 2d$, the ſecond $= 2f$, the unknown Quantities $CK$, or $CD = x$, $AK$ or $BD$ or $GX$ or $KH = y$; then will $DK$ or $BH = 2x$, $DG$ or $BX = c + x$, $GK = x - c$, $TK = \frac{ffx}{dd}$, and $AT$ or $\sqrt{\overline{TK}^2 + \overline{AK}^2} = \sqrt{yy + \frac{f^4xx}{d^4}} = \frac{f}{dd}\sqrt{ddxx + ffxx - d^4}$ by ſubſtituting * $\frac{ffxx}{dd} - ff$ for $yy$. But becauſe the Triangles $B \times F$, $BHE$, $TKV$, are ſimilar, therefore $BX\,(c + x) : XF\,(b - y) :: BH\,(2x) : HE = \frac{2bx - 2xy}{c + x} :: TX\left(\frac{ffx}{dd}\right) : KV = \frac{bffx - ffxy}{cdd - ddx}$; whence $AE$ or $AK + KH + HE = \frac{2bx + 2cy}{x + c}$, and $AV$ or $AK - KV = \frac{cddy + ddxy + ffxy - bffx}{cdd + dax}$. But becauſe the Triangles $AVT$, $AEQ$, and $ATK$, $QTR$, are ſimilar, therefore $AV : AT :: AE : AQ = \frac{2bfx + 2cfy\sqrt{ddxx + ffxx - d^4}}{ddxy + ffxy - bffx + cddy}$, and $AT : TK :: QT : TR :: AT + TQ$, or $AQ : KT + TR$, or $KR = \frac{2bff}{}$

* *Art.* 81.

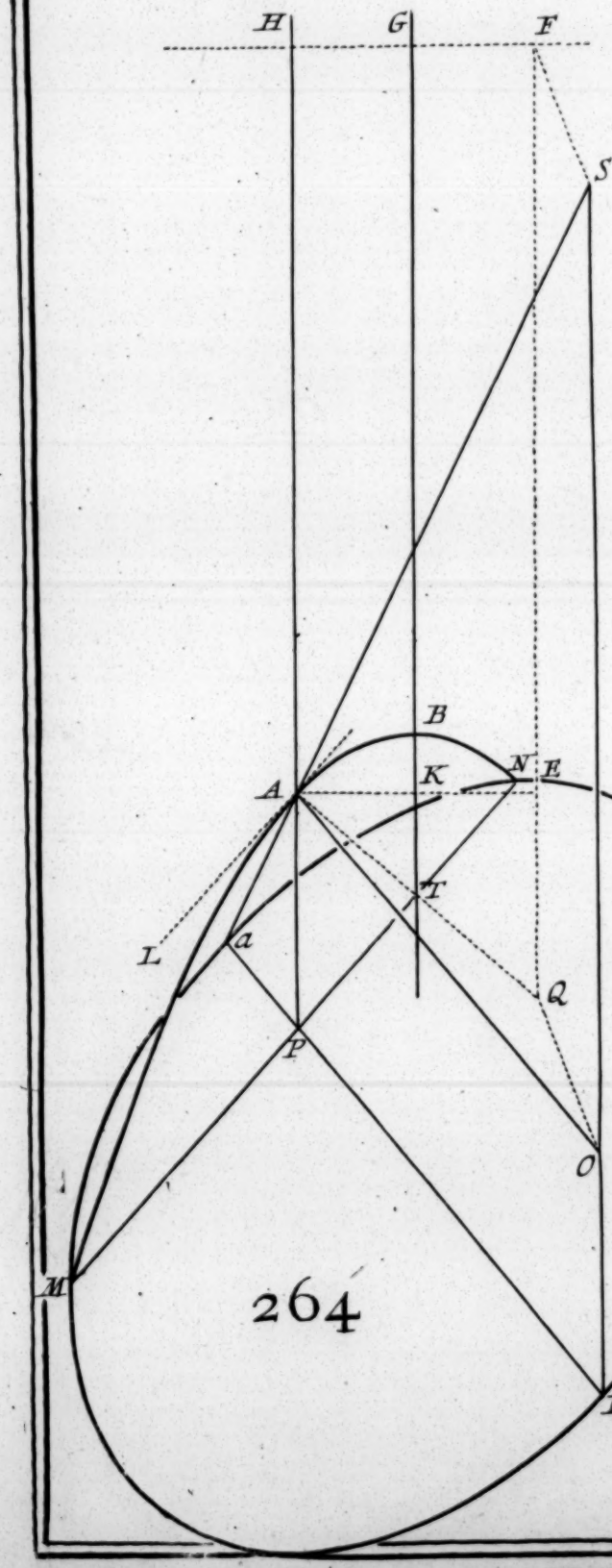
Plate 30 page 316.



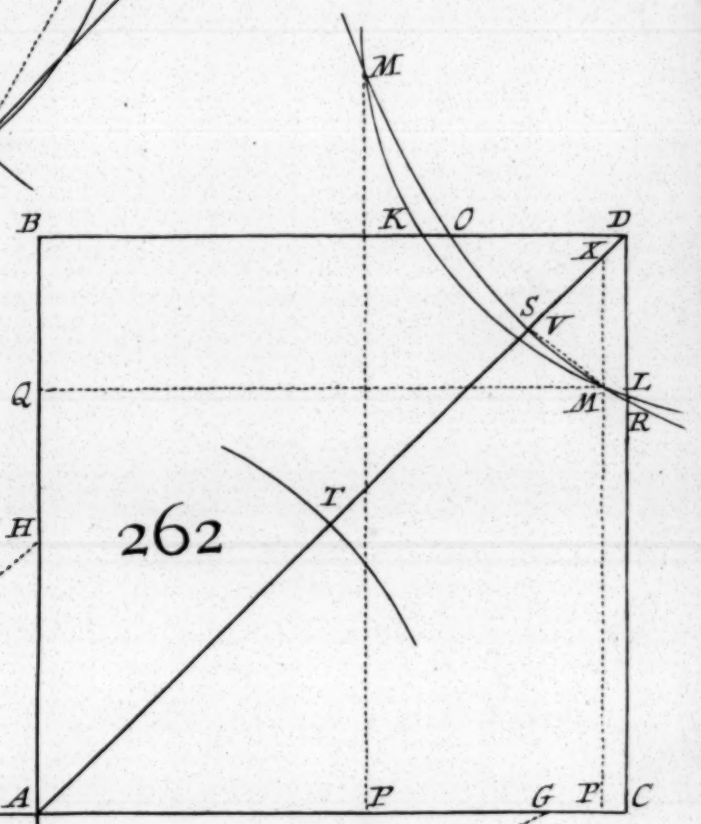

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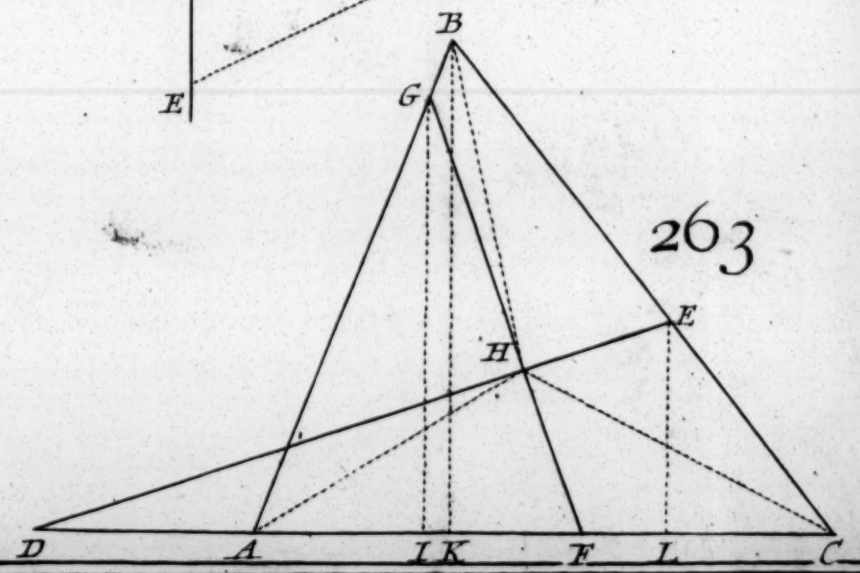

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$\frac{2bffxx+2cffxy}{ddxy+ffxy-bffx+cddy}$. Whence $\frac{GR \text{ or } DK+KR \times SF}{DG} = \frac{ddxy+ffxy+bffx-cddy}{ddxy+ffxy-bffx+cddy}$, and $\frac{DR \text{ or } DK+KR \times FS}{DG} = \frac{2addxy+2affxy}{ddxy+ffxy-bffx+cddy} = 2O$, becauſe $DG : DR :: BF : B2 :: FS : 2O$; and ſince the Triangle $A2O$ is right-angl'd, therefore $\overline{AO}^2$ or $\overline{A2}^2 + \overline{2O}^2 = = =$ $\frac{\overline{2bfx+2cfy}^2 \times \overline{ddxx+ffxx-d^4}+\overline{2addxy+2affxy}^2}{\overline{ddxy+ffxy-bffx+cddy}^2}$. Moreover, $SO : SB :: F2 : FB :: GR : GD$, and $vv = {}^*4xx + 4yy + 4ff - 4dd$, or $4xx - 4dd + \frac{4ffxx}{dd}$. *Art. 125.

Now if theſe literal Values be put for the Lines equal to them in the Proportion $SO : SB :: \overline{AO}^2 : vv$, and the Means and Extremes be multiplied, then we ſhall get this Equation $\overline{\overline{2bfx+2cfy}^2 \times \overline{ddxx+ffxx-d^4}+\overline{2addxy+2affxy}^2} = \overline{\overline{ddxy+ffxy+bffx-cddy} \times \overline{ddxy+ffxy-bffx+cddy} \times 4xx + \frac{4ffxx}{dd} - 4dd}$, in which all the Terms affected with $y$ do go out; and if $\frac{ffxx}{dd} - ff$ be put for $yy$, then will $bbd^4x^4 + 2bbddffx^4 + bbf^4x^4 - bbd^6xx - ccd^4xx - bbffd^4xx - ccffd^4xx + ccffd^6 + ccd^8 = \overline{\overline{ddxx+ffxx-d^4-aadd-ccdd} \times ddx^4+2ffx^4+\frac{f^4}{dd}x^4-d^4xx - 2ddffxx-f^4xx}$. Now if you make (for Brevity's Sake) $dd+ff = mm$, $bb+cc = nn$, $aa+dd+cc = rr$, the Equation may be brought to this $bbm^4x^4 - mmnnd^4xx + ccmmd^6 = \overline{mmxx-ddrr \times \frac{m^4}{dd}x^4 - m^4xx}$ and laſtly, making $xx = dz$, we ſhall have the following Equation of the third Degree.

$$z^3 \left\{ \begin{matrix} -d \\ -\frac{drr}{mm} \\ -\frac{dbb}{mm} \end{matrix} \right\} zz \quad \left\{ \begin{matrix} +\frac{ddrr}{mm} \\ +\frac{nnd^4}{m^4} \end{matrix} \right\} z - \frac{ccd^5}{m^4} = 0;$$

one of the Roots whereof; *viz.* that which is greater than $d$, being ſuch that if a mean Proportional be taken between that Root and $d$, the half of the firſt Axis; this mean Proportional does expreſs the Value of $CK$, by means of which the ſought Point $A$ is determin'd. Note, the Roots of this Equation may be found by means of (even) the given

given Hyperbola, according to the 396th, and 399th Articles.

When $CG\ (c) = o$, that is, when the Point $F$ falls in the ſecond Axis; then it is plain, that the aboveſaid Equation will become one of the ſecond Degree; becauſe the laſt Term being nothing, the Equation is diviſible by $z$. But when $F\,G\ (b) = o$, which happens when the Point $F$ does fall in the firſt Axis; then the Term $\frac{dbbzz}{mm}$ in the ſaid Equation will be nothing, and the Term $bb + cc$ will become $cc$; and ſo the Equation may be divided by $z - d$, and conſequently likewiſe brought down to this of the ſecond Degree $zz - \frac{drr}{mm}z + \frac{ccd^4}{m^4} = o$. Laſtly, if in this later Equation you make $c = o$; which muſt needs happen, when the Point $F$ falls in the Centre $C$, becauſe in this Caſe the Lines $b$ and $c$ do both become nothing; then we ſhall have $z = \frac{drr}{mm}$, and therefore $dz$ or $xx$ is $= \frac{ddrr}{mm}$, and $x = \frac{dr}{m} = d\sqrt{\frac{aa+dd}{dd+ff}}$.

It is needleſs to take notice, that the Problem is reſolved after the ſame way in the Ellipſis, only with the Alteration of ſome Signs. But the ſecond Caſe may be brought always to the firſt, after the following manner.

FIG. *A.* *Plate* 31. Through any Point ($B$) of one of the given Hyperbola's draw a Tangent $BG$, and through this Tangent and the Vertex ($S$) of the Cone, let the Plane $GBS$ paſs; and any how draw another Plane $HKL$ parallel to this Plane $GBS$. I ſay, that the Section $HKL$ of this Plane and the Conick Superficies, which is generated by the Motion of an indefinite right Line about the oppoſite Hyperbola $MAN$ always paſſing thro' the fixed Point $F$, is a Parabola; and ſo the whole Difficulty is reduced to the firſt Caſe. For ſuppoſing the Circle $DHMNL$ to be the Baſe of the Cone, having the Point $S$ for the Vertex, and the Hyperbola $MAN$, with that oppoſite to it, for a Section; it is manifeſt, that the Plane $GBS$ touching the oppoſite Superficies having that Circle for a Baſe, in the Side $BSD$, ſhall form a ſtraight Line $DE$ in the Plane of the Baſe, which will touch the Circle in the Point $D$. Now ſince this Line is the Directrix with reſpect to the Section $HKL$, therefore, according to the ſecond Definition of Book VI. the ſaid Section is a Parabola.

This was a very celebrated Problem in the Time of M. *Deſcartes*. A Solution thereof may be found among his Manuſcripts; which Solution is printed at the End of the 75th Letter of the third *Tome*. If any one has a mind to be at the Trouble of comparing *Deſcartes*'s Solution with mine, mine will be found much more natural, and leſs com-

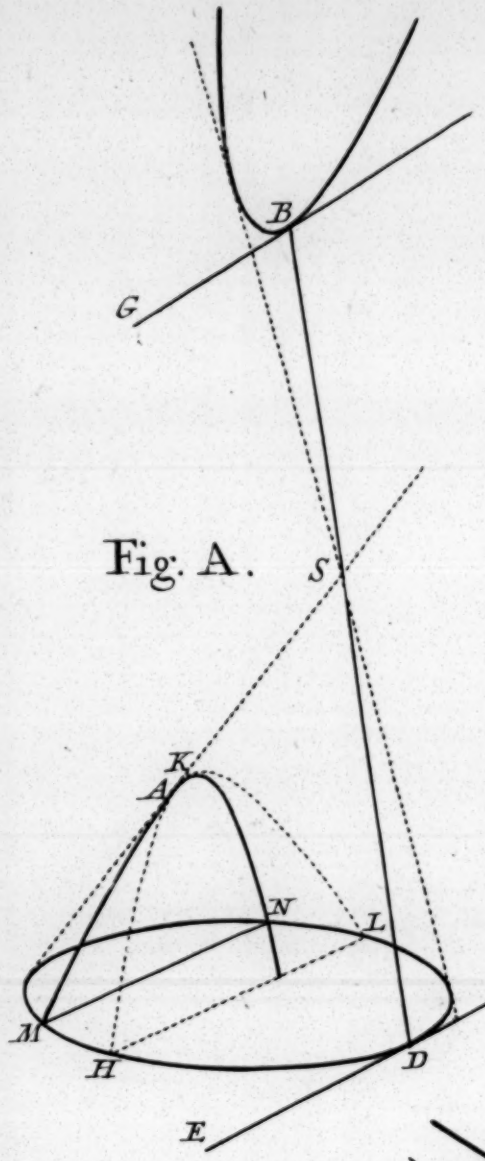



Fig. A.

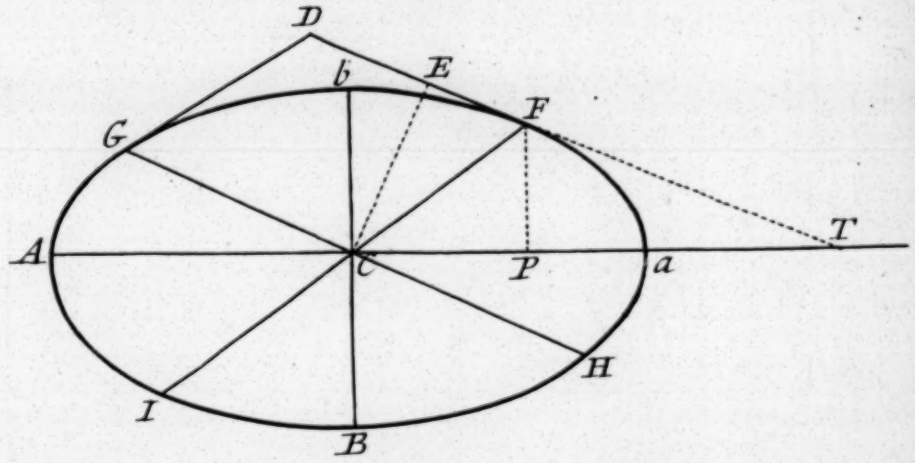



Fig. B.

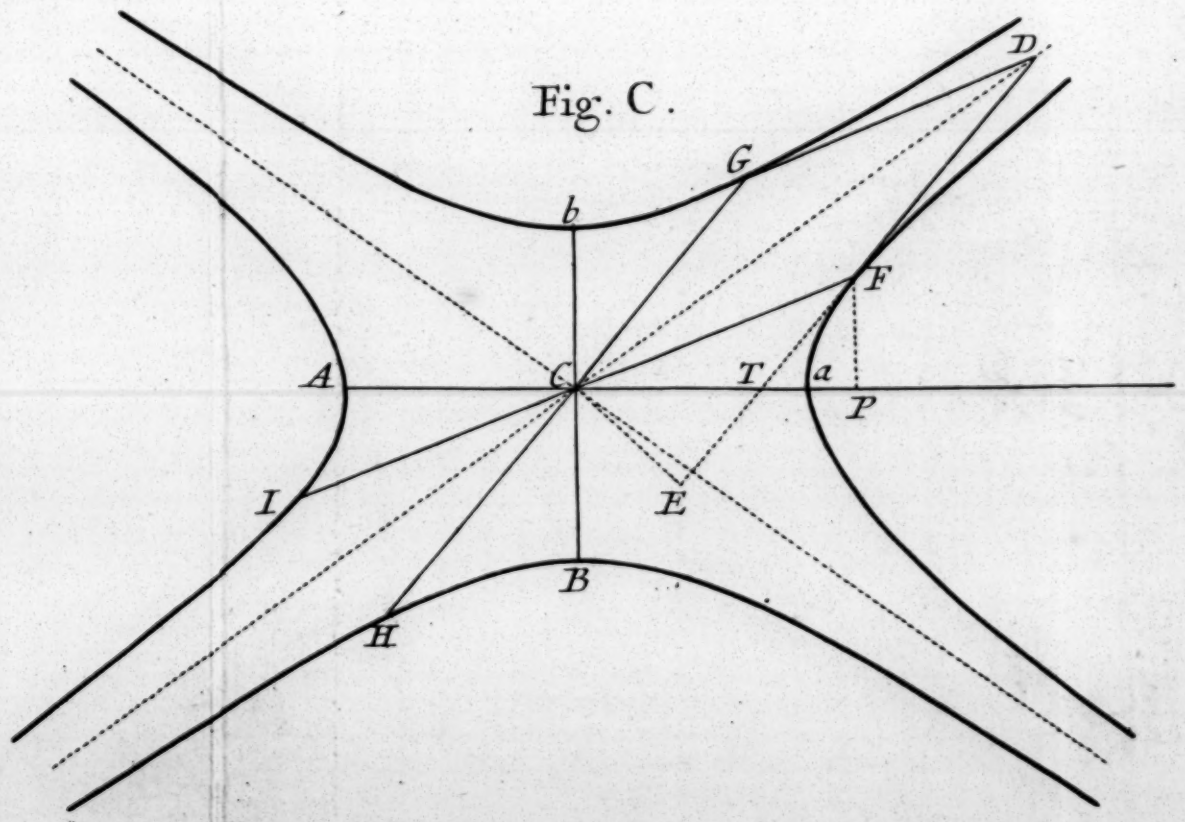



Fig. C.

complex than his: Neither does he give the Analyſis of the Caſe in which the Section is an Ellipſis or Hyperbola; and he thought it enough to aſſure himſelf, that the Equation, including the Conditions of the Problem, muſt not exceed the fourth Degree.

## LEMMA I.

442. *IF from* B *the End of the Diameter* A B [*of a Circle*] *there be drawn any Chord* B D *terminating the Arc* A D *leſs than the Semi-circumference; and if any two contiguous Arcs* E F, F G, *be taken any where at pleaſure, each equal to the Arc* A D, *and the Chords* B E, B F, B G, *be drawn. I ſay the middle Chord* B F, *is to the Sum or Difference of the two Chords* B E, B G, *next to it, as the Radius* C B *is to the Chord* B D: *viz. to the Sum when* B *the common Origin of the Chords* B D, B E, B F, B G, *falls in neither of the Arcs* E F, F G; *and to the Difference when the ſame is in one of theſe Arcs.* FIG. 266, 267, 268.

For about *F* as a Centre, with the Radius *F B*, deſcribe the Arc of a Circle cutting the Chord *B G* produced (if neceſſary) to *H*, that ſo we may have an Iſoſceles Triangle *B F H* which ſhall be ſimilar to the Iſoſceles Triangle *D C B*; becauſe the Meaſure of the Angle *F B H* is half of the *F G* equal to the Arc *A D*, the half of which is alſo the Meaſure of the Angle *C B D*. Therefore *F B* : *B H* : : *C B* : *B D*, and ſo it only remains to prove that the Line *B H* is the Sum of the Chords *B E*, *B G*, in the former Caſe, and their Difference, in the latter. To do this,

Draw the Chords *E F*, *F G*, then we ſhall have two ſimilar and equal Triangles *B E F*, *F H G*. For in the former Caſe the Angle *F H B* or *F H G*, is equal to *F B H* = *F B E*, ſince the Arcs *F G*, *F E*, are equal; likewiſe the Angle *B E F* is equal to the Angle *F G H*, becauſe the Half of the ſame Arc *B F* is the Meaſure of them both; and therefore the Angle *G F H* is equal to the Angle *E F B*. But the Sides *F E*, *F G*, and *F B*, *F H*, are equal to one another. Therefore the Side *G H* ſhall be equal to the Side *B E*. Whence, &c. FIG. 266. FIG. 268, 269.

In the latter Caſe we prove almoſt after the ſame Manner that the Triangles *F H G*, *F B E*, are ſimilar and equal; and ſo the Line *B H* is the Difference between the Chords *B G*, *B E*.

## LEMMA II.

443. *IF there be a Table of Expreſſions wherein the Number* 2 *being the firſt Term in the firſt Horizontal Row, and* x *the ſecond; the* 3*d Horizontal Row* xx—2 *is the Product of the ſecond, by* x *minus the firſt; the* 4*th* $x^3 - 3x$, *the Product of the third by* x *minus the ſecond; the fifth,* $x^4 - 4xx + 2$, *the Product of the fourth by* x *minus the third, and ſo on infinitely. Moreover, if there be any Arc* A R *of a Circle, divided into any Number of equal* FIG. 269, 270.

*equal Parts at pleaſure in the Points* D, E, F, G, &c. *I ſay, if the firſt Term* 2 *of the Horizontal Row in the Table does expreſs the Diameter* B A, *and the ſecond* x, *the firſt Chord* B D; *the third Row* x x — 2 *ſhall expreſs the Value of the ſecond Chord* B E; *the fourth Row* $x^3 - 3x$ *the Value of the third Chord* B F, *and ſo on to the laſt* B R; *where it muſt be obſerved that theſe Chords do become negative, when they are drawn on the other Side the Point* B.

*A Table for dividing the Arcs of a Circle into equal Parts.*

| | |
|---|---|
| 1 | 2 |
| 2 | $1x$ |
| 3 | $1xx - 2$ |
| 4 | $1x^3 - 3x$ |
| 5 | $1x^4 - 4xx + 2$ |
| 6 | $1x^5 - 5x^3 + 5x$ |
| 7 | $1x^6 - 6x^4 + 9xx - 2$ |
| 8 | $1x^7 - 7x^5 + 14x^3 - 7x$ |
| 9 | $1x^8 - 8x^6 + 20x^4 - 16xx + 2$ |
| 10 | $1x^9 - 9x^7 + 27x^5 - 30x^3 + 9x$ |
| 11 | $1x^{10} - 10x^8 + 35x^6 - 50x^4 + 25xx - 2$ |
| 12 | $1x^{11} - 11x^9 + 44x^7 - 77x^5 + 55x^3 - 11x$ |
| 13 | $1x^{12} - 12x^{10} + 54x^8 - 112x^6 + 105x^4 - 36xx + 2$ |
| 14 | $1x^{13} - 13x^{11} + 65x^9 - 156x^7 + 182x^5 - 91x^3 + 13x$ |

Fig. 269. For (1.) When the Arc *A R* is leſs than the Semi-circumference *A D B*; if any Chord, as *B F*, be multiply'd by *x*, and the Chord *B E* next before it be taken from the Product, the Remainder will be the Chord *B G* next after *B F*, becauſe by the aforegoing Lemma *C B* (1) : *B D* (*x*) : *B F* : *B E* + *B G* = *x B F*, and therefore *B G* is = *x BF* — *B E*. Whence, &c.

Fig. 270. 2. When the Arc *AR* is greater than the Semi-circumference *ADB*; then it is plain that *B* the common Origin of all the Chords will neceſſarily fall in one of the equal Parts (as *G H*) wherein the Arc *A R* is divided: And it may be proved as in (Caſe 1.) that the third Row in the Table does expreſs the Value of *B E*, the fourth the Value of *B F*, and ſo on to *B G*: But it remains to demonſtrate that the Row next after that expreſſing the Chord *B G*, will not expreſs the Value of + *B H* but — *B H*; and moreover, that the Row next after this laſt does expreſs the Value of — *B I*, and ſo on to — *B R*.

According to the Formation of the Table, the Row next after that expreſſing *B G* is *x B G* — *B F*. But by the Lemma *C B* (1) : *B D* (*x*) : : *B G* : *B F* — *B H*, and therefore — *B H* is = *x B G* — *B F*; that is, — *B H* is expreſſed by the parallel Row of the Table next after that expreſſing the Value of *B G*. But according to the Formation

mation of the ſame Table, the Row next after that expreſſing $-BH$ is $-x\,BH-BG$ the Value of $BI$, becauſe by the Lemma $x\,BH = BI-BG$: And moreover, the Row next after that expreſſing $-BI$ according to the Formation of the Table, is $-x\,BI+BH$ the Value of the negative Chord $-BL$, becauſe according to the Lemma $x\,BI$ is $=BL+BH$. And the ſame may be ſaid of all the Chords from $BL$ to $BR$; and this is what remained to be demonſtrated.

### COROLLARY I.

444. HENCE if the Arc $AR$ be divided into five equal Parts, the ſixth Row $x^5-5x^3+5x$ ſhall expreſs the Value of the Chord $BR$, ſubtending the Arc $BR$ the Difference between the Arc $AB$ and the Semi-circumference $ABD$; if the Arc $AR$ was divided into ſeven equal Parts, the eighth Row would expreſs the Value of $BR$; and generally the Number of equal Parts muſt be augmented by Unity, in order to get that Row of the Table expreſſing $BR$: Suppoſing the Radius $CB=1$, the firſt Chord $BD=x$, and that the laſt Chord $BR$ is negative, when the Arc $AR$ is greater than the Semi-circumference. FIG. 269, 270.

### COROLLARY II.

445. FROM the Compoſition of the Table, it appears, 1. That the Number 2 is the firſt Term of every perpendicular or upright Row. 2. That the Coefficients of all the other Terms of the firſt perpendicular Row are equal to Unity. 3. That the Coefficient of any Term of whatſoever perpendicular Row, is always equal to the Coefficient of a like Term in the perpendicular Row, to the left thereof *plus* the Coefficient of that Term which is above it: For example, the Coefficient 14 of the fourth Term $14x^3$ of the third perpendicular Row, is equal to the Coefficient 5 of the fourth Term $5x^3$ of the ſecond Perpendicular which is on the left, *plus* the Coefficient (9) of the Term $9\,x\,x$ which is above the Term $14x^3$.

### SCHOLIUM.

446. IF you ſhould continue on dividing the Circumference into Parts equal to the Arcs $AD$, $DE$, &c. beyond the Point $R$; then it is manifeſt that thoſe Horizontal Rows of the Table following that which does expreſs $-BR$, would ſtill orderly expreſs all the negative Chords that would follow $BR$, until you got again beyond the Point $B$, when the Chords would again become negative: And ſo on alternately poſitive and negative, as often as you ſhould paſs the Point $B$ *ad infinitum*. FIG. 270.

### EXAMPLE I.

447. TO cut a given circular Arc $A\,R$ into any Number of equal Parts $A\,D$, $D\,E$, $E\,F$, $F\,G$, &c.

FIG. 269, 270. Draw the Diameter $A\,B$, and the Chord $B\,R$, and call the given Radius $C\,A$ or $C\,B$, 1; and the given Chord $B\,R$, $a$; then form an equality between that Horizontal Row in the Table exceeding the Number of equal Parts the Arc is to be divided into by Unity, and $\mp a$; $a$ being affirmative when the Arc $A\,R$ is leſs than the Semi-circumference, and negative when the ſame is greater. Now it is manifeſt by the 443d Article, that the Root $x$ of this Equation will give us the Value of $B\,D$ being ſuch, that a Circle deſcribed about $B$, as a Centre, with the Diſtance $B\,D$, ſhall cut off $A\,D$ from the given Arc $A\,R$ the firſt of the equal Parts ſought.

For Example, if the given Arc $A\,R$ is to be divided into three equal Parts; then will $x^3 - 3x = \mp a$, and one of the Roots ($B\,D$) of this Equation ſhall terminate the firſt of the three equal Parts requir'd. If the Arc $A\,R$ is to be divided into five equal Parts, then will $x^5 - 5x^3 + 5x = \mp a$; in like manner if that Arc be to be divided into ſeven equal Parts, then will $x^7 - 7x^5 + 14x^3 - 7x = \mp a$: So that all the Difficulty conſiſts in finding the Roots of the Equations. Which may be done as is directed in the laſt Book. Therefore, &c.

Here obſerve that the ſaid Equations are the moſt ſimple poſſible, when the Number of equal Parts is a prime Number. But when it is compounded of two or more prime Numbers, the given Arc muſt be firſt divided into the ſame Number of equal Parts as one of theſe Numbers contains Unities, and then the firſt of theſe Parts into the ſame Number of equal Parts as one of the remaining Numbers contains Unities, and ſo on until all the prime Numbers are exhauſted, whoſe Product forms the given Number; and this gives us the firſt of the equal Parts ſought: For Example, If the Arc $A\,R$ is to be divided into 30 equal Parts, you muſt firſt divide the ſame into five equal Parts, and then the firſt of theſe five Parts into three; and finally, the firſt of theſe laſt Arcs into two Parts, that ſo we may get the thirtieth Part required, for $30 = 5 \times 3 \times 2$.

### SCHOLIUM I.

FIG. 270, 271, 272, &c. 448. TWO given Points $A$, $R$, in the Circumference of a Circle do not only determine two Arcs thereof, as $A\,R$, $A\,B\,R$, the one leſs, and the other greater than the Semi-circumference: But likewiſe an infinite Number of Portions, ſome whereof are the whole Circumference *plus* the Arc $A\,R$, twice the Circumference *plus* the Arc $A\,R$, three times the Circumference *plus* the Arc $A\,R$, &c. and others

others the whole Circumference *plus* the Arc *ABR*, twice the Circumference *plus* the Arc *ABR*, three times the Circumference *plus* the Arc *ABR*, &c. The Reaſon of which is, becauſe the Circumference of a Circle does return into it ſelf, therefore it may be conſider'd as a Curve making an Infinity of Revolutions about it ſelf; therefore if the Arc *AR* be called *d*, the whole Circumference *c*, the Arc *ABR* ſhall be *c*—*d*, and we ſhall have the two following Series:

1. $d, c+d, 2c+d, 3c+d, 4c+d, 5c+d, 6c+d, 7c+d, 8c+d$, &c.

2. $c-d, 2c-d, 3c-d, 4c-d, 5c-d, 6c-d, 7c-d, 8c-d$, &c.

orderly expreſſing all the Portions of the Circumference terminated by the two Points *A*, *R*. This being premis'd,

If the Arc *AD* be any *aliquot* Part of the Arc *AR*, which is leſs than the Semi-circumference; and a Polygon *DEFGH*, &c. (beginning at the Point *D*) being inſcribed in the Circle, having the ſame Number of Sides as the Number of Times the Arc *AR* contains the Arc *AD*; and if from *B*, the End of the Diameter *AB*, you draw the Chords *BD*, *BF*, *BG*, *BH*, &c. to the Angles of the Polygon: I ſay, theſe Chords do determine like *aliquot* Parts of all the Terms of the two Series, whereof the fix'd Origin is always at the Point *A*.

For Example: Let $AD = \frac{1}{5} d$; then it is evident, that the Arc *ADE* is $= \frac{c+d}{5}$, the Arc *ADEF* $= \frac{2c+d}{5}$, the Arc *ADEFG* $= \frac{3c+d}{5}$, the Arc *ADEFGH* $= \frac{4c+d}{5}$, which are the fifth Parts, or the like *aliquot* Parts of the five firſt Terms of the firſt Series. And if any other Term thereof be divided by 5, then it is evident, that the Quotient will contain exactly the whole Circumference ſome Number of times *plus* one of the precedent five Fractions. Whence ſince the Chord terminating an Arc, whoſe Origin is in *A*, is the ſame as the Chord that terminates that Arc *plus* the Circumference taken any Number of times at pleaſure; therefore the Chords *BD*, *BE*, *BF*, *BG*, *BH*, do determine fifth Parts of all the Terms of the firſt Series. After the ſame manner we prove, that the Arcs *AH*, *AHG*, *AHGF*, *AHGFE*, *AHGFED*, are fifth Parts of the five firſt Terms of the ſecond Series, and ſo the Chords *BH*, *BG*, *BF*, *BE*, *BD*, do terminate fifth Parts of all the Terms of the ſecond Series. But it is plain, that this Demonſtration may be apply'd to any other *aliquot* Part whatſoever of the Arc *AR*. Whence, *&c.* FIG. 271.

Hence, if the two preceding Series be brought to one only, *viz.* $d, c \mp d, 2c \mp d, 3c \mp d$, *&c.* the two Chords next on each Side the greateſt (*BD*) bounding the Arc *AD*, an *aliquot* Part of the Arc *AR* leſs than the Semi-circumference, ſhall bound Arcs being like *aliquot* Parts FIG. 273, 274.

of the ſecond Term $(c \mp d)$ of the Series; the two Chords immediately next to theſe here ſhall determine like *aliquot* Parts of the third Term $2c \mp d$ of the Series; and ſo on by two's to the two laſt, when the *aliquot* Part is odd, and to the laſt when it is even. So when the Arc $AD = \frac{1}{5} AR$; the Chords $BE$, $BH$, do bound the two Arcs $ADE$, $AH$, being both fifth's Parts of the ſecond Term $(c \mp d)$ of the Series, that is, of the Circumference *plus* the Arc $AR$, and of the Circumference *leſs* that Arc; the Chords $BF$, $BG$, immediately next to theſe, do bound the Arcs $ADEF$, $AHG$, each being fifth Parts of the third Term $(2c \mp d)$ of the Series. Moreover, when the Arc $AD$ is $= \frac{1}{6} AR$; then the Chords $BE$, $BK$, immediately on each Side the longeſt Chord $BD$, do bound the Arcs $ADE$, $AK$, being each one ſixth Part of the ſecond Term $c \mp d$; the Chords $BF$, $BH$, immediately next to theſe here, do bound the Arcs $ADEF$, $AKH$, each being one ſixth Part of the third Term $2c \mp d$; and $BG$ the laſt Chord of all does bound the Arcs $ADEFG$, $AKHG$, each being one ſixth Part of the fourth Term $3c \mp d$.

In the following Scholiums, by *odd Chords*, we underſtand thoſe (taken on both Sides the firſt or longeſt one $BD$) whoſe Number, together with the ſaid longeſt Chord, is odd; and by *even Chords*, thoſe whoſe Number, together with that, is even. So when the Arc $AD$ is $= \frac{1}{5} AR$, the Chords $BD$, $BF$, $BG$, are odd Chords, and the Chords $BE$, $BH$, even ones: In like manner, when the Arc $AD$ is $= \frac{1}{6} AR$; then the Chords $BD$, $BF$, $BH$, are odd Chords, and the Chords $BE$, $BK$, $BG$, even ones.

## SCHOLIUM II.

FIG. 273, 274. 449. IF the Arc $AD$ be any *aliquot* Part of the Arc $AR$ leſs than the Semi-circumference $ARB$; and if a Polygon $DEFGH$, &c. be inſcrib'd in the Circle, being at the Point $D$, having its Number of Sides equal to the Number of times the Arc $AD$ is contain'd in $AR$, and if from $B$ the Extremity of the Diameter $AB$, you draw the Chords $BD$, $BE$, $BF$, $BG$, $BH$, &c. I ſay, the odd Chords, when the Arc $AD$ is an odd *aliquot* Part of the Arc $AR$, and their Squares, when it is an even *aliquot* Part, does expreſs the affirmative Roots of an Equation form'd by making an Equality between the poſitive Magnitude $+ a$, and that parallel or horizontal Row of Expreſſions in the Table, whoſe Exponent is greater by an Unit than the Number of the Sides of the Polygon; and the even Chords in the former Caſe, and their Squares in the latter, do expreſs the affirmative Roots of the other Equation form'd by making an Equality between the negative Magnitude $- a$, and the ſame parallel or horizontal Row of Expreſſions in the Table.

For

For Example: Let the Arc $AD$ be $= \frac{1}{5} AR$; I ſay, the odd Chords $BD$, $BF$, $BG$, are the affirmative Roots of this Equation $x^5 - 5x^3 + 5x = a$, and the even Chords $BE$, $BH$, are the affirmative Roots of this other Equation $x^5 - 5x^3 + 5x = -a$. If the Arc $AD$ be $= \frac{1}{7} AR$; then the Squares of the odd Chords $BD$, $BF$, $BH$, ſhall be the affirmative Roots of this Equation $x^6 - 6x^4 + 9xx - 2 = a$, and the Squares of the even Chords $BE$, $BK$, $BG$, ſhall be the Roots of this other Equation $x^6 - 6x^4 + 9xx - 2 = -a$. FIG. 271, 273. FIG. 272, 274.

For if it be propos'd to divide the whole Circumference repeated ſome Number of times *plus* or *minus* the Arc $AR$, into equal Parts, ſo that the firſt may be leſs than the Semi-circumference; then, by the 444th Article it is plain, that the ſame Table muſt be form'd, as was for the Diviſion of the Arc $AR$; but here the Chords muſt neceſſarily change their Signs once (before the laſt $BR$ be had) when the Circumference is but once repeated, becauſe $B$ the common Origin of them all does fall in one of the equal Parts; twice, when the Circumference is repeated twice, becauſe the Origin $B$ will neceſſarily be found in two of the equal Parts; three times, when the Circumference is repeated thrice, becauſe the Origin $B$ is found in three equal Parts, and ſo on. Therefore the Chord $BR$ will be poſitive, when the Arc $AR$, and the Circumference repeated ſome even Number of times *plus* or *leſs* the Arc $AR$, is to be divided into equal Parts; and negative, when the Circumference is repeated an odd Number of times; that is, in the former Caſe, the parallel Row of the Table muſt be made equal to $+a$; and conſequently the odd Chords, or their Squares, ſhall be the affirmative Roots of the other Equation, whereof one of the Members is $-a$. *W. W. D.*

## SCHOLIUM III.

450. THE ſame Things being premis'd, if the Arc $AD$ be an odd *aliquot* Part of the Arc $AR$; then by the bare Inſpection of the Table it appears, that all the even Terms, that is, the ſecond, fourth, ſixth, *&c.* except the laſt Term $a$, are always wanting in the two Equations found according to the preceding *Scholium*. But it is ſhewn in Algebra, that if the Signs of the even Terms of an Equation be chang'd, all that is done by this, is only changing the affirmative Roots into negative ones, and the negative ones into affirmative ones. Whence the even Chords being the affirmative Roots of the Equation, having $-a$ for one Side thereof, will become the negative Roots of the other Equation, having $+a$ for one Side of it. For Example; if the Arc $AD$ be $= \frac{1}{5} AR$; then the odd Chords $BD$, $BF$, $BG$, ſhall be the affirmative Roots of the Equation $x^5 - 5x^3 + 5x = a$, and the even Chords $BE$, $BH$, the negative Roots thereof. FIG. 271, 273.

From

From theſe two latter Scholiums may be gotten ſeveral new Theorems touching the Inſcription of regular Polygons; in obſerving that the known Magnitude of the ſecond Term of an Equation is equal to the Sum of its Roots; that of the third Term, equal to the Sum of the alternate Products of the Roots, that of the fourth Term equal to the Sum of the alternate Solids, *&c.* and the laſt Term is equal to the Product of all the Roots into one another. Four of the Principal of which I ſhall lay down, after having firſt premiſed the following Scholium, which perhaps may be of uſe.

### SCHOLLIUM IV.

FIG. 271. 273. 451. THE ſame things being laid down as in the preceeding Scholium, wherein we ſuppoſe that the *aliquot* Arc ( *A D* ) of the Arc *A R* is odd; I ſay, that among the Chords contain'd in the Semi-circumference *A R B*, whereof the Arc *A R* is a part, the laſt or ſhorteſt *B F* ſubtends an Arc *B F*, being to the Arc *B R*, in the Ratio of the Arc *A D* to the Arc *A R*.

For let the Arc *A D* be a fifth Part of the Arc *A R*, and ſo the Arc *D E* the fifth Part of the Circumference; then it is plain that the Semi-circumference *ARB* ſhall contain the Arc *D E* twice and an half, that is, twice the Arc *DE*, or elſe the Arc *D E F plus* the fifth Part of the Semi-circumference. Therefore the Arc *A D plus* the Arc *B F*, is equal to the fifth Part of the Semi-circumference *A R B*. And ſo becauſe *A D* is the fifth Part of the Arc *A R*, the Arc *B F* ſhall be alſo the fifth Part of the Arc *B R*, which is the Complement of the Arc *A R* to two right Angles. But what we have demonſtrated here, is of the ſame Importance, whether the Arc *A D* be a fifth Part of the Arc *A R*, or any other odd *aliquot* Part. Therefore, *&c.*

Hence if the Chord *B R* of any Arc *B R* leſs than the Semi-circumference be called *b*, and the Radius 1; and if an Equation be form'd one Side of which is *b*, and the other that horizontal Row of Expreſſions in the Table greater by an Unit than the Number of equal Parts the Arc *BR* muſt be divided into; one of the Roots of this Equation will be (*B F*) the Chord of the firſt of its Parts, and conſequently the Chord (*B G*) of the firſt of a like Number of equal Parts of the Arc *B A R*, being the Complement of *B R* to four right Angles, will be another of its Roots.

### THEOREM I.

FIG. 271. 452. *IF any regular Polygon* D E F G H, *&c. of an odd Number of Sides, be inſcribed in a Circle; and from any Point* B *in the Circumference be drawn the Chords* B D, B E, B F, B G, B H, *&c. to all the Angles of the Polygon*: I *ſay,*

1. *The Sum of the odd Chords* B D, B F, B G, *&c. beginning at the firſt and largeſt* B D, *ſhall always be equal to the Sum of the even Chords* B E, B H,

B H, &c. *that is, the ſhorteſt Chord* B F — B E + B D — B H + D G &c. = *o.*

For draw the Diameter *B A*, and take the Arc *A R* containing the Arc *A D* the ſame Number of times as the Polygon has Sides; then it is evident, as we have ſhewn already, if the Chord *B R* be call'd *a*; and the Radius *C A* or *C B*, 1; that the odd Chords *B D*, *B F*, *BG*, &c. ſhall be the affirmative Roots, and the even Chords *B E*, *B H*, &c. the negative Roots of the Equation, whereof one Side is + *a*. And ſince the ſecond Term, which is equal to the Sum of the Roots, as is prov'd in *Algebra*, is always wanting in that Equation; therefore, &c.

2. *If the Diameter* B A *be drawn, and the Arc* A R *be taken, containing the Arc* A D *the ſame Number of times as the Polygon has Sides, and the Chord* B R *be drawn: The Product* B D × B E × B F × B G × B H, &c. *of all the Chords* B D, B E, B F, B G, B H, &c. *into one another, ſhall always be equal to the Product of the Chord* B R *into the Radius* C A, *raiſed to a Power leſs by Unity than the Number of Chords.*

For this laſt Product is equal to the Member *a*; ſince *B R* = *a*, and the Radius *C A* is taken for Unity in the Table. And becauſe the Term *a* is always the laſt Term of the Equation, whoſe Roots are the Chords *B D*, *B E*, *B F*, *B G*, *B H*. &c. and the laſt Term of an Equation, as is proved in *Algebra*, is always equal to the Product of all its Roots; therefore, &c.

## THEOREM II.

453. IF *any Semi-circumference* A E B *be divided into an odd Number of equal Parts, whereof the two firſt is the Arc* A E; *the four firſt, the Arc* A E F, *and ſo on by Pairs to the laſt; and if the Chords* B E, B F, &c. *be drawn: I ſay,* FIG. 273.

1. *That* B E *the firſt of thoſe Chords, minus the ſecond* B F, *plus the third, minus the fourth, and ſo on to the laſt incluſively, is always equal to the Radius.*

2. *That the Product* [B E × B F, &c.] *of all the Chords into one another, is equal to an anſwerable Power of the Radius. So in this Example wherein the Number of Diviſions is* 5, *and conſequently there are but two Chords* B E, B F, *we ſhall have* 1. B E — B F = C A. 2. B E × B F = $\overline{\text{C A.}}^{2}$

For in the whole Circle inſcribe the regular Polygon *E F G H*, having the ſame Number of Sides as there are Diviſions, beginging from the Point *A*; and from the other End (*B*) of the Diameter *A B* draw the Chords *B D*, *B E*, *B H*, *B F*, *B G*, &c. to all the Angles of the Polygons; and then it is manifeſt, 1. That *B D* the largeſt of theſe Chords is equal to the Diameter *B A*, and ſo the Arc *A D* being

= *o*,

$= a$, the Arc $AR$ ſhall be ſo alſo; and conſequently the Chord $BR$ ſhall likewiſe be equal to the Diameter $BA$. 2. That the Chords $BE$, $BH$; $BF$, $BG$, &c. being taken by Pairs, are equal to one another. And this being premis'd, if the aforegoing Theorem be apply'd to this particular Caſe, you will perceive this Theorem to ariſe from it. Whence, *&c.*

## THEOREM III.

FIG. 272. 454. *IF any regular Polygon* DEFGHK, &c. *be inſcrib'd in a Circle, of an even Number of Sides; and if from any Point* B *in the Circumference, there be drawn the Chords* BD, BE, BF, BG, BH, BK, *&c. to all the Angles of the Polygon. I ſay,*

1. *The Sum of the Squares of the odd Chords* BD, BF, BH, *or elſe of the even Chords* BE, BG, BK, *is equal to the Square of the Radius* CB, *taken the ſame Number of times as the Polygon has Sides.*

Draw the Diameter $BA$, and take the Arc $AR$ containing the Arc $AD$ the ſame Number of times as the Polygon has Sides; then if the Chord $BR$ be called $a$, and the Radius $CA$ or $CB$, 1; the Squares of the odd Chords $BD$, $BF$, $BH$, &c. ſhall * be the affirmative Roots of the Equation having one Side equal to $+ a$; and the Squares of the even Chords $BE$, $BK$, $BG$, &c. the affirmative Roots of the other Equation having one Side equal to $- a$. But the Coefficient of the ſecond Term of both the aforeſaid Equations being equal to the Sum of their Roots, is always equal to the Square of the Radius taken the ſame Number of times as the Polygon has Sides, as appears in the Table. Therefore, *&c.*

* *Art.* 449.

2. *If the Diameter* BA *be drawn, and the Arc* AR *taken as many times containing the Arc* AD *as the Polygon has Sides, and the Chord* BR *be drawn; the Product* $[\overline{BD}^2 \times \overline{BF}^2 \times \overline{BH}^2$ &c.$]$ *of all the Squares of the odd Chords, is equal to the Product of* $BA \mp BR$ *into an anſwerable Power of the Radius,* viz. $BA + BR$, *when the Number of the Sides of the Polygon is barely even, and* $BA - BR$ *when it is evenly even, that is, diviſible by* 4; *and the Product* $\overline{BE}^2 \times \overline{BG}^2 \times \overline{BK}^2$, &c. *of the even Chords, is equal to the Product of* $BA \mp BR$ *into the ſame Power of the Radius,* viz. $BA - BR$ *in the former, and* $BA + BR$ *in the latter Caſe.*

For call $BR$, $a$; and the Radius $CA$, 1; then it is plain, that the Squares of the odd Chords $BD$, $BF$, $BH$, &c. are the Roots of an Equation, whoſe laſt Term will always be $2 \mp a$, that is, $BA \mp BR$; and that the Squares of the even Chords $BE$, $BG$, $BK$, &c. are the Roots of an Equation, whoſe laſt Term will be always $2 \mp a$, that is, $BA \mp BR$. And becauſe the laſt Term of an Equation is equal always to the Product of all its Roots, therefore, *&c.*

COROL-

COROLLARY.

455. HENCE it is evident, 1. That the Sum of the Squares of all the Chords, as well even as odd, is equal to the Square of the Radius drawn into double the Number of Sides of the Polygon, that is, $\overline{BF}^2 + \overline{BE}^2 + \overline{BD}^2 + \overline{BK}^2 + \overline{BH}^2 + \overline{BG}^2 = 12\,\overline{CA}^2$. 2. That the Difference between the Squares of the even Chords and the odd ones, is always equal to nothing; that is, $\overline{BF}^2 - \overline{BE}^2 + \overline{BD}^2 - \overline{BK}^2 + \overline{BH}^2 - \overline{BG}^2 = 0$. 3. That the Product of the Squares of the odd Chords *plus* the Product of the Squares of the even Chords, is equal to the Quadruple of an answerable Power of the Radius, that is, $\overline{BF}^2 \times \overline{BD}^2 \times \overline{BH}^2 + \overline{BE}^2 \times \overline{BK}^2 \times \overline{BG}^2 = 4\,\overline{CA}^6$. 4. That the Difference between the two Products, is equal to the Double of the Chord $BR$ multiply'd by an answerable Power of the Radius; taking notice at the same time, that the Product of the Squares of the odd Chords is greater than that of the Squares of the even Chords, when the Number of the Sides of the Polygon is barely even, and less when the Number is evenly even; that is, $\overline{BF}^2 \times \overline{BD}^2 \times \overline{BH}^2 - \overline{BE}^2 \times \overline{BK}^2 \times \overline{BG}^2 = 2\,BR \times \overline{CA}^5$. 5. That the Product of the Squares of all the Chords, both even and odd, shall be always equal to the Product of $\overline{BA}^2 - \overline{BR}^2 = BA \mp BR \times BA \pm BR = \overline{AR}^2$, (because $ARB$ is a right Angle) by an answerable Power of the Radius; that is, by extracting the square Root of both Sides, the Product of all the Chords is equal to the Product of the Chord $AR$ by a Power of the Radius less by Unity than the Number of Chords; that is, $BF \times BE \times BD \times BK \times BH \times BG = AR \times \overline{CA}^5$.

THEOREM IV.

456. *IF the Semi-circumference* ADB *be divided into any even Number of equal Parts, the first whereof let be the Arc* AD; *the first three, the Arc* ADE; *the first five, the* ADEF, *and so on by two's to the last; and if the Chords* BD, BE, BF, &c. *be drawn; I say,* FIG. 275.

1. *The Sum of the Squares of these Chords is equal to the Square of the Radius taken as many times as there are Divisions in the Semi-circumference; that is, the Number of Divisions being here* 6, $\overline{BD}^2 + \overline{BE}^2 + \overline{BF}^2$ *will be* $= 6\,\overline{CA}^2$.

2. *That the Product of the Squares of the Chords by one another, is equal to twice an answerable Power of the Radius. So* $\overline{BD}^2 \times \overline{BE}^2 \times \overline{BF}^2 = 2CA^6$. *And consequently* $BD \times BE \times BF = CA^3 \times \sqrt{2}$.

For if a regular Polygon $DEFGHK$ be inſcrib'd in the whole Circle of the ſame Number of Sides as there are Diviſions, beginning at $D$; and if Chords, as $BD$, $BK$, $BE$, $BH$, $BF$, $BG$, &c. be drawn from the Extremity ($B$) of the Diameter $AB$, to all the Angles of the Polygon; then it is plain, that the Chords $BD$, $BK$, $BE$, $BH$, $BF$, $BG$, &c. being taken two and two, are equal, and therefore if the firſt and third Numbers of the laſt Corollary be apply'd to this particular Caſe, you will perceive this Theorem to ariſe from it.

## EXAMPLE XII.

457. TO inſcribe any regular Polygon in a given Circle, that ſhall have any given Number of Sides.

FIG. 275. This Problem is only a particular Caſe of the laſt Example; for if the Chord $BR$ be ſuppos'd to become nothing, the Arc $AR$ terminating it, will become the Semi-circumference. But if it be propos'd to divide the whole Circumference into any Number of equal Parts; if the Semi-circumference be divided into the ſame Number of Parts, and the ſecond Chord be taken, it is plain, that this will terminate FIG. 276. the firſt of the Parts requir'd. For Example: If the Semi-circumference $ADB$ be divided into the ſeven equal Parts $AD$, $DE$, $EF$, $FG$, $GH$, $HI$, $IB$; the ſecond Chord $BE$ ſhall terminate the Arc $AE$, being the ſeventh Part of the whole Circumference. Therefore if that horizontal Row of the Table exceeding the Number of Sides of the Polygon by Unity be made equal to $o$, we ſhall by this means form an Equation, whoſe greater Root $x$ ſhall expreſs the Value of the Chord $BD$ terminating the Arc $AD$, being the half of $AE$ which is ſought. But * $CB\,(1) : BD\,(x) :: BD\,(x) : BE + BA$, and conſequently if the ſecond Chord $BE$ be called $z$, $xx$ will be $= z + 2$. Whence, if by means of this Equation you get out the unknown Quantity $x$ from the former one, we ſhall have a new Equation, whoſe greateſt Root $z$ ſhall expreſs the Chord $BE$, bounding the ſought Arc $AE$. So in our Example, the eighth horizontal Row in the Table being made equal to $o$, and divided by $x$, will be this, $x^6 - 7x^4 + 14xx - 7 = o$, wherein ſubſtituting $z + 2$ for $xx$, and that Equation will be changed iuto this $z^3 - zz - 2z + 1 = o$, whoſe greateſt Root $z$ does expreſs the Value of the Chord $BE$ terminating the Arc $AE$, being the ſeventh Part of the whole Circumference.

*Art. 442.

We ſhall now ſhew a general Way of preſently finding all theſe Equations, when the Number of the Sides of the Polygon is odd, which is the only Caſe neceſſary; becauſe when even, it can always, by dividing by 2, be reduced to an odd Number; in which having divided the Circle by the Biſection of one of theſe equal Parts, repeated as often as is neceſſary, we ſhall have the Arc requir'd.

Make

Make a Table, wherein the firſt horizontal Row being 1, and the ſecond $z - 1$; let the third $z\,z - z - 1$ be equal to the Product of the ſecond by $z$ leſs the firſt; the fourth $z^3 - z\,z - 2z + 1$, equal to the Product of the third by $z$, leſs the ſecond, and ſo on. Then form an Equation, one Side of which being nothing, let the other be that horizontal Row of Quantities in the Table, whoſe Exponent is half the Number of Sides of the Polygon *plus* 1: I ſay, the greateſt Root $z$ of this Equation ſhall terminate an Arc, whoſe Chord ſhall be the Side ſought of the Polygon.

*A Table for the Inſcription of regular Polygons in a Circle.*

| | |
|---|---|
| 1 | $1$ |
| 2 | $z - 1$ |
| 3 | $zz - z - 1$ |
| 4 | $z^3 - zz - 2z + 1$ |
| 5 | $z^4 - z^3 - 3zz + 2z + 1$ |
| 6 | $z^5 - z^4 - 4z^3 + 3zz + 3z - 1$ |
| 7 | $z^6 - z^5 - 5z^4 + 4z^3 + 6zz - 3z - 1$ |
| 8 | $z^7 - z^6 - 6z^5 + 5z^4 + 10z^3 - 6zz - 4z + 1$ |
| 9 | $z^8 - z^7 - 7z^6 + 6z^5 + 15z^4 - 10z^3 - 10zz + 4z + 1$ |
| 10 | $z^9 - z^8 - 8z^7 + 7z^6 + 21z^5 - 15z^4 - 20z^3 + 10zz + 5z - 1$ |

For Example, It is requir'd to inſcribe an Heptagon in a Circle. Take the 4th Horizontal Row of Quantities in the Table, becauſe four is greater than half ſeven by 1, and making it equal to nothing, we have $z^3 - zz - 2z + 1 = 0$, and the greateſt Root $z$ of this Equation ſhall expreſs the Value of the Chord $BE$, terminating the Arc $AE$ the ſeventh Part of the whole Circumference, which may be proved thus.

Let $AR$ be an Arc of a Circle leſs than the Semi-circumference di- FIG. 275. vided into any odd Number of equal Parts in the Points $D$, $E$, $F$, $G$, &c. and from $B$ the Extremity of the Diameter $BA$, draw the Chords $BD$, $BE$, $BF$, $BG$, &c. Moreover, take the Arc $AS$ equal to the Arc $AD$, draw the Chord $BS$, and call the firſt Chord $BD$, or $BS$, $x$; and the ſecond $BE$, $z$. Then by the Lemma, $CB\,(1) : BE\,(z) :: BD\,(x) : BF + BS$. And conſequently $BF = xz - x$. Moreover, $CB\,(1) : BE(z) :: BF : BD + BH$. And conſequently $BH = zBF - BD$: In like manner $CB\,(1) : BE\,(z) :: BH : BF + BR$, and therefore $BR = z\,BH - BF$: That is, the fifth Chord $BH$ is equal to the Product of the third $BF$ by $z$ *minus* the firſt $BD$; the ſeventh $BR$ is equal to the Product of the fifth $BH$ by $z$ *minus* the third $BF$, and ſo on. Whence it appears, that if a Table be made wherein the firſt Horizontal Row [of Quantities] is $x$, and the ſecond $zz - x$; the third, is $xzz - xz - x$ equal to the Product of the ſecond by $z$ *minus* the firſt; the fourth $xz^3 - x\,z\,z - 2xz + 1$ equal to the Product of the third by $z$ *minus* the ſecond, and ſo on: Then ſhall the Rows of Quantities of

this Table orderly expreſs all the odd Chords *B D*, *BF*, *B H*, *B R*, of the Arc *A R*. But the ſaid Rows of Quantities being the ſame as in the aforegoing Table, each multiply'd by $x$; therefore ſuppoſing the laſt Chord *B R* to become equal to nothing, (which happens when the Arc *AR* becomes the Semi-circumference) and proceeding as we have directed, we ſhall get an Equation whoſe unknown Quantity $z$ ſhall expreſs the ſecond Chord *B E* terminating the Arc *A E*, which is contain'd the ſame Number of times in the whole Circumference) as the Arc *A D* being the half thereof, is contain'd in the Semi-circumference.

FIG. 276.

Here we muſt obſerve, 1. That the Equations found after this manner are the moſt ſimple poſſible, when the Number of the Sides of the Polygon is a prime Number. But when it is a Number compoſed of two or more prime Numbers, then the whole Circumference muſt be firſt divided into the ſame Number of equal Parts as the greater of thoſe Numbers contains Units; and then again, one of theſe laſt Parts into the ſame Number of equal Parts as one of the remaining Numbers contains of Unites, and continue ſo doing, until all the prime Numbers which do compoſe the given Number be exhauſted. 2. That among the Chords drawn from the Point *B*, which are contain'd in the Semi-circumference *A E B*; the odd ones reckoning the longeſt *BE* the firſt, are the affirmative, and the even ones the negative Roots of the Equation found by this Method: So the Chords *B E*, *B I*, are the two affirmative Roots of the Equation $z^3 - zz - 2z + 1 = 0$, and the Chord *B G* is the negative Root thereof. 3. That among the Roots of theſe kind of Equations, the ſmalleſt is the Chord of an Arc, being the $\frac{1}{7}$ of that ſought: That is, in this Example, the ſmalleſt Root *B I*, of the Equation $z^3 - zz - 2z + 1 = 0$, is the Chord of an Arc *B I*, being the 14th Part of the Circumference.

## SCHOLIUM.

458. IN this Table it appears plain that the Coefficient of all the Terms of the firſt and ſecond perpendicular Rows is Unity; the Coefficients of the third and fourth Rows, are the natural Numbers 1, 2, 3, 4, 5, *&c.* being form'd by the continual Addition of an Unit; the Coefficients of the fifth and ſix Rows, the Triangular Numbers 1, 3, 6, 10, *&c.* form'd by the continual Addition of the natural Numbers; the Coefficients of the ſeventh and eighth Rows, the pyramidal Numbers 1, 4, 10, *&c.* which are form'd by the continual Addition of Triangular Numbers; and ſo on, *ad infinitum*, the Coefficients of any Perpendicular Row being always formed by the continual Addition of thoſe of the Row immediately before that.

LEMMA

## LEMMA I.

459. IF A D, E F, *be two equal Arcs of the Semi-circle* A E B, *one of them, as* A D *being taken from* A, *the end of the Diameter* A B, *and the other any where at pleaſure; and if the Chords* B D, B E, B F, *and* A D, A E, A F, *be drawn: I ſay,* 1. A B × B F = B D × B E — A D × A E. 2. A B × A F = B D × A E + A D × B E. Fig. 277.

For the three right-angl'd Triangles *A D G* (the Point *G* being, here the Interſection of the Chords *B D*, *A F*) *A E B*, *B F G*, are ſimilar to each other, becauſe the Angle *A G D* or *B G F*, having half of the two Arcs *B F*, *A D*, for the Meaſure thereof, is equal to the Angle *B A E*, having likewiſe the half of the two Arcs *B F*, *F E*, or *A D*, for the Meaſure thereof. Whence if the Diameter *A B* be called 1; the Chords *B D*, $x$; *A D*, $y$; *B E*, $v$; *A E*, $z$; then will *B E* $(v)$ : *E A* $(z)$ :: *A D* $(y)$ : *D G* $= \frac{yz}{v}$; and therefore *B G* or *B D* — *D G* $= x - \frac{yz}{v}$. 2. And *A B* (1) : *B E* $(v)$ :: *B G* $\left(x - \frac{yz}{v}\right)$ : *B F* $= vx - yz$, that is (becauſe *A B* = 1) *A B* × *B F* = *B D* × *B E* — *A D* × *A E*. Which was in the firſt place to be demonſtrated.

Now *B E* $(v)$ : *B A* (1) :: *A D* $(y)$ : *A G* $= \frac{y}{v}$. And *A B* (1) : *A E* $(z)$ :: *B G* $\left(x - \frac{yz}{v}\right)$ : *G F* $= xz - \frac{zzy}{v}$; and therefore *A G* + *G F* or *A F* $= xz - \frac{zzy}{v} + \frac{y}{v} = xz + vy$, becauſe the Triangle *A E B* being right-angl'd; $1 - zz = vv$; that is, *A B* × *A F* = *B D* × *A E* + *A D* × *B E*. Which was what remained to be demonſtrated.

## LEMMA II.

460. LET *there be form'd a Table, the firſt Horizontal Row whereof conſiſting of two Parts* x *and* y; *let all the others be ſo likewiſe according to this Law,* viz. *That the firſt Part of any Horizontal Row be form'd by multiplying the firſt Part of that Horizontal Row immediately, before it by* x minus *the ſecond Part of the ſame multiply'd by* y; *and the ſecond Part form'd by multiplying the ſaid firſt one by* y, plus *that ſecond Part multiply'd by* x. *Moreover, let there be any Arc* (A R) *of a Circle, leſs than half the Circumference, which ſuppoſe to be divided into any Number of equal Parts at pleaſure, in the Points* D, E, F, G, *&c. I ſay, if the Diameter* AB = 1, *and the two firſt Chords* BD = $x$, *A D* = $y$; *then all the other Chords* BE, BF, BG, *&c. ſhall be expreſs'd by the firſt Parts of the ſecond,* Fig. 278.

*third, fourth, &c. Horizontal Row of Terms, and the other correſpondent Chords* A E, A F, A G, *&c. by the ſecond Parts of the ſame Rows. So* B G *being the fourth Chord, will be expreſſed by* $x^4 - 6yyxx + y^4$ *the firſt Part of the fourth Horizontal Row, and its Correſpondent* A G *by the ſecond Part* $4yx^3 - 4y^3x$ *of the ſame Row.*

| | | |
|---|---|---|
| 1 | $x$ | $y$ |
| 2 | $xx - yy$ | $2yx$ |
| 3 | $x^3 - 3yyx$ | $3yxx - y^3$ |
| 4 | $x^4 - 6yyxx + y^4$ | $4yx^3 - 4y^3x$ |
| 5 | $x^5 - 10yyx^3 + 5y^4x$ | $5yx^4 - 10y^3xx + y^5$ |
| 6 | $x^6 - 15yyx^4 + 15y^4xx - y^6$ | $6yx^5 - 20y^3x^3 + 6y^5x$ |
| 7 | $x^7 - 21yyx^4 + 35y^4x^3 - 7y^6x$ | $7yx^7 - 35y^3x^4 + 21y^5xx - x^7$ |

For by the foregoing Lemma it is plain that the Product of any Chord *B F* by the firſt Chord *B D* ($x$), *minus* the Product of the correſpondent Chord *A F*, by the other firſt Chord *A D* ($y$), does expreſs the Value of the Chord *B G*, being that next after *B F*; and ſo the Chord *A G* is $= BF \times AD\,(y) + AF \times BD\,(x)$. Whence, *&c.*

## COROLLARY.

461. IF the two Parts of each horizontal Row of Terms in the aforegoing Table be added together, and all the Terms be order'd according to the different Powers of $x$; the following new Table will thereby be form'd, orderly containing all the Powers of the Binomial $x + y$; where the firſt and ſecond Terms muſt be affirmative, the third and fourth negative, and ſo of every two alternately to the laſt. Thus the third horizontal Row of Terms is $x^3 + 3yxx - 3yyx - y^3$; that is, the Cube of the Binomial $x + y$, the two firſt Terms whereof are affirmative, and the two others negative. In like manner, the 5th horizontal Row will be $x^5 + 5yx^4 - 10yyx^3 - 10y^3xx + 5y^4x + y^5$, which is the fifth Power of the Binomial $x + y$, whereof the firſt and ſecond Terms are affirmative, the third and fourth negative, and the fifth and ſixth affirmative.

| | |
|---|---|
| 1 | $x + y$ |
| 2 | $xx + 2xy - yy$ |
| 3 | $x^3 + 3yxx - 3yyx - y^3$ |
| 4 | $x^4 + 4yx^3 - 6yyxx - 4y^3x + y^4$ |
| 5 | $x^5 + 5yx^4 - 10yyx^3 - 10y^3xx + 5y^4x + y^5$ |
| 6 | $x^6 + 6yx^5 - 15yyx^4 - 20y^3x^3 + 15y^4xx + 6y^5x - y^6$ |
| 7 | $x^7 + 7yx^6 - 21yyx^5 - 35y^3x^4 + 35y^4x^3 + 21y^5xx - 7y^6x - y^7$ |

For

For if Regard be had to the manner of the Formation of the precedingTable, it will appear, that all the Terms of every horizontal Row are form'd by thoſe of that horizontal Row next before it, multiply'd by $x$ and $y$, and ſo connected together by the Signs + and —, ſo that the Terms of the two Parts compoſing every horizontal Row, being put in order, according to the different Powers of the unknown Quantity $x$, there is an alternate Succeſſion of two affirmative, and then two negative Signs.

### SCHOLIUM.

462. IN this latter Table it is manifeſt, that the Coefficients of every Term of the firſt perpendicular or upright Row, is Unity; of the Coefficients of the ſecond Row, the natural Numbers 1, 2, 3, 4, *&c.* being form'd by the continual Addition of Unities; the Coefficients of the third Row, the triangular Numbers, 1, 3, 6, 10, *&c.* being form'd by the continual Addition of the natural Numbers; the Coefficients of the fourth Row, the Piramidal Numbers 1, 4, 10, 20, *&c.* being form'd by the continual Addition of triangular Numbers; and ſo on from Row to Row towards the right Hand, the Numbers of a ſuperior Order being formed by the continual Addition of thoſe of the Order immediately going before.

### EXAMPLE XII.

463. AN Arc of a Circle $AR$ being given; to divide it into any Number of equal Parts at pleaſure in the Points $D, E, F, G$, *&c.* after a different manner from that in the 10th Example. FIG. 278.

Call the Diameter $AB$, 1; the given Chords $BR$, $AR$ (terminating the given Arc $AR$) $a$ and $b$; and the unknown Chords $BD$, $AD$, (terminating the ſought Arc $AD$) $x$ and $y$, and raiſe the Binomial $x + y$ to ſuch a Power, that its Exponent be equal to the Number of Diviſions. Then form two Equalities; one between the given Quantity $a$, and all the odd Terms of that Power of $x + y$ connected by the Signs + and — alternately; and the other, between the given Quantity $b$, and all the remaining Terms of the ſaid Power of the Binomial $x + y$, connected by the alternate Signs. This being done, get out one of the unknown Quantities $x$ or $y$ by means of the Equations $xx = 1 - yy$, or $yy = 1 - xx$, ariſing from the right-angl'd Triangle $ADB$; and then we ſhall have an Equation, having only one unknown Quantity in it, which being reſolv'd, will give us $BD$ or $AD$, terminating the Arc ſought $AD$.

For Example: It is requir'd to divide the given Arc $AR$ into ſeven equal Parts in the Points $D, E, F, G, H, I$. Take the ſeventh Power of the Binomial $x + y$, *viz.* $x^7 + 7yx^6 + 21yyx^5 + 35y^3x^4 + 21y^5xx + 7y^6x$

+

$+y^7$. And from it form thoſe two Equations $a = x^7 - 21yyx^5 + 35y^4x^3 - 7y^6x$, and $b = 7yx^6 - 35y^3x^4 + 21y^5xx - y^7$. Then if the unknown Quantity $y$ be gotten out of the former of theſe Equations, or the unknown Quantity $x$ from the latter, by means of $yy = 1 - xx$, or $xx = 1 - yy$, and one of theſe new Equations will be formed, *viz.* $a = 64x^7 - 112x^5 + 56x^3 - 7x$, or $b = 7y - 56y^3 + 112y^5 - 63y^7$, having but one unknown Quantity in it, which being reſolved according to Directions in the Ninth Book, will give us *B D* or *A D*, determining the firſt of the equal Parts required. All this is but a Continuation of the two foregoing Articles.

Here you muſt obſerve, that if the Arc *A R* ſhould be greater than half the Circumference, the latter of theſe Equations will equally ſolve the Problem in this Caſe, without altering any thing in it: But in order for the former to do it, *a* muſt be made negative; the reaſon of which is, that the Chord *B R* as falling below or on the other Side the Point *B* does become negative, whereas the Chord *A R* does always remain poſitive.

## LEMMA I.

464. *IF in any Square of Cells, the firſt Horizontal Row be filled up with* a', *and the firſt upright or perpendicular Row with* b', *the firſt Cell excepted; and afterwards if all the other Cells be fill'd up according to this Law*; viz. *That any Cell be equal to that which is next above it*, plus *that being to the left thereof: By this means we ſhall have the following Square of Cells:*

| | 1. | 2. | 3. | 4. | 5. | 6. | 7. |
|---|---|---|---|---|---|---|---|
| 1. | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ |
| 2. | $b$ | $a+b$ | $2a+b$ | $3a+b$ | $4a+b$ | $5a+b$ | $6a+b$ |
| 3. | $b$ | $a+2b$ | $3a+3b$ | $6a+4b$ | $10a+5b$ | $15a+6b$ | $21a+7b$ |
| 4. | $b$ | $a+3b$ | $4a+6b$ | $10a+10b$ | $20a+15b$ | $35a+21b$ | $56a+28b$ |
| 5. | $b$ | $a+4b$ | $5a+10b$ | $15a+20b$ | $35a+35b$ | $70a+56b$ | $126a+84b$ |
| 6. | $b$ | $a+5b$ | $6a+15b$ | $21a+35b$ | $56a+70b$ | $126a+126b$ | $252a+210b$ |
| 7. | $b$ | $a+6b$ | $7a+28b$ | $35a+56b$ | $84a+126b$ | $210a+252b$ | $462a+462b$ |

*Wherein I ſay, that any Cell is equal to that to the left thereof,* plus *all thoſe that are above it. For Example, the fourth Cell* 4 a + 6 b *of the third upright Row, is equal to the Cell* a + 3 b *to the left thereof, which is conſequently the fourth Cell of the ſecond upright Row,* plus a + 2 b, a + b a, *being the Cells above it in the ſecond upright Row.*

For

For ſuppoſing $a$, $c$, $d$, $e$, to expreſs the four firſt Cells of the ſecond upright Row, and $a, f, g, h$, the four firſt Cells of the third Row; then by the Formation of the Cells, we ſhall have $h = e + g$, $g = d + f$, $f = c + a$, and therefore $h = e + d + c + a$; which is the thing to be prov'd. And it is manifeſt, that this Demonſtration extends to any Number of Cells whatſoever of two adjoining upright Rows. Whence, *&c.*

### COROLLARY.

465. BEcauſe all the Cells, the firſt horizontal and upright Rows being excepted, are compoſed of two Terms, the Letter $a$ being in the firſt, and $b$ in the laſt Term; therefore, 1. The Term wherein is the Letter $a$, is equal to the Term wherein is likewiſe found the Letter $a$ in the next Cell to the left, *plus* all the Terms wherein it is, in the Cells being above this here. 2. The Term wherein is the Letter $b$, is equal to the Term wherein is likewiſe the Letter $b$, in the Cell to the left, *plus* all the Terms wherein it is found in the Cells above it. So the Term $15a$ of the fifth Cell of the fourth upright Row, is equal to the Term $5a$ of the Cell to the left, *plus* the Terms $4a$, $3a$, $2a$, $1a$, which are in the Cells above it; and $20b$ is equal to the Term $10b$ of the Cell to the left, *plus* the Terms $6b$, $3b$, $1b$, of all the Cells above it.

### LEMMA II.

466. IF *the Term wherein is the Letter* a, *in any Cell, be multiplied by the Sum of the Index's of its horizontal and upright Rows* minus 2, *and the Product be divided by the Index of its perpendicular Row* minus 1; *I ſay, the Quotient ſhall be equal to that Term,* plus *all them that be above it. For Example: If the Term* $15a$ *in the fifth Cell of the fourth upright Row, be multiply'd by* $5 + 4 - 2 = 7$, *and the Product divided by* $4 - 1 = 3$, *the Quotient* $35a$ *ſhall be equal to the Term* $15a$, plus *the Terms* $10a$, $6a$, $3a$, $1a$, *being above it.*

This appears plain in all the Cells of the ſecond upright Row, ſince they all do contain the ſame Term $1a$. And I ſhall demonſtrate, that ſuppoſing this Property to happen in any upright Row whatſoever, it muſt needs be ſo likewiſe in the Cell to the right of that; from whence it will follow, that becauſe it happens in the ſecond upright Row, it ſhall be alſo in the third; and becauſe in the third, it ſhall be in the fourth; and ſo on *ad infinitum.* And to prove this,

Let $a, b, c, d, e, f$, &c. be any Number of the Terms wherein is the Letter $a$, in any upright Row; and $a, g, h, k, l$, &c. the like Number of Terms of the Row to the right thereof. Alſo let $m$ be equal to the Sum of the Indexes (*minus* 2.) of the upright and horizontal Rows

wherein is the Term $f$; and $r$ equal to the Index *minus* 1 of the upright Row, in which that Cell is. Now by Suppoſition $\frac{m}{r} f = f + e$ $+ d + c + a = {}^{*}l$, $\frac{m-1}{r} e = e + d + c + a = k$, $\frac{m-2}{r} d = d + c + a$ $= b$, $\frac{m-3}{r} c = c + a = g$, $\frac{m-4}{r} a = a$. Whence $l+k+b+g+a = \frac{m}{r} f +$ $\frac{m-1}{r} e + \frac{m-2}{r} d + \frac{m-3}{r} c + \frac{m-4}{r} a = \frac{m}{r} \times f + e + d + c + a$, $-$ $\frac{1}{r} \times 1e+2d+3c+4a = \frac{m}{r} l$, $-\frac{1}{r} \times k + b + g + a$, by ſubſtituting $l$ for $f + e + d + c + a$, and $k + b + g + a$, for $1e + 2d + 3c + 4a$; then if $l$ and $-\frac{1}{r} \times k+b+g+a$ be tranſpos'd, we ſhall have $\frac{r+1}{r} \times k + b+g$ $+ a = \frac{m-r}{l} l$; and multiplying both Sides by $r$, dividing by $r + 1$, and adding $l$ to both Sides; there ariſes $\frac{m+1}{r+1} l = l + k + b + g + a$.

*Art. 464.

But becauſe the Index of that upright Row, wherein is the Cell, whoſe Term is repreſented by $l$, does exceeed the Index of that wherein is the Cell, whoſe Term is expreſs'd by $f$, and theſe Cells are both in the ſame horizontal Rows; therefore the Property to be prov'd in every Term, having the Letter $a$ in any upright Row, does alſo agree to the Term $l$ in the upright Row to the right thereof. Farther, becauſe this Demonſtration is the ſame, be the Number of Terms of two adjoining Rows what they will, therefore what we have demonſtrated with regard to the Term $l$, will be likewiſe true of any Term in the ſame Row.

Now if we ſuppoſe $n$ to expreſs generally the Index of any horizontal Row except the firſt, then the firſt Cell of this Row will not contain any Term wherein is the Letter $a$; the ſecond will always contain $1a$; and if you multiply $1a$ by $\frac{n+2-2}{2-1} = \frac{n}{1}$, we ſhall get $\frac{n}{1} a$ for the Term wherein is found the Letter $a$ in the third Cell; and if you multiply $\frac{n}{1} a$ by $\frac{n+3-2}{3-1} = \frac{n+1}{2}$, we ſhall have $\frac{n}{1} a \times \frac{n+1}{2}$ for the Term wherein is the Letter $a$ in the fourth Cell; ſo that this Series $o$, $1a$, $\frac{n}{1} a$, $\frac{n}{1} \times \frac{n+1}{2} a$, $\frac{n}{1} \times \frac{n+1}{2} \times \frac{n+1}{3} a$, $\frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \times$ $\frac{n+3}{}$

$\frac{n+3}{4}a$, *&c.* will orderly expreſs all the Terms wherein is the Letter *a* in the horizontal Row of Cells, whoſe Index is *n*. So if $n = 5$, the Series *o*, 1 *a*, 5 *a*, 15 *a*, 35 *a*, *&c.* will orderly expreſs all the Terms, wherein is the Letter *a* in the 5th horizontal Row of Cells.

## LEMMA III.

467. *IF the Term, wherein is the Letter* b *in any Cell, be multiply'd by the Sum of the Indexes of its horizontal Row, and its perpendicular Row* minus 2, *and if the Product be divided by the Index of its perpendicular Row: I ſay, the Quotient ſhall be equal to that Term* plus, *all thoſe being above it. For Example; If the Term* 10 b *of the fifth Cell in the third upright Row be multipy'd by* $5 + 3 - 2 = 6$, *and the Product be divided by* 3, *then will* 20 b *be the Sum of the Term* 10 b, *together with all the Terms* 6 b, 3 b, 1 b, *above it.*

It is manifeſt, that this Property happens in the firſt perpendicular Row, wherein all the Cells do contain 1 *b*, except the firſt, which has not the Letter *b* in it. By means of which we can demonſtrate, as in the laſt *Lemma* with regard to the Terms multiplying *a*, that the ſame will likewiſe happen in the ſecond upright Row, the third, the fourth, and ſo in all the others *ad infinitum*. From whence we conclude, that if *n* expreſſes the Index of any horizontal Row, except the firſt, the Series $1\,b, \frac{n-1}{1}b, \frac{n-1}{1}\times\frac{n}{2}b, \frac{n-1}{1}\times\frac{n}{2}\times\frac{n+1}{3}b, \frac{n-1}{1}\times\frac{n}{2}\times\frac{n+1}{3}\times\frac{n+2}{4}b$, *&c.* ſhall orderly expreſs all the Terms, wherein is the Letter *b* in the parallel Row of Cells whoſe Index is *n*. So if $n = b$, the Series 1 *b*, 4*b*, 10*b*, 20*b*, 35 *b*, *&c.* ſhall orderly expreſs all the Terms wherein *b* is, in the fifth parallel Row.

## COROLLARY.

468. IT follows from the two laſt Lemmata, if all the Terms of this Series be added to the Terms of that in the laſt Lemma, that thereby we ſhall form this, $1\,b, 1\,a, +\frac{n-1}{1}b, \frac{n}{1}a + \frac{n-1}{1}\times\frac{n}{2}b, \frac{n}{1}\times\frac{n+1}{2}a + \frac{n-1}{1}\times\frac{n}{2}\times\frac{n+1}{3}b, \frac{n}{1}\times\frac{n-1}{2}\times\frac{n+2}{3}a \times \frac{n-1}{1}\times\frac{n}{2}\times\frac{n+1}{3}\times\frac{n+2}{4}b$, *&c.* or by abbreviating the Expreſſion, $b, a+\frac{n-1}{1}b, a + \frac{n-1}{2}b\times\frac{n}{1}a + \frac{n-1}{3}b\times\frac{n}{1}\times\frac{n+1}{2}, a + \frac{n-1}{4}b\times\frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}$ *&c.* which ſhall orderly expreſs all the Cells of that horizontal Row of the Table whoſe Index is *n*.

Whence you ſee that by means of this Series, we can at once find any Cell at pleaſure, the Exponent of the upright and horizontal Rows being given; becauſe taking in the general Series the Term anſwering the Exponent of the upright Row, that is, the fourth Term, if the upright Row be the fourth, the fifth, if it be the fifth, *&c.* and ſubſtituting the Exponent of the horizontal Row for $n$, and the Cell ſought will be had. For Example, ſuppoſe it be requir'd to find the fifth Cell of the fourth upright Row; ſubſtitute the Exponent 5 of the horizontal Row the Cell is in for $n$, in the fourth Term $a + \frac{n-1}{3} b \times \frac{n}{1} \times \frac{n+1}{2}$, and then ſhall we have $a + \frac{4}{3} b \times 15$, or, $15a + 20b$ for that Cell. The ſame muſt be underſtood for others.

## LEMMA IV.

469. IF a *be made* $= 2$, and $b = 1$, *in the Square of Cells of Article* 464. *then ſhall that Square of Cells be changed into this: Wherein, I ſay, that the firſt horizontal Row of Numbers does contain continually the firſt Terms of all the upright Rows of the Table in Article* 143; *the ſecond horizontal Row, the ſecond Terms; the third, the third Terms, and ſo on continually to Infinity.*

| | 1. | 2. | 3. | 4. | 5. | 6. | 7. |
|---|---|---|---|---|---|---|---|
| 1. | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 2. | 1 | 3 | 5 | 7 | 9 | 11 | 13 |
| 3. | 1 | 4 | 9 | 16 | 25 | 36 | 49 |
| 4. | 1 | 5 | 14 | 30 | 55 | 91 | 140 |
| 5. | 1 | 6 | 20 | 50 | 105 | 196 | 336 |
| 6. | 1 | 7 | 27 | 77 | 182 | 378 | 714 |
| 7. | 1 | 8 | 35 | 112 | 294 | 672 | 1386 |

This is a natural Continuation of the 445th Article, and the Formation of the Square of Cells of Article 464. explain'd in that ſame Article, and the 465th one following it.

## COROLLARY.

470. IF you make $b = 1$, and $a = 2$, in the general Series of Article 468, *viz.* $b$, $a + \frac{n-1}{1} b$, $a + \frac{n-1}{2} b \times \frac{n}{1}$, $a + \frac{n-1}{3} b \times \frac{n}{1} \times \frac{n+1}{2}$, $a + \frac{n-1}{4} b \times \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3}$, *&c.* then will it be changed into this

this 1, $\frac{n+1}{1}$, $\frac{n+3}{2}\times\frac{n}{1}$, $\frac{n+5}{3}\times\frac{n}{1}\times\frac{n+1}{2}$, $\frac{n+7}{4}\times\frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}$, &c. by means whereof at once may be found the Coefficient of the Table in *Art.* 443. its upright Row, and the Exponent of the horizontal Row being given. The Rule is this.

Take the Term in this Series anſwering to the given upright Row, that is, the third, if it be the third Row; the fourth, if it be the fourth, *&c.* and having ſubſtituted the Number expreſſing the parallel Row the Term is in, for $n$ in that Term, you will have the Coefficient ſought. For Example: If you want the Coefficient of the fourth Term $14x^3$ of the third upright Row, ſubſtitute the Number 4 for $n$ in the third Term $\frac{n+3}{2}\times\frac{n}{1}$, and 14 will be the Coefficient ſought.

For, the Exponent of the upright Row of Coefficients in the Table of *Article* 443, is the ſame as the Exponent of the upright Row in the Square of Cells of the laſt *Article*; and the Exponent of the horizontal Row, is equal to the Exponent of the horizontal Row of the Square of Cells. Whence it is manifeſt, that this Rule is only an Application of that of *Article* 468, to this particular Caſe wherein $a = 2$ and $b = 1$.

## LEMMA V.

471. *IF* 1 *be ſubſtituted for* b, *in the Square of Cells of Article* 464; *then will that be chang'd into this, wherein I ſay that the upright Rows do orderly contain all the Numbers call'd Figurate,* viz. *the firſt Row, the Numbers of the firſt Order, which are Units; the ſecond Row, the natural Numbers, or thoſe of the ſecond Order; the third Row, the triangular Numbers, or thoſe of the third Order, being form'd by the continual Addition of the natural Numbers; the fourth, the Pyramidal Numbers, or thoſe of the fourth Order, being form'd by the continual Addstion of the triangular Numbers; and ſo on to Infinity.*

| | 1. | 2. | 3. | 4. | 5. | 6. | 7. |
|---|---|---|---|---|---|---|---|
| 1. | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2. | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 3. | 1 | 3 | 6 | 10 | 15 | 21 | 28 |
| 4. | 1 | 4 | 10 | 20 | 35 | 56 | 84 |
| 5. | 1 | 5 | 15 | 35 | 70 | 126 | 210 |
| 6. | 1 | 6 | 21 | 56 | 126 | 252 | 462 |
| 7. | 1 | 7 | 28 | 84 | 110 | 462 | 924 |

For by the 464th Article, every Cell is equal to that to the left thereof *plus*, all the others being above it.

Mr. *Paſchal*

Mr. *Paſchal* has written a Treatiſe, entitled, Triangular Arithmetick, wherein the Properties of theſe Numbers are handled, and their great Uſe in many Arithmetical Queſtions is ſhewn.

### COROLLARY.

472. IF you make $a = 1$ and $b = 1$ in the general Series of Article 468. *viz.* $b, a + \frac{n-1}{1} b, a + \frac{n-1}{2} b \times \frac{n}{1}, a + \frac{n-1}{3} b \times \frac{n}{1} \times \frac{n+1}{2}, a + \frac{n-1}{4} b \times \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3}$, *&c.* the ſame will be changed to this, $1, n, \frac{n}{1} \times \frac{n-1}{2}, \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}, \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4}$, by means of which we can at once find any figurate Number requir'd, its Order being given, and the Number expreſſing the Place of it. The Rule is thus:

Take that Term in this laſt Series anſwering to the given Order, *viz.* the third, if it be the third Order; the fourth, if it be the fourth Order, *&c.* and having ſubſtituted the Number expreſſing the Place of the figurate Number for $n$, that is, 4 if it be 4, 5 if it be 5, and ſo one and the ſaid Number will be had. For Example: To find the fifth Number of the fourth Order, ſubſtitute 5 for $n$ in the fourth Term $\frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3}$ of the Series, and 35 will be the Number ſought.

This is only the Application of the Rule in Article 468. to this particular Caſe.

### PROBLEM I.

473. *IT is requir'd to find a general Series orderly expreſſing all the Terms of the Table for dividing of Arcs in the* 443*d Article.*

Becauſe the third Term of any upright Row of that Table, always anſwers to the firſt Term of the Row to the right; therefore if $m + 1$ generally expreſſes the Exponent of an horizontal Row, you muſt find the Coefficient of the Term, whoſe Place is expreſs'd by $m + 1$ in the firſt upright Row; the Coefficient of the Term, whoſe Place is expreſſed by $m + 1 - 2$, or $m - 1$, in the ſecond upright Row; the Coefficient of the Term, whoſe Place is expreſs'd by $m - 1 - 2$, or $m - 3$, in the third upright Row, and ſo on. The Expreſſion of the Place continually leſſening by 2, as the upright Row goes on to the right. Whence, according to the 470th Article, you muſt ſubſtitute the Number $m-1$ for $n$, in the ſecond Term $\frac{n+1}{1}$; the Number

$m-3$

$m-3$ for $n$ in the third Term $\frac{n+3}{2} \times \frac{n}{1}$; the Number $m-5$ for $n$ in the fourth Term $\frac{n+5}{3} \times \frac{n+1}{2}$; the Number $m-7$ for $n$ in the fifth Term $\frac{n+7}{4} \times \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3}$, *&c.* and ſo we ſhall have the following Series of Coefficients, $1$, $m$, $\frac{m}{2} \times \frac{m-3}{1}$, $\frac{m}{3} \times \frac{m-5}{1} \times \frac{m-4}{2}$, $\frac{m}{4} \times \frac{m-7}{1} \times \frac{m-6}{2} \times \frac{m-5}{3}$, *&c.* and becauſe the Signs of any horizontal Row of Terms in the Table are always alternate; and the firſt Term is always the unknown Quantity $x$ raiſed to ſuch a Power, whoſe Exponent is leſs by Unity than that of the horizontal Row; and ſince all the other Terms do contain Powers of $x$, whoſe Exponents continually leſſen by 2, ſuppoſing $x^0 = 1$. Therefore $x^m - m\,x^{m-2} + \frac{m}{2} \times \frac{m-3}{1} x^{m-4} - \frac{m}{3} \times \frac{m-5}{1} \times \frac{m-4}{2} x^{m-6} + \frac{m}{4} \times \frac{m-7}{1} \times \frac{m-6}{2} \times \frac{m-5}{3} x^{m-8}$, *&c.* ſhall be a general Expreſſion for that horizontal Row in the Table, whoſe Index is $m + 1$.

When the firſt Terms of theſe Series are had, you will eaſily perceive the Law that they go on with, and ſo may continue them as far as you pleaſe. For Example, ſuppoſe in this Series, that $r$ expreſſes the Place of the Term, whoſe Coefficient is required, then ſhall the Coefficient be expreſs'd by the general Fraction $\frac{m \times m-r \times m-r-1 \times m-r-2, \&c.}{r-1 \times r-2 \times r-3 \times r-4, \&c.}$ where the Numerator and Denominator muſt each have as many Terms, as the Number $r-1$ contains Units. So if $r = 5$, the Coefficient of the fifth Term will be $\frac{m \times m-5 \times m-6 \times m-7}{4 \times 3 \times 2 \times 1}$: and if $r = 4$, then we ſhall have $\frac{m \times m-4 \times m-5}{3 \times 2 \times 1}$.

Here it muſt be obſerved that the Number of Terms of the aforeſaid Series is always determinate, it being equal to the half (*plus* 1) of the Exponent of the Horizontal Row it expreſſes when that Exponent is odd, and to the half, when the ſame is even. For Example, it will have but three Terms, when it expreſſes the fifth or ſixth horizontal; four, when it expreſſes the ſeventh or eight, *&c.*

## PROBLEM II.

474. *TO find a general Series, orderly expreſſing all the Terms of any horizontal Row, in the Table of Article* 457, *for inſcribing of regular Poligons.*

Becauſe the ſecond Term of every upright Row anſwers to the firſt of that being to the right; therefore if *$m + 1$ be the Exponent of any horizontal Row in that Table, the Coefficients of the four firſt Terms of that Row ſhall be 1, 1, $m - 1$, $m - 2$, the Coefficient of the fifth Term ſhall be the triangular Number whoſe place is expreſſed by $m-3$, *viz.* $\frac{m-3}{1} \times \frac{m-2}{2}$; that of the ſixth Row the triangular Number, whoſe Place is expreſs'd by $m-4$, *viz.* $\frac{m-4}{1} \times \frac{m-3}{2}$; that of the ſeventh Term ſhall be the piramidal Number, whoſe Place is expreſs'd by $m-5$, *viz.* $\frac{m-5}{1} \times \frac{m-4}{2} \times \frac{m-3}{3}$; that of the eighth Term the piramidal Number, whoſe Place is expreſs'd by $m-6$, *viz.* $\frac{m-6}{1} \times \frac{m-5}{2} \times \frac{m-4}{3}$; that of the ninth Term ſhall be a Number of the 5th Order, whoſe Place is expreſs'd by $m - 7$, *viz.* $\frac{m-7}{1} \times \frac{m-6}{2} \times \frac{m-5}{3} \times \frac{m-4}{2} \times$; and ſo on, *ad infinitum.* Now if the proper Powers of $z$ be join'd to theſe Coefficients, and you prefix the Sign — to the ſecond and third Terms; the Sign + to the fourth and fifth; and ſo alternately; then we ſhall get the following general Series, $z^m - z^{m-1} - \overline{m-1}z^{m-2} + \overline{m-2}z^{m-3} + \frac{m-3}{1} \times \frac{m-2}{2} z^{m-4} - \frac{m-4}{1} \times \frac{m-3}{2} z^{m-5} - \frac{m-5}{1} \times \frac{m-4}{2} \times \frac{m-3}{3} z^{m-6} + \frac{m-6}{1} \times \frac{m-5}{2} \times \frac{m-4}{3} z^{m-7} + \frac{m-7}{1} \times \frac{m-6}{2} \times \frac{m-5}{3} \times \frac{m-4}{4} z^{m-8}$, *&c.* which expreſſes all the Terms in that horizontal Row of the Table, in Art. 457. whoſe Exponent is $m + 1$: Where you muſt obſerve to take the ſame Number of Terms, as there are Units contain'd in $m + 1$.

*Art.458.

*Art.472.

## PROBLEM III.

475. *TO find a general Series orderly expreſſing the Coefficients of all the Terms in any horizontal Row of the Table of Article* 460. *(or which is the ſame) of any Power of the Binomial* $x + y$.

Let $m$ in general be the Exponent of any horizontal Row of that Table; then it is manifeſt, that the Coefficients of the two firſt Terms of that Row ſhall be * 1, $m$, and becauſe the ſecond Term of every upright Row (beginning at the ſecond) anſwers to the firſt Term of

*Art.462.

of the Row being to the right, therefore the Coefficient of the third Term of the horizontal Row ſhall * be the triangular Number, whoſe Place is expreſs'd by $m-1$, *viz.* * $\frac{m-1}{1} \times \frac{m}{2}$; that of the fourth Term ſhall be the piramidal Number, whoſe Place is expreſs'd by $m-2$, *viz.* $\frac{m-2}{1} \times \frac{m-1}{2} \times \frac{m}{2}$; that of the fifth Term ſhall be a Number of the fifth Order, whoſe Place is expreſs'd by $m-3$, *viz.* $\frac{m-3}{1} \times \frac{m-2}{2} \times \frac{m-1}{3} \times \frac{m}{4}$; and ſo on *ad infinitum.* Whence the general Series requir'd will be 1, $m$, $\frac{m}{1} \times \frac{m-1}{2}$, $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}$, $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$, *&c.*

*Art.462.

COROLLARY.

476. HENCE $x \mp y^n = x^m \mp \frac{m}{1} y x^{m-1} + \frac{m \times m-1}{1 \times 2} yy x^{m-2} \ldots \mp \frac{m \times m-1 \times m-2}{1 \times 2 \times 3} y^3 x^{m-3} + \frac{m \times m-1 \times m-2 \times m-3}{1 \times 2 \times 3 \times 4} y^4 x^{m-4} \mp \frac{m \times m-1 \times m-2 \times m-3 \times m-4}{1 \times 2 \times 3 \times 4 \times 5} y^5 x^{m-5}$, *&c.*

PROBLEM IV.

477. TO *find a general Equation for dividing a given Arc* (AR) *of a Circle into any Number of equal Parts.*

Let $m$ generally expreſs the Number of equal Parts, and $AD$ the firſt of the equal Parts. Draw the Diameter $AB$, and the Chords $BD$, $BR$; and make $CA$ or $CB = 1$, the given Chord $BR = a$, and the ſought Chord $BD = x$. Then will * $\mp a = x^m - m x^{m-2} + \frac{m \times m-3}{2 \times 1} x^{m-4} - \frac{m \times m-5 \times m-4}{3 \times 2 \times 1} x^{m-6} + \frac{m \times m-7 \times m-6 \times m-5}{4 \times 1 \times 2 \times 3} x^{m-8}$, *&c.* (Where $a$ is affirmative, when the given Arc $AR$ is leſs than half the Circumference, and negative when it is greater) be the general Equation requir'd; wherein you muſt take the Number of Terms equal to the Units contain'd in half the Number $m$ when it is even, or equal to thoſe in its half *plus* 1, when odd; becauſe what follows will be equal to $o$.

FIG. 279.

*Art. 444, and 473.

As ſuppoſe $m$ be $= 5$; then will $a$ be $= x^5 - 5x^3 + 5x$; and if $m = 7$, then $\mp a = x^7 - 7x^5 + 14x^3 - 7x$.

*Another Way.*

Draw the Diameter $AB$, and the Chords $BR$, $AR$, $BD$, $AD$, bounding the given Arc $AR$, and the ſought Arc $AD$. Let the Number of equal Parts be $m$, the Diameter $AB = 1$, the given Chords $BR = a$, $AR = b$, and the unknown Chords $BD = x$, $AD = y$. Then we ſhall * have theſe two general Series $\mp a =$

*Art. 463, 475.

$$x^m - \frac{m \times m-1}{1 \times 2} y\,y\,x^{m-2} + \frac{m \times m-1 \times m-2 \times m-3}{1 \times 2 \times 3 \times 4} y^4 x^{m-4}, \text{ \&c. } b = \frac{m}{1} y\,x^{m-1} - \frac{m \times m-1 \times m-2}{1 \times 2 \times 3} y^3 x^{m-3} + \frac{m \times m-1 \times m-2 \times m-3 \times m-4}{1 \times 2 \times 3 \times 4 \times 5} y^5 x^{m-5}, \text{ \&c.}$$

Wherein ſubſtituting the Number of equal Parts, the Arc $AR$ is to be divided into, for $m$, and we ſhall get two particular Equations, which being reſolved will give us the ſought Value of the Chord $BD$ $(x)$ or $AD$ $(y)$. For Example: If $m = 7$, then will $\mp a = x^7 - 21\,yy\,x^5 + 35y^4x^3 - 7\,y^6x$, and $b = 7\,yx^6 - 35\,y^3x^4 + 12\,y^5\,x\,x - y^7$, be two Equations, which muſt be proceeded with as in *Art.* 462.

## PROBLEM V.

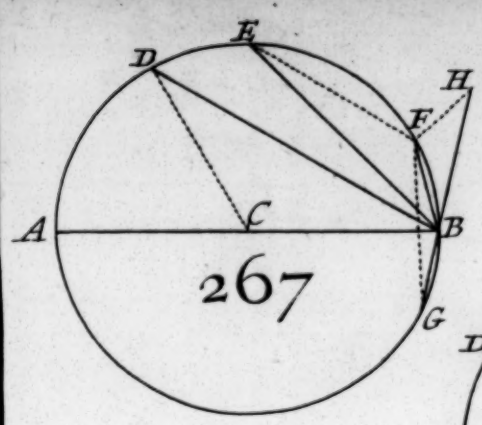
FIG. 280. 478. *TO find a general Equation, for the Inſcription of any regular Polygon* A D E F G H K, *&c. in a Circle.*

Draw the Diameter $AB$, and the Chord $BD$, terminating the firſt Side of the Polygon, and let the given Radius $CA$ or $CB = 1$ the unknown Chord $BD = z$, and in general $m$ be equal to half the Number of Sides of the Polygon, which I ſuppoſe to be odd. Then we ſhall

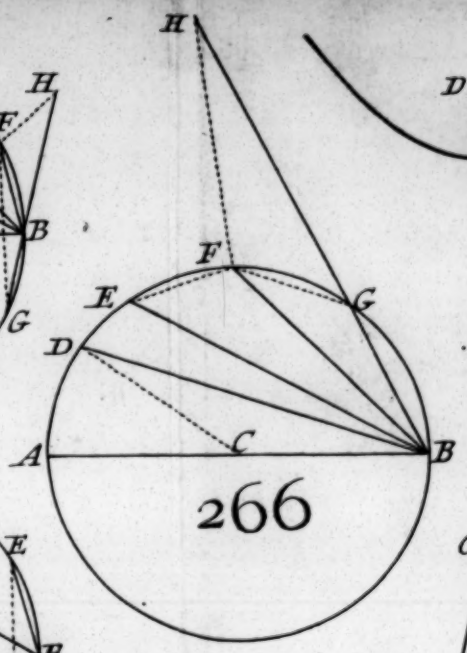
*Art. 457, 474.

have * $o = z^m - z^{m-1} - \overline{m-1}z^{m-2} + \overline{m-2}z^{m-3} + \frac{m-3}{1} \times \frac{m-2}{2} z^{m-4} - \frac{m-4}{1} \times \frac{m-3}{2} z^{m-5} - \frac{m-5}{1} \times \frac{m-4}{2} \times \frac{m-3}{3} z^{m-6} + \frac{m-6}{1} \times \frac{m-5}{2} \times \frac{m-4}{3} z^{m-7} + \frac{m-7}{1} \times \frac{m-6}{2} \times \frac{m-5}{3} \times \frac{m-4}{4} z^{m-8}$, &c. for the general Equation requir'd: of which you muſt take the ſame Number of Terms, as the Number $m + 1$ contains Units; becauſe what follows will be $= o$.

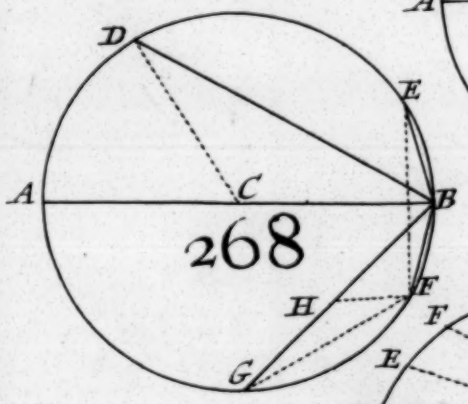
For Example, let 7 be the Number of Sides of the Polygon to be inſcribed, then will $m = 3$, and ſo $o = z^3 - zz - 2z + 1$, and the greateſt Root $(z)$ thereof ſhall expreſs the Chord $BD$, bounding the Arc $AD$, whoſe Chord is $AD$ the firſt Side of the Polygon. In like manner, if the Number of Sides be 11, then will $m = 5$, and ſo the



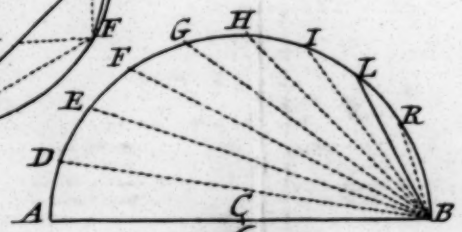
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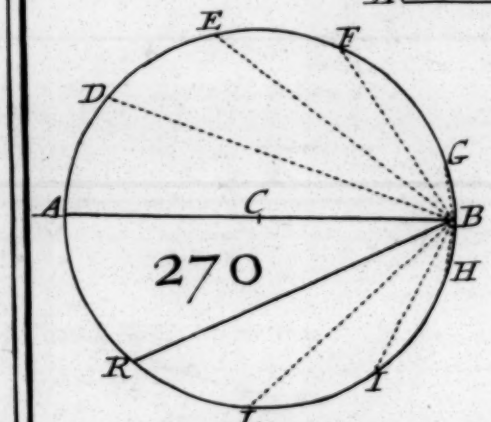
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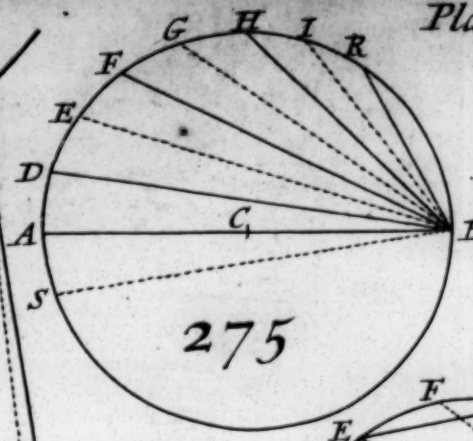
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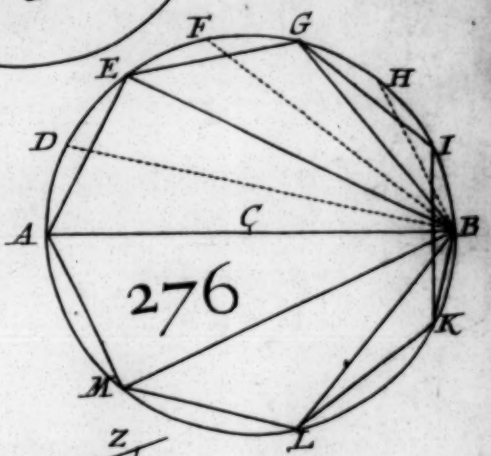
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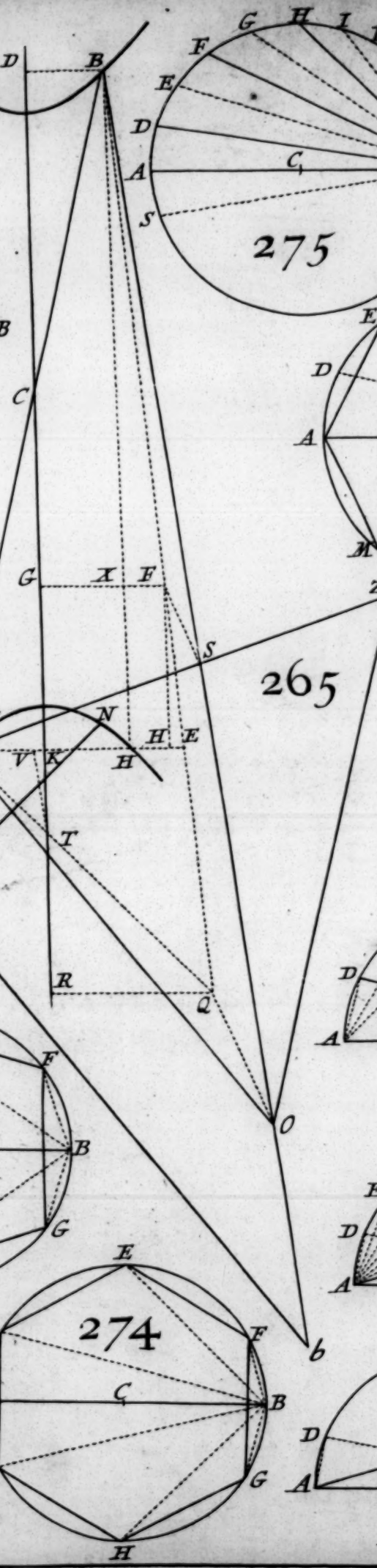
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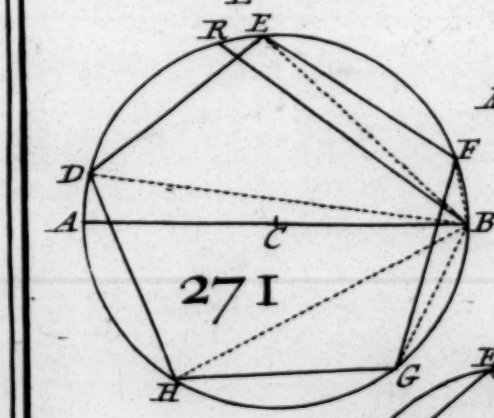
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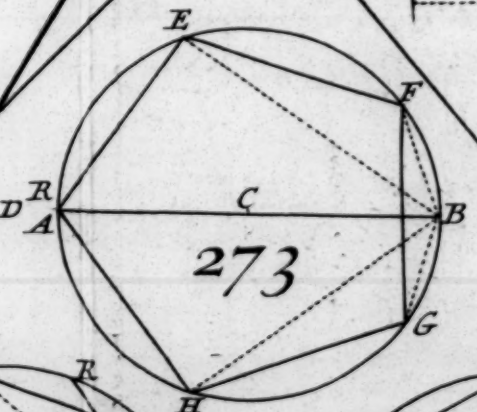
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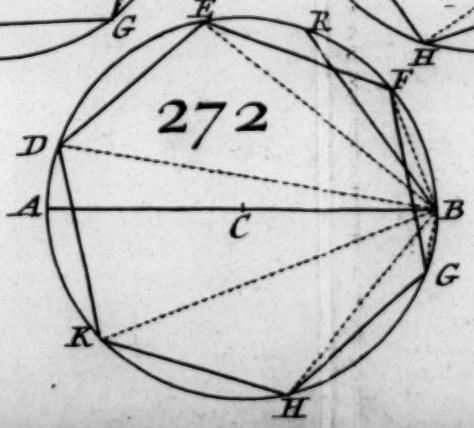
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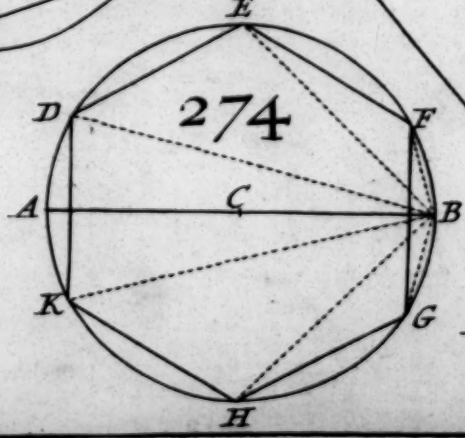
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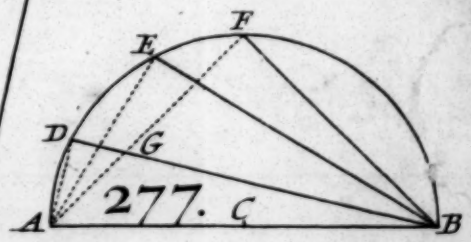
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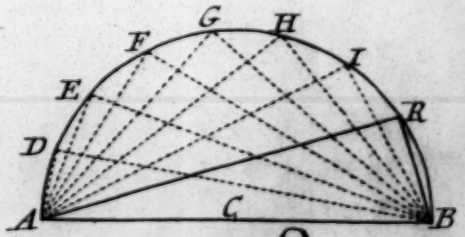
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277.



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the general Equation will become $o = z^5 - z^4 - 4z^3 + 3zz + 3z - 1$, whoſe greateſt Root $z$ is $= BD$.

## PROBLEM VI.

479. *TO divide a given Angle into any odd Number of equal Parts, by means of an Inſtrument.*

1. It is required to divide the given Angle $ECF$ into three equal parts. Take a Rhombus $ABCD$, whoſe four Sides let be moveable about the four Angles, and two of them as $AD$, $AB$, let be indefinitely continued out to $X$ and Z: faſten the Angle $C$ of this Rhombus in the Vertex $C$ of the given Angle $ECF$. And on the Sides $CE$, $CF$, mark the Points $E$, $F$, ſuch, that $CE$, and $CF$ be each equal to any one Side of the Rhombus. This being done, open or ſhut the Sides ($AX$, $AZ$,) of the Angle $BAD$, ſo that they paſs through the Points $E$, $F$; then ſhall the Angle $BAD$ be the third Part of the Angle $ECF$. FIG. 281, 282.

For the Triangles $ABC$, $BCE$ being Iſoſcelles, the external Angle $CBE$ or $CEB$ equal to it, which is equal to the internal and oppoſite Angles $BAC$, $BCA$, ſhall be the double of the Angle $BAC$; and in the Triangle $ECA$, the external Angle $ECT$, which is equal to the Angle $CEA$ *plus* the Angle $BAC$, ſhall be the triple of the Angle $BAC$. After the ſame manner we prove that the Angle $FCT$ is the triple of the Angle $DAC$. Whence the given Angle $ECF$, is triple of the Angle $BAD$. *W. W. D.*

2. It is required to divide the given Angle $HGK$ into five equal Parts. To the Angle $C$ of the aforeſaid Rhombus $ABCD$, faſten another Rhombus $CEGF$, having the Sides equal to thoſe of the other Rhombus, and likewiſe moveable about their Angles. Then fix the Angle $G$ of this latter Rhombus, in the Vertex $G$ of the given Angle $HGK$, and having taken the Parts $GH$, $GK$, in the Sides of the Angle each equal to $GE$ one Side of the Rhombus, open or ſhut the Anlge $XAZ$, moveable about the Point $A$, ſo, that its Sides $AX$, $AZ$, touch the Angle $E$, $F$, and at the ſame time paſs through the Points $H$, $K$. I ſay the Angle $XAZ$ or $BAD$, ſhall be the fifth Part of the given Angle $HGK$. For the Diagonal $AC$ being drawn in the Rhombus $ABCD$, and produced at pleaſure towards $T$; this will paſs through the Point $G$, becauſe the Angle $ECT$, $FCT$, being tripple of the equal Angles $BAC$, $DAC$, ſhall be equal to one another. But in the Triangle $EGA$, the external Angle $HEG$, which is equal to the internal and oppoſite ones $BAC$, $EGA$, or $ECT$ (becauſe $CEG$ is an Iſoſcelles Triangle) ſhall be quadruple of the Angle $BAC$. And therefore in the Triangle $AHG$, the external Angle $HGT$, which is equal to the internal and oppoſite ones $BAC$, FIG. 283, 284.

$GHA$, or $GEH$, (becauſe $EGH$ is an Iſoſcelles Triangle) ſhall be five times the Angle $BAC$. After the ſame manner we prove that the Angle $KGT$ ſhall be five times the Angle $DAC$; whence it is manifeſt that the whole Angle $HGK$ ſhall be five times the whole Angle $BAD$ or $XAZ$.

If a given Angle is to be divided into ſeven equal Parts, then to the two preceeding Rhombus's you muſt join a third, being equal, and made after the ſame manner: and ſo on. For the Operation and Demonſtration will be the ſame.

### EXAMPLE.

480. *TO find any Number of mean Proportionals between two given Lines* a *and* b.

Let $x$ be the firſt of the mean Proportionals ſought, and then we ſhall have this Geometrical Progreſſion $a, x, \frac{xx}{a}, \frac{x^3}{a^2}, \frac{x^4}{a^3}, \frac{x^5}{a^4}$, &c. that Term whereof whoſe place exceeds the Number of mean Proportionals by 2 to be found, muſt be made equal to $b$. And ſo we ſhall get an Equation, which being reſolved, will give us the Value of $x$, the firſt of the mean Proportionals requir'd.

For Example, to find two mean Proportionals between $a$ and $b$. Take the fourth Term $\frac{x^3}{aa}$ and make it equal to $b$, and then $x^3 = aab$; moreover, if four mean Proportionals be required, then will $x^5 = a^4b$. Whence is is manifeſt, that if $n$ generally expreſſes the Number of mean Proportionals to be found between $a$ and $b$, then will $x^{n+1} = a^n b$ be the Equation to be ſolv'd. This being premiſed,

1. Let $x^{16} = a^{16}b$ be an Equation for finding 16 mean Proportionals. Multiply both Sides of this Equation by $x^3$, that ſo $x^{20} = a^{16}bx^3$, where the greateſt Dimenſion 20 is the Product of two Numbers 4 and 5 immediately following one another. Take $x^5 = a^4y$, each Side of which Equation being rais'd to the fourth Power, and then will $x^{20} = a^{16}y^4 = a^{16}bx^3$, from whence we get another Equation $y^4 = bx^3$, whoſe Locus being drawn, as alſo that of the ſuppoſed Equation $x^5 = a^3y$, the Interſection of theſe two Loci will give the Value of $x$. Or elſe take the Equation $x^4 = a^3y$, and raiſe it to the fourth Power, and then multiply it by $x$, and $x^{17} = a^{12}y^4x = a^{16}b$, and ſo $y^4x = a^4b$, and the Locus of this Equation and that of this here $x^4 = a^3y$, will give us the Value of $x$.

2. Let $x^{31} = a^{30}b$, be an Equation for finding 30 mean Proportionals. Multiply both Sides by $x^5$, in order to have $x^{36} = a^{30}bx^5$, whoſe great Dimenſions 36 is the Square of 6: Whence making $x^6 = a^5y$, and raiſing both Sides to the ſixth Power, we get $x^{36} = a^{30}y^6 = a^{30}b^5$, and ſo $x^6 = by^5$, the Locus of which, as alſo that of the aſſum'd Equation

tion being drawn, their Interſection will give the Value of $x$. Or elſe the Equation $x^6 = a^5 y$ being taken as before, raiſe it to the fifth Power, and afterwards multiply it by $x$, and then will $x^{31} = a^{25} y^5 x = a^{10} b$, and ſo $y^5 x = a^5 b$. Whence the Locus of the Equation $x^6 = a^5 y$ being conſtructed, together with the Locus of this $y^5 x = a^5 b$, ſhall ſolve the propoſed Equation $x^{31} = a^{10} b$, ſo that you may take that of the two Loci you judge moſt ſimple. The ſame muſt be underſtood of any other Examples.

Here you muſt obſerve, that if the Dimenſion of the unknown Quantity $x$ be not a prime Number, the propoſed Equation may be always brought lower. For Example: If $x^9 = x^8 b$, be an Equation for finding of mean Proportionals, then by extracting the Cube Root of both Sides, you will have $x^3 = \sqrt[3]{a^8 b}$. But that the Number $a^8 b$ may be a Cube, you muſt find a Line $z$, whoſe Cube $z^3 = a\,a\,b$, or, which is the ſame thing, two mean Proportionals between $a$ and $b$; for if $z^3$ be ſubſtituted for $a\,a\,b$, we ſhall have $x^9 = a^6 z^3$, or $x^3 = a^2 z$, ſo that if this Equation $z^3 = a\,a\,b$, be firſt ſolv'd, and then this $x^3 = a\,a\,z$, we ſhall have the Value of $x$ being the firſt of eight mean Proportionals between $a$ and $b$. In like manner, if $x^{14} = a^{13} b$ be an Equation for finding 13 mean Proportionals between $a$ and $b$, by extracting the ſquare Root of both Sides we ſhall have $x^7 = \sqrt{a^{13} b}$. But that $\sqrt{a^{13} b}$ may be a Square, you muſt find a Line $z$, whoſe Square $zz = a\,b$; for if $zz$ be ſubſtituted for $a\,b$ in the propoſed Equation, then will $x^{14} = a^{12} zz$, or $x^7 = a^6 z$; whence we have only theſe two Equations to reſolve $zz = a\,b$, and $x^7 = a^6 z$.

You muſt obſerve farther, that theſe Equations for finding of mean Proportionals, when the Dimenſion of the unknown Quantity is a prime Number, have but one real Root; becauſe but one Line only can be the firſt of the mean Proportionals ſought.

## SCHOLIUM.

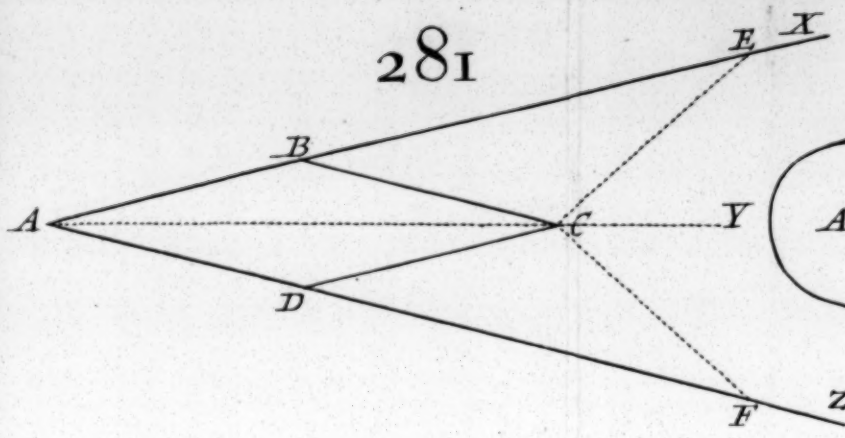
481. THE aforegoing Problem may be ſolv'd by means of an Inſtrument, whoſe Conſtruction is thus. Let there be two indefinite ſtraight Lines $X\Upsilon$, $\Upsilon Z$, moveable about the Point $\Upsilon$, ſo that they may open and ſhut. Then on ſome Point $B$ in the Side $\Upsilon X$, fix the indefinite Perpendicular $BC$, which, at the Point $C$, wherein it cuts the other Side $\Upsilon Z$ during the opening of the Sides $X\Upsilon$, $Z\Upsilon$, moves the indefinite Perpendicular $CD$ along the Side $\Upsilon Z$; and the indefinite Perpendicular $CD$, at the Point $D$, wherein it meets the Side $\Upsilon X$, moves the indefinite Perpendicular $DE$ along the Side $X\Upsilon$; and this laſt Perpendicular, at the Point $E$, wherein it meets the Side $\Upsilon Z$, moves the indefinite Perpendicular $EF$ along the Side $\Upsilon Z$; and this laſt Perpendicular at the Point $F$, wherein it meets the Side $\Upsilon X$, moves the indefinite Perpendicular $FG$ along the Side $\Upsilon X$; and this Perpendicular $FG$ at the Point $G$, wherein it cuts the Side $\Upsilon Z$, moves the FIG. 285.

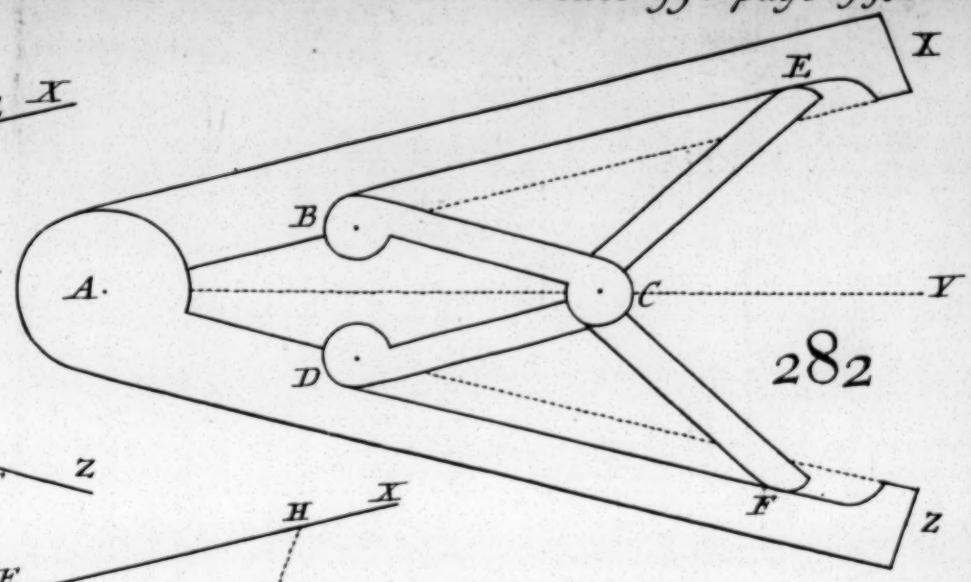
the indefinite Perpendicular $GH$ along the Side $TZ$; and ſo on *ad infinitum*, augmenting the Number of Perpendiculars on the Sides $TX$ and $TZ$. Now let it be requir'd to find four mean Proportionals between the two given Lines $a$ and $b$. In the Side $TZ$, take $TG$ a fourth Proportional to the three Lines $a, b, TB$, and open the Side $XT$ of the Inſtrument, until the fifth Perpendicular $FG$ (becauſe four mean Proportionals are ſought) paſſes through the Point $G$; and then the Lines $TC, TD, TE, TF$, ſhall be the four mean Proportionals between $TB, TG$; and therefore the fourth Proportional to the Lines $TB, TC, a$, ſhall be the firſt of the four mean Proportionals ſought.

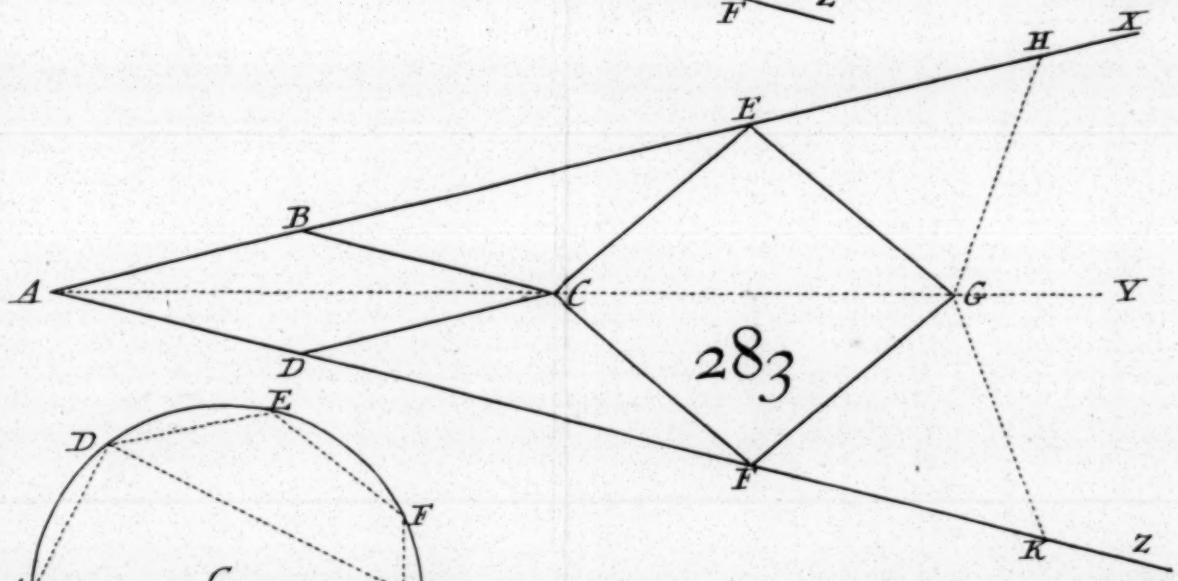
For the right-angl'd Triangles $TBC$, $TCD$, $TDE$, $TEF$, $TFG$, being all ſimilar; their Sides $TB, TC, TD, TE, TF, TG$ ſhall be continually proportional. Whence, *&c.*

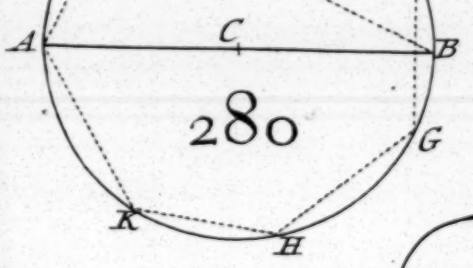
It is manifeſt, that while the Angle $XTZ$ is opening, the Point $B$ deſcribes ($AB$) the Arc of a Circle; and the continual Interſections ($D, F, H$,) of the Perpendiculars $CD, EF, GH$, with the Side $TX$, do deſcribe the Curve Lines $AD, AF, AH$, by means of which may be found any Number of mean Proportionals. For Example: If a Semicircle be deſcrib'd on the Diameter $TE$, it will cut the Curve $AD$ in the Point $D$, being ſuch that $TD$ is the ſecond of two mean Proportionals $TB$ or $TA$ and $TE$; and moreover, if a Semicircle be deſcrib'd on the Diameter $TG$, it ſhall cut the Curve $AF$ in the Point $F$ ſuch, that $TE$ is the laſt of four mean Proportionals between $TA$ and $TG$, &c. Whence you muſt obſerve, that the Curve $AD$ is of the fourth Degree, the Curve $AF$ of the eighth, the Curve $AH$ of the ſixteenth, *&c.* which may be proved thus.

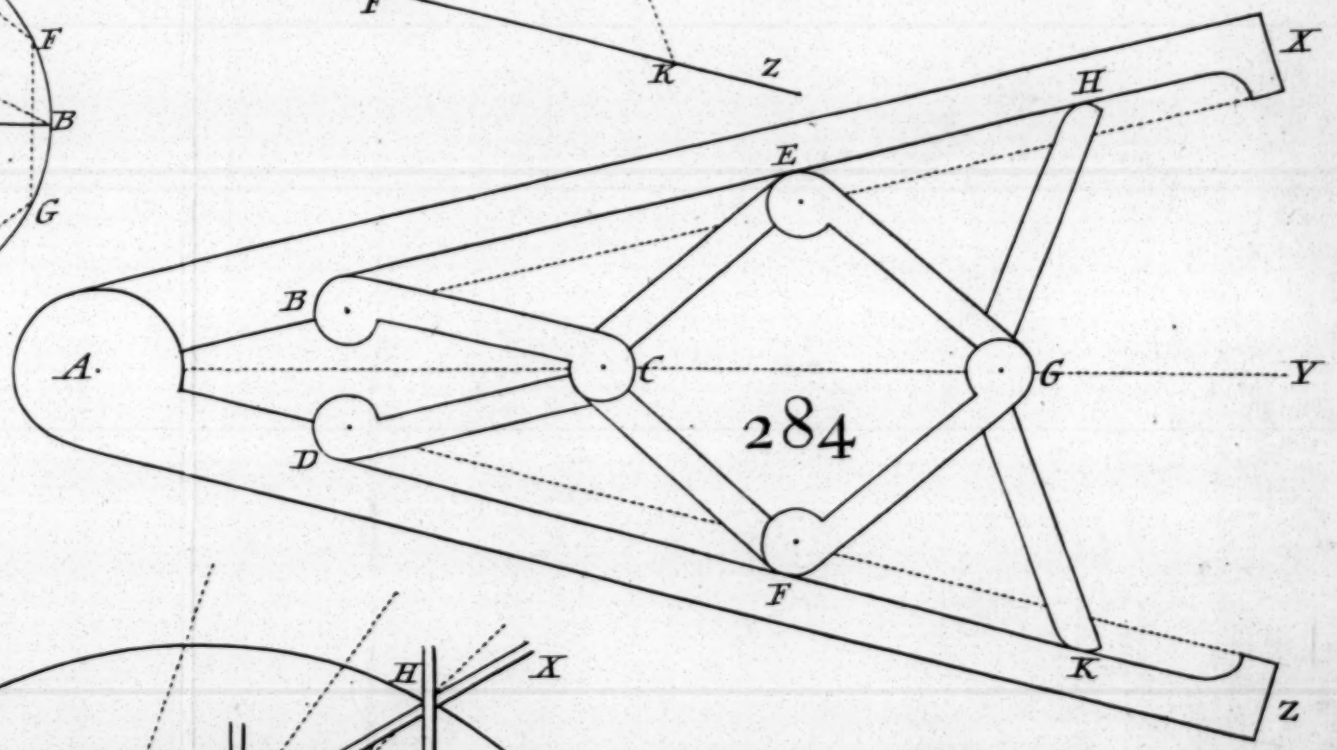
1. Let the unknown and indeterminate Quantities $TC = x$, $CD = y$, $TD = z$, and the known Quantity $TA$ or $TB = a$; then becauſe the right-angl'd Triangles $TCD$, $TBC$, are ſimilar, we ſhall have this Equation, $TB\ (a) = \frac{xx}{z}$, and becauſe $TCD$ is a right-angl'd Triangle, therefore we have this ſecond Equation $yy + xx = zz$, in which ſubſtituting $\frac{xx}{a}$ for $z$, and then will $aayy = x^4 - aaxx$, which ſhews that the Curve $AD$ is of the fourth Degree. 2. Let the unknown and indeterminate Quantities be $TE = x$, $EF = y$, $TF = z$, and the known Quantity $TA$ or $TB = a$; then becauſe the right-angl'd Triangles $TFE$, $TED$, $TDC$, $TCB$, are ſimilar, we have this Equation $TB\ (a) = \frac{x^4}{z^3}$, and becauſe $TEF$ is a right-angl'd Triangle, we get this ſecond Equation, $yy + xx = zz$, from which getting out the unknown Quantity $z$ by means of the former Equation, and afterwards freeing the Equation from Surds, and then $aay^6 + 3aaxxy^4 + 3aax^4yy + aax^6 = x^8$; therefore the Curve $AF$ is a Locus of the eighth

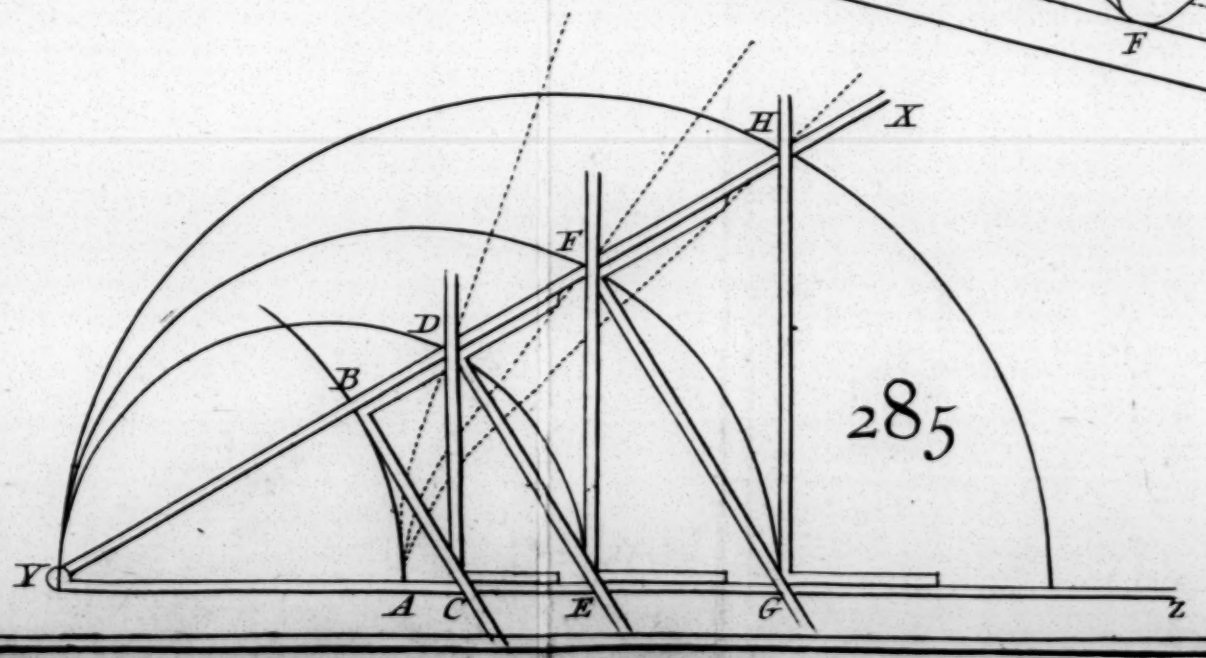

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eighth Degree. After the ſame manner we prove, that the Curve *AH* is a Locus of the ſixteenth Degree, *&c.*

Now ſince according to the Example, we can find two mean Proportionals by only two Lines of the ſecond Degree; four mean Proportionals by a Locus of the ſecond Degree, and another of the third; whereas here, in the firſt Caſe, there is requir'd a Locus of the fourth Degree, being the Line *AD*, and a Locus of the ſecond Degree being the Circle *TDE*; and in the ſecond Caſe, a Locus of the eighth Degree, *viz.* the Curve *AF*, and a Locus of the ſecond Degree, *viz.* the Circle *TFG*; therefore theſe Curves *AD*, *AF*, *AH*, are too much compounded for ſolving the Problem, yet the Facility of their Conſtruction and Demonſtration will in ſome meaſure make up the Deficiency.

*F I N I S.*

BRITISH MUSEUM

# ERRATA.

PAGE 7. line 13. for N O read *MO*. p. 8. l. 13. f. $\frac{a\,y}{x\,x}$ r. $\frac{a\,y}{2\,x}$. ibid. l. 20. f. $b \times a$. r. $b \pm a$. p. 9. l. 30. f. *FM* r. *PM*. p. 11. l. 32. dele *not*. p. 14. l. 37. f. *QA* (*u*) : *AL* (*y*) : r. *QA* (*u*) : : *AL* (*y*) : p. 15. l. 1. f. *from* r. *form*. p. 20. l. 12. f. *MPF* r. *MPF*, *MPf*. ibid. l. 20. f. $4cc = xx$ r. $4ccxx$. p. 23. l. 28. f. *becauſe* r. *becauſe there is*. p. 26. l. penult. f. $\frac{2ayy}{ss}$ r. $\frac{aayy}{ss}$. p. 31. l. 23. for $-\overline{OQ}$ r. $-\overline{OQ}^2$. p. 34. l. 29. f. *MS* r. *RS*. p. 35. l. 29. f. *ASC* r. *ACS*. p. 37. againſt l. penult. in Marg. r. *Fig.* 35. p. 43. l. 4. f. *ME* r. *MF*. p. 44. l. penult. for $\pm^1 pt$ r. $\pm\frac{1}{2}pt$. p. 49. l. 6 & 7. f. *NI* r. *Nl*. p. 50. l. 23, and 24. f. *NI* r. *Nl*. p. 52. l. ult. f. *Part* r. *Point*. p. 57. l. 31. f. *Dl* r. *Dd*. p. 66. l. penult. f. *CG* r. *Cg*. p. 67. l. 34. f. *CE*, *CE*, r. *EE*, *EE*. p. 68. l. 19. f. *FE* r. *PE*. p. 78. l. 12. f. *given Point* r. *Point*. ibid. l. 29. f. *AP* r. *AT*. p. 84. l. 36. f. *AE* r. *AF*. p. 85. l. ult. f. *PK* r. *PR*. p. 86. l. 5. f. *intermediate* r. *indeterminate*. ibid. l. 17. f. *nutt* r. *nntt*. p. 95. l. 20. f. *AB* : *DE* r. $\overline{AB}^2 : \overline{DE}^2$. p. 96. l. 26. f. *CB* r. *CD*. p. 99. l. 13. f. *do* r. *does*. p. 101. l. 23. f. *by in*. r. *in the*. p. 125. l. 5. f. $a^{m \times n}$ r. $a^{m+n}$. p. 128. l. 23. f. $\frac{2}{4}a\,a$ r. $\frac{2}{4}a$. p. 141. l. 30. f. *PA* r. *Pa*. p. 143. l. 17. f. *SD*, *CH*. r. *SDE*, *CHK*. p. 145. l. 21. f. *HK* r. *EK*. p. 169. l. 22. f. *multiply'd by* r. *multiplying*. p. 174. l. 27. f. *in from* r. *from*. p. 185. in marg. f. *Fig.* 178. r. *Fig.* 177, 178. p. 208. l. 35. f. $s + t\,p$. r. $S + t = p$. p. 242. l. 37. f. *taken for*. r. *taken*. p. 280. l. 28. f. BH r. EH. p. 321. l. 12. f. *AB* r. *AR*. p. 329. l. 29. f. *the ADEF*. r. *the Arc ADEF*.